Chapter 5

Assembly Line Balancing by Stochastic Programming Approach

5.1 Introduction

Minimization of balancing loss or cost of assignment was the only important consideration for the previous researchers. These methods can be best used in the case of transfer lines because in a transfer line
elements are preferably performed by machines. Assembly lines involving human elements have another pressing problem. “The losses resulting from workers’ variable operation times” is known as System loss (see Ray Wild, 2004) and this loss is perhaps more important than the losses resulting from uneven allocation of work elements to workstations. Consequently, the problem of line design is not only the equal division of work among the stations or the adaptation of tasks to the speed of the workers but also to provide some amount of slackness in each workstation to take care of the variability of the elemental times.

Our objective in this current work is to design an assembly line where dual objectives of minimization of balancing loss and system loss can be met by switching over from the domain of deterministic set-up to the domain of stochastic set-up. We propose an optimization method based on stochastic programming approach for that purpose.

5.2 Notation

\[ E(.) \] statistical expectation operator

\[ K \] number of jobs
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- $N$: number of workstations
- $N(\mu, \sigma^2)$: normal distribution with mean $\mu$ and variance $\sigma^2$
- $\mu_i$: expected task time of $i^{th}$ job
- $\sigma_i^2$: variance task time of $i^{th}$ job
- $\tau_{\alpha_j}$: the upper $\alpha_j$ point of $N(0,1)$
- $t_i$: random task time or assembly time of $i^{th}$ job
- $W_j$: $j^{th}$ workstation
- $a(i,j)$: binary measure taking value 1 for assignment of task $i$ to workstation $j$
- $L_j$: variable idle time of $j^{th}$ workstation
- $N_{\text{min}}$: minimum number of workstation for a given cycle time
- $C$: cycle time
- $C_t$: trial cycle time
- $C_{\text{min}}$: minimum cycle time for a given $K$
- $S_t$: slackness for trial cycle time $C_t$, i.e., $S_t = C - C_t$
- $B$: balancing loss
- $V$: variance of idle times, $L_1, L_2, \ldots, L_N$. 
5.3 Methodology

The main cause of balancing loss, as pointed out earlier, is the uneven allocation of work to different workstations. Generally, to examine the efficiency of an assembly line one uses the concept of balancing loss, B. We propose a completely different approach. Our proposed work is a multi-objective one. We have taken into consideration system loss as well as balancing loss. This system loss arises out of workers' variable operation time (Ray Wild, 2004). But no standard measure has been proposed so far to examine the extent of system loss. We propose to consider a measure for system loss. As we know, system loss arises out of workers' variable operation time, so any configuration where one workstation has high idle time and another workstation has no idle time will create high disruption. In the deterministic set up that will lead us to consider the variance of the idle times (V) as a measure for system loss for the system. The stability of the total system will be maximum when this variance will be minimum (Roy and Khan, 2010). Under stochastic task times the objective of our proposed method is minimization of
5.4 Mathematical Formulation

Let us consider the binary variable \( a(i, j) \) such that

\[
a(i, j) = \begin{cases} 
1 & \text{if } i \in W_j \text{ th task is assigned to } W_j, \\
0 & \text{if } i \notin W_j \text{ th task is assigned to } W_j,
\end{cases}
\]

and is true for \( i = 1, 2, \ldots, K, \quad j = 1, 2, \ldots, N. \)

Then, under the condition that the ith task can be assigned to only one workstation, the following condition must hold for all \( i = 1, 2, \ldots, K. \)

\[
\sum_{j=1}^{N} a(i, j) = 1 \quad (1)
\]

Further, according to precedence constraints if task \( i' \) is to be assigned before assigning task \( i \), that is \( i' \prec i \), then

\[
a(i, j) \leq \sum_{r=1}^{j-1} a(i', r) \quad \forall \quad i' \prec i \quad (2)
\]

Human beings are involved in completion of tasks involved in assembly line. So, depending upon variations in human skills and behavior, the task
The time of each job will become a random variable. Therefore, we should consider both expected time for completion of each job and the extent of variability. Let $\mu_i$ be the expected time for completing the $i^{th}$ job. Then, the expected balancing loss of the system should be

$$E(B) = E\left(\frac{NC - \sum t_i}{NC}\right) \times 100\%,$$

or,

$$Or,(B) = \frac{(NC - \sum \mu_i)}{NC} \times 100\%.$$

At the same time, the measure of system loss should be calculated taking the expectation of the variance of random idle times, i.e. $E(V)$. By definition,

$$V = \frac{1}{N} \sum (L_j - \bar{L})^2$$

$$= \frac{1}{N} \sum \left[ C - \sum_{i=1}^{K} t_i a(i,j) - C + \frac{1}{N} \sum_{i=1}^{K} t_i \right]^2$$

Or,

$$V = \frac{1}{N} \left( \sum_{i=1}^{K} t_i (a(i,j) - \frac{1}{N}) \right)^2$$

Hence the expectation of $V$ can be simplified as

$$E(V) = \frac{1}{N} \sum_{j=1}^{N} E\left( \sum_{i=1}^{K} t_i (a(i,j) - \frac{1}{N}) \right)^2$$

$$- \frac{1}{N} \sum_{j=1}^{N} \left[ \sum_{i=1}^{K} \sum_{i'(i\neq i')} E(t_i t_{i'}) \left( a(i,j) - \frac{1}{N} \right) \left( a(i',j) - \frac{1}{N} \right) + \sum_{i=1}^{K} \sigma_i^2 + \mu_i^2 \right] (a(i,j) - \frac{1}{N})^2$$
Thus, since the task times are random variables, the condition for completion of tasks in a workstation within the assigned cycle time can be best described in terms of chance constraints

\[
Pr. \left[ \sum_{i=1}^{K} t_i a(i, j) \leq C \right] \geq 1 - \alpha_j , \quad \text{where } 0 \leq \alpha_j \leq 1, \; j = 1, 2, \ldots, N.
\]

Equivalently it can be expressed as

\[
Pr. \left[ L_j \geq 0 \right] \geq 1 - \alpha_j
\]
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\[ Pr \left[ \frac{L_j - E(L_j)}{\sqrt{\text{var}(L_j)}} \leq - \frac{E(L_j)}{\sqrt{\text{var}(L_j)}} \right] \geq 1 - \alpha_j \]

or,

\[ 1 - \Phi \left( - \frac{E(L_j)}{\sqrt{\text{var}(L_j)}} \right) \geq \Phi(\tau_{\alpha_j}) \]

or,

\[ \Phi \left( \frac{E(L_j)}{\sqrt{\text{var}(L_j)}} \right) \geq \Phi(\tau_{\alpha_j}) \]

So,

\[ E(L_j) \geq \tau_{\alpha_j} \sqrt{\text{var}(L_j)} \quad (5) \]

But,

\[ E(L_j) = E[C - \sum_{i=1}^{K} t_i a(i,j)] \]

\[ = [C - \sum_{i=1}^{K} E(t_i) a(i,j)] \]

\[ = [C - \sum_{i=1}^{K} \mu_i a(i,j)] \quad (6) \]

and,

\[ \text{Var}(L_j) = E\left( \sum t_i a(i,j) - \sum E(t_i) a(i,j) \right)^2 \]

\[ = E(\sum (t_i - E(t_i)) a(i,j))^2 \]

\[ = \sum E(t_i - E(t_i))^2 a^2(i,j) \]

\[ = \sum a^2(i,j) \text{var}(t_i) \]

\[ = \sum a^2(i,j) \sigma_i^2 \quad (7) \]
Now with the help of equation (6) and equation (7), equation (5) can be rewritten as,

\[ E(L_f) \geq \tau_{\alpha_f} \sqrt{\text{Var}(L_f)} \]

Or, \[ [C - \sum_{i=1}^{K} \mu_i a(i,j)] \geq \tau_{\alpha_f} \sqrt{\text{Var}(L_f)} \]

Or, \[ C \geq \sum_{i=1}^{K} \mu_i a(i,j) + \tau_{\alpha_f} \sqrt{\sum a^2(i,j) \sigma_i^2} \] (8)

Thus, chance constraints regarding cycle time can be reduced to the following deterministic constraints

\[ \sum_{i=1}^{K} \mu_i a(i,j) + \tau_{\alpha_f} \sqrt{\sum a^2(i,j) \sigma_i^2} \leq C \] (9)

Finally, combining (1), (2), (3), (4) and (9) a deterministic problem for stochastic model formulation of the optimization problem can be written as:

\[
\text{Minimize } E(B) = \frac{(NC - \sum \mu_i)}{NC}
\]
Minimize \( E(V) = \frac{1}{N} \sum_{j=1}^{N} \left[ \sum_{i=1}^{K} \mu_i \left( \frac{a(i,j)}{N} \right) \right]^2 + \)
\[
\frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{K} \sigma_i^2 \left( \frac{a(i,j)}{N} \right)^2
\]

Subject to constraints,

(i) \( \sum_{j=1}^{N} a(i,j) = 1 \quad \forall \ i \)

(ii) \( a(i,j) \leq \sum_{i'=1}^{j} a(i',r) \quad \forall \ i' < i \)

(iii) \( a(i,j) = 0,1 \quad \forall \ i, j \)

(iv) \( \sum_{i=1}^{K} \mu_i a(i,j) + \tau a_j \sqrt{\sum a^2(i,j) \sigma_i^2} \leq C \quad \forall \ j \quad (10) \)

To assign equal importance to each workstation we consider \( \alpha_j = \alpha \)
\( \forall \ j \).

Further \( \alpha_j \) may be considered 0.05 for which \( \tau_{\alpha_j} = 1.6449 \). One way of dealing with dual objective is to combine them with weights or priorities.

In that case the reduced objective can be written as:

Minimize
\[
Z = w_1 \left( \frac{(NC-\sum_{i=1}^{N} \mu_i)}{NC} \right) + w_2 \left[ \frac{1}{N} \sum_{j=1}^{N} \left[ \sum_{i=1}^{K} \mu_i \left( \frac{a(i,j)}{N} \right) \right]^2 \right] + \]
\[\sum_{j=1}^{N} \sum_{i=1}^{K} \sigma_i^2 \left( \frac{a(i,j)}{N} \right)^2 \],
where $0 \leq w_1, w_2 \leq 1$, and $w_1 + w_2 = 1$. However, we prefer to sequentially undertake the task of minimization by generating in the first instant feasible solutions under the objective of minimization of $E(B)$ under $w_1 = 1$ and $w_2 = 0$ and then obtaining the final solution by imposing the second objective of minimization of $E(V)$ with $w_1 = 0$ and $w_2 = 1$. To generate the set of feasible solutions we consider a sequential approach of assigning trial cycle time and resulting in slack time, to be assigned to each workstation meeting the optimality condition arising out of the first objective of (10).

5.5 The Algorithm

1. Calculate the theoretical minimum number of workstations, $N_{\text{min}}$, following the formula

$$\sum_{i=1}^{K} \frac{\mu_i}{C} \leq N_{\text{min}} \leq \sum_{i=1}^{K} \frac{\mu_i}{C} + 1$$

2. Calculate minimum cycle time, $C_{\text{min}}$, using the relation, $C_{\text{min}} = \left[\sum_{i=1}^{K} \frac{\mu_i}{N_{\text{min}}} + 1\right]$.

3. Set the cycle time at $C_{\text{min}}$. 
4. Make an attempt to get feasible solution following the algorithm of Roy and Khan (2010) with usual cycle time constraints replaced by (10)(iv).

5. If no feasible solution is obtained then increase $N_{\text{min}}$ by 1 and go to step 3.

6. Within the generated set calculate $E(V)$ for each set and save the $E(V)$ value.

7. Compare the $E(V)$ with the previous value of $E(V)$. If the current $E(V)$ is lower than the previous one then save the current value of $E(V)$.

8. When all the feasible sets are over we get the final solution to the optimization problem.

5.6 Worked Out Example

We consider in Figure 5.1 an assembly line balancing problem from Ray Wild (2004) for the purpose of explaining how the proposed model works. Task number is represented by the figure within a circle.
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This problem is summarized in a tabular form in terms of work elements, immediate predecessor(s), expected task durations and their variability is given in Table 5.1. For this particular problem, following the algorithm, we first get minimum number of workstation with cycle time $C = 35$. $N_{\text{min}}$ works out as 5. So, minimum trial cycle $C_{\text{min}}$ comes out as $C_{\text{min}} = \left[ \sum_{i=1}^{K} \frac{\mu_i}{N_{\text{min}}} + 1 \right]$, i.e. $C_{\text{min}} = 29$. Since Cycle time $C$ is 35, the trial cycle time starts with 29 and goes up to 35. But among the initial trial cycle times as 29, 30, 31, 32, 33, 34, 35 we get the first feasible solution at $C_t = 33$. Thus, our trial cycle times of 33, 34 and 35. The final optimum configuration has been obtained from trial cycle time

![Figure 5.1: Precedence diagram of workstations.](image-url)
as $34$ with slackness $s_t = c - c_t = 1$. This optimum configuration is presented in Table 5.2.

<table>
<thead>
<tr>
<th>Work Element</th>
<th>Immediate Predecessor</th>
<th>Expected Activity Time</th>
<th>Variance of activity time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>6</td>
<td>0.09</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>5</td>
<td>0.0625</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>8</td>
<td>0.16</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>9</td>
<td>0.2025</td>
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<tr>
<td>5</td>
<td>1, 2</td>
<td>5</td>
<td>0.0625</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>4</td>
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</tr>
<tr>
<td>7</td>
<td>3</td>
<td>5</td>
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<td>3</td>
<td>6</td>
<td>0.09</td>
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<td>8</td>
<td>6</td>
<td>0.09</td>
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<td>2</td>
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</tr>
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<td>15</td>
<td>9, 11, 14</td>
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<td>15</td>
<td>10</td>
<td>0.25</td>
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<tr>
<td>17</td>
<td>16</td>
<td>5</td>
<td>0.0625</td>
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<td>15</td>
<td>0.5625</td>
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<td>19</td>
<td>16</td>
<td>10</td>
<td>0.25</td>
</tr>
<tr>
<td>20</td>
<td>17</td>
<td>5</td>
<td>0.0625</td>
</tr>
<tr>
<td>21</td>
<td>18, 19, 20</td>
<td>6</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Table 5.1: Precedence relation and task times of work elements.

<table>
<thead>
<tr>
<th>C</th>
<th>Work Station 1</th>
<th>Work Station 2</th>
<th>Work Station 3</th>
<th>Work Station 4</th>
<th>Work Station 5</th>
<th>E(V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>34</td>
<td>2, 3, 6, 7, 8</td>
<td>1, 4, 5, 11</td>
<td>9, 10, 12, 13, 14</td>
<td>15, 16, 19</td>
<td>17, 8, 20, 21</td>
<td>6.70</td>
</tr>
</tbody>
</table>

Table 5.2: Final Optimum Configuration
This optimum configuration speaks of 5 workstations with work elements 2, 3, 6, 7, 8 assigned to workstation 1, work elements 1, 4, 5, 11 assigned to workstation 2, work elements 9, 10, 12, 13, 14 assigned to workstation 3, in workstation 4 work elements 15, 16, 19 are assigned and work elements 17, 18, 20, 21 assigned to workstation 5 for trial cycle time 34. Finally, the optimum value of $E(V)$ comes out as 6.707.

5.7 Conclusion

A mathematical programming approach is presented here for balancing an assembly line with twin objectives of minimization of balancing loss and system loss. As system loss arises out of variations in human behavioral, stochastic setup is needed for describing the situation, representing of the problem and arriving at the optimum solution of the same. Reduction of the stochastic setup into deterministic constraints has been indicated under normality assumption. A sequential approach has been installed to arrive at the final solution. Our approach being a generic one, it is capable
of solving different line assembly problem with reasonable computation time. Final choice can be made based on optimum number of workstations and minimum value of expected variance of the idle times, proposed herein as a measure of system loss.