Chapter 4

Split Domination Number

4.1 Introduction

The concept of split domination was introduced by Kulli and Janakiram [19]. In this chapter, we obtain some results about the split domination number and its upper bound. In the first section of this chapter, we obtain some results concerning split domination set. In the second section, we characterize the class of graphs for which $\gamma_s(G) = p - 2$ and vice versa. The contents of this chapter have been communicated to the "Journal of Applied Mathematics Letters".

4.2 Characterization of Split Dominating Set

In this section, we obtain certain characterizations about the split domination set.
**Definition 4.2.1** A domination set $S$ of a graph $G$ is a split dominating set if the induced subgraph $(V - S)$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set of $G$ and the corresponding set is called as $\gamma_s$-set of $G$.

It may be noted that the split dominating set cannot be defined for complete graphs and hence hereafter by a graph we mean a non complete connected simple graph with $p$ vertices and $q$ edges.

**Proposition 4.2.2**

1. For any graph $G$, $1 \leq \gamma_s(G) \leq p - 2$.

2. $\gamma_s(P_p) = \lfloor p/3 \rfloor, p > 2$ where $P_p$ is a path of length $p - 1$.

3. $\gamma_s(K_{m,n} - \{e\}) = 2$ where $(K_{m,n} - \{e\})$ is the complement of $K_{m,n} - \{e\}$ and $e$ is an edge in $K_{m,n}$.

4. $\gamma_s(mK_2) = 2m - 2, m > 1$, where $mK_2$ is the $m$ copies of $K_2$.

**Theorem 4.2.3** Let $G$ be a graph. Then $\gamma_s(G) = 1$ if and only if there exists only one cut vertex $v$ in $G$ with degree $p - 1$.

**Proof.** Let $\gamma_s(G) = 1$ and $S = \{v\}$ is a $\gamma_s$-set of $G$. One can see that $(V - \{v\})$ is disconnected and $v$ dominates all the other vertices of $G$. Hence $v$ is a cut vertex with degree $p - 1$. Suppose there exists another cut vertex $u$ of degree $p - 1$, then $u$ is adjacent to all the remaining vertices of $G$. In this case, $(V - \{v\})$ is connected, a contradiction to $S = \{v\}$ is a $\gamma_s$-set of $G$. Converse part is obvious. □
This result has the following immediate corollary.

**Corollary 4.2.4** If the graph $G$ has no cut vertices, then $\gamma_s(G) \geq 2$.

**Remark 4.2.5** Converse of Corollary 4.2.4 need not be true as one can see from the graph $G$ given in Figure 4.1. Here $\gamma_s(G) = 4$ and $\{5, 6, 7, 8\}$ is a set of cut vertices.

![Figure 4.1](image)

**Example 4.2.6** The lower bound and strict inequality in Corollary 4.2.4 are attained. For the graph given in Figure 4.2, one can verify that it has no cut vertices and $\gamma_s = 2$.

For the graph given in Figure 4.3, one can verify that it has no cut vertices and $\gamma_s = 3 > 2$.

**Proposition 4.2.7** Let $G_1$ and $G_2$ be two connected graphs and $G_1 \circ G_2$ be the corona of $G_1$ and $G_2$. Then $\gamma(G_1 \circ G_2) = \gamma_s(G_1 \circ G_2) = |V(G_1)|$.

**Proof.** The set of all vertices in $G_1$ is a $\gamma_s$-set for $G_1 \circ G_2$ which is also a $\gamma$-set for $G_1 \circ G_2$ and hence the proposition follows. \qed 

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Theorem 4.2.8 A tree \( T \) has a support adjacent to more than one pendant vertex or \( T \) has a non support if and only if every \( \gamma \)-set of \( T \) is also a \( \gamma_s \)-set of \( T \).

Proof. Let \( S \) be a \( \gamma \)-set of a tree \( T \).

Case (i) Suppose \( T \) has a support \( u \) adjacent to more than one pendant vertex. Then \( u \) must be in \( S \) and so \( S \) is a \( \gamma_s \)-set of \( T \).

Case (ii) Suppose \( T \) has a non support \( v \). Then \( S \) contains either \( v \) or at least one support or a non support adjacent to \( v \). In this case \( (V - S) \) is disconnected and so \( S \) is a \( \gamma_s \)-set of \( T \). Converse part is obvious. \( \Box \)
**Theorem 4.2.9** Let $v$ be a vertex in a graph $G$ with degree $k$ and $\langle N(v) \rangle$ disconnected. Then $\gamma_s(G) \leq p - k$.

**Proof.** Since $v$ is of degree $k$ and $\langle N(v) \rangle$ disconnected, $V - N(v)$ is a split dominating set of $G$. Therefore $|V - N(v)| \geq \gamma_s(G)$ and hence the theorem follows. □

**Theorem 4.2.10** For any $p > 4$, the following are true:

(i) $\gamma_s(C_p) = p - 3$.

(ii) $\gamma_s(P_p) = p - 3$.

**Proof.** One can notice that $\gamma(C_p) = 2$ for $p > 4$. Further any subgraph of $C_p$, induced by more than three vertices is connected and so $\gamma_s(C_p) > p - 4$. But the induced subgraph of three consecutive vertices in $C_p$ is disconnected in $C_p$. Therefore $\gamma_s(C_p) = p - 3$. Similarly we can prove (ii). □

**Remark 4.2.11** Theorem 4.2.10 is not true if $p = 4$. Since $C_4$ is disconnected and $\gamma_s(P_4) = 2 = p - 2 \neq p - 3$.

**Theorem 4.2.12** Let $\bar{G}$ be a connected complement of a bipartite graph $G$ with partition $(X, Y)$ and $|X| \leq |Y|$ and $k \geq 1$ vertices in $X$ are adjacent to all the vertices in $Y$. Then $2 \leq \gamma_s(\bar{G}) \leq |X| + 1 - k$.

**Proof.** Let $x_1, x_2, \ldots, x_k$ in $X$ be the $k$ vertices adjacent to all the vertices in $Y$. Since $\bar{G}$ is connected, $1 \leq k < |X|$. Now one of the vertices in $Y$
together with \( X - \{x_1, x_2, \ldots, x_k\} \) form a split domination set of \( \tilde{G} \). Hence 
\[ |X| - k + 1 \geq \gamma_s(\tilde{G}) \]
Therefore \( \gamma_s(\tilde{G}) \leq |X| + 1 - k \). Also we observe that 
\( \gamma(\tilde{G}) = 2 \) and so \( 2 = \gamma(\tilde{G}) \leq \gamma_s(\tilde{G}) \leq |X| + 1 - k \).

**Proposition 4.2.13** If \( S \) is a \( \gamma_s \)-set of a graph \( G \), then \( V - S \) is a dominating set of \( G \) and hence \( \gamma(G) + \gamma_s(G) \leq p \).

**Proof.** Suppose \( V - S \) is not a dominating set of a graph \( G \), then there exists a vertex \( v \) in \( S \) which is not adjacent to any of the vertices in \( V - S \). Thus \( S - \{v\} \) is a split dominating set of \( G \), a contradiction to the minimality of \( S \). Further \( V - S \) is a dominating set of \( G \) and so \( |V - S| \geq \gamma(G) \). This implies that inequality stated is true. \( \Box \)

**Remark 4.2.14** There are graphs for which the strict inequality holds good in Proposition 4.2.13.

For the graph \( G \) given in Figure 4.4, one can verify that \( p = 9 \), \( \gamma(G) = 2 \), \( \gamma_s(G) = 3 \) and \( \gamma(G) + \gamma_s(G) = 5 < p \).

**Theorem 4.2.15** Let \( G \) be a graph with \( \delta = 1 \). Then \( \gamma(G) + \gamma_s(G) = p \) if and only if \( G = H \circ K_1 \) for some connected graph \( H \).

**Proof.** Suppose that \( \gamma(G) + \gamma_s(G) = p \). Then \( \gamma(G) = \gamma_s(G) = p/2 \). By Theorem 2.2 [27], \( G = H \circ K_1 \) where \( H \) is a connected graph. Converse part is obvious. \( \Box \)
Lemma 4.2.16. Let $G$ be any graph with $\delta > 1$. Then $\gamma(G) + \gamma_s(G) = p$ if and only if $\gamma_s(G) = p - 2$ and $\gamma(G) = 2$.

Proof. Suppose that $\gamma(G) + \gamma_s(G) = p$. If $\gamma_s(G) < p/2$, then $\gamma(G) \leq \gamma_s(G) < p/2$ and so $\gamma(G) + \gamma_s(G) < p$, a contradiction to the assumption. Therefore $p/2 \leq \gamma_s(G) \leq p - 2$ (by Proposition 4.2.2) and so $2 \leq \gamma(G) \leq p/2$.

If $\gamma(G) = p/2$, then $\gamma_s(G) = p/2$ and by Theorem 2.2 [27], $G = C_4$. This implies that $p = 4$, then $\gamma_s(G) = p - 2$ and $\gamma(G) = 2$.

If $p/2 < \gamma_s(G) \leq p - 2$ and so $2 \leq \gamma(G) < p/2$. Claim that $\gamma(G) = \beta_o(G)$. If not $\gamma(G) < \beta_o(G)$. Complement of a $\beta_o(G)$-set is a split dominating set with cardinality $p - \beta_o(G)$. By the assumption $p - \gamma(G) > p - \beta_o(G)$. That is, $\gamma_s(G) > p - \beta_o(G)$ which is a contradiction to the minimality of split dominating set. Hence $\gamma(G) = \beta_o(G)$. So there exists a split dominating $S$ with its complement $V - S$ is an independent $\gamma(G)$-set. To prove that every vertex in $V - S$ is adjacent to every vertex in $S$. If not, there exists a vertex $u$ in $V - S$ which is not adjacent to a vertex $x$ in $S$. We note that $\langle S \rangle$ is connected, otherwise $V - S$ is a split
dominating set with lesser number of elements than in $S$. Clearly $S - \{ x \}$ is a split dominating set which is a contradiction to the minimality of $S$. Hence $G$ contains a complete bipartite spanning subgraph and so $\gamma(G) \leq 2$ and hence $\gamma(G) = 2$ and $\gamma_s(G) = p - 2$. Converse part is obvious. \hfill \Box

**Proposition 4.2.17** If a dominating set $S$ of $G$ is also a split dominating set, then there exists two vertices $v_1, v_2$ in $V - S$ such that $d(v_1, v_2) \geq 2$.

**Proof.** If not, assume that for any two vertices $v_1, v_2$ in $V - S$, $d(v_1, v_2) = 1$. Then $(V - S)$ is connected which is a contradiction to $S$ is a split dominating set of $G$. \hfill \Box

**Remark 4.2.18** There exist graphs with a dominating set $S$ such that the complement $V - S$ contains two vertices with distance greater than or equal to 2, but $S$ need not be a split dominating set. For consider the graph $G$ given in Figure 4.5. One can verify that $S = \{ v_1, v_5 \}$ is a dominating set $v_2, v_4 \in V - S$ and $d(v_2, v_4) = 2$ but $S$ is not a split dominating set of $G$.

![Figure 4.5](image.png)

Figure 4.5:
Proposition 4.2.19 Let $S$ be a $\gamma_s$-set of a graph $G$. If $V - S$ is a split dominating set of $G$, then $\gamma_s(G) \leq p/2$.

Proof. Since $V - S$ is a split dominating set of $G$, $|V - S| \geq \gamma_s(G)$ and hence Proposition follows. \qed

Remark 4.2.20 The converse of the above Proposition 4.2.19 need not be true and the same can be seen from the graph given in Figure 4.6.

In this graph, $S = \{1, 4\}$ is a $\gamma_s$-set with $\gamma_s < p/2$. But $V - S = \{2, 3, 5\}$ is not a split dominating set.
Theorem 4.2.21 If $S$ is an independent $\gamma_s$-set of a graph $G$, then $V - S$ is a split dominating set of $G$.

Proof. Since $S$ is independent and dominating set of $G$, $V - S$ is a dominating set of $G$. Further $(S) = (V - (V - S))$ is disconnected. Hence $V - S$ is a split dominating set of $G$. \qed

Theorem 4.2.22 If $S$ is an independent $\gamma_s$-set of a graph $G$ with $\delta > 1$. Then $V - S$ is a npn-set and $\gamma_s(G) + \gamma_{npn}(G) \leq p$.

Proof. Let $S$ be an independent $\gamma_s$-set of a graph $G$ with $\delta > 1$. Then every vertex in $S$ is adjacent to at least two vertices in $V - S$. Since $V - S$ is a dominating set of $G$ and it has no private neighbours in $S$, $V - S$ is a npn-set of $G$. Hence $|V - S| \geq \gamma_{npn}(G)$ and so $\gamma_s(G) + \gamma_{npn}(G) \leq p$. \qed

Theorem 4.2.23 Let $S$ be a maximal independent set of a graph $G$. Then $V - S$ is a split dominating set of $G$. In particular $\gamma_s(G) + \beta_o(G) \leq p$.

Proof. Let $S$ be a maximal independent set of $G$. Then $V - S$ is a split dominating set of $G$. This gives that $|V - S| \geq \gamma_s(G)$. In particular, if $|S| = \beta_o(G)$, then $p - \beta_o(G) \geq \gamma_s(G)$ and hence $\gamma_s(G) + \beta_o(G) \leq p$. \qed

Example 4.2.24 The upper bound and strict inequality in Theorem 4.2.23 are attained. For consider, the graph $K_p - e$, where $e$ is an edge in $K_p$, $\gamma_s(K_p - e) = p - 2$, $\beta_o(K_p - e) = 2$ and $\gamma_s(K_p - e) + \beta_o(K_p - e) = p$. 78
Further, for the graph $G$ given in Figure 4.7, $\gamma_s(G) = 2$, $\beta_o(G) = 2$ and $\gamma_s(G) + \beta_o(G) = 4 < p$.

Figure 4.7:

**Theorem 4.2.25** For a graph $G$ with $p$ vertices and $q$ edges, $[p/(\Delta + 1)] \leq \gamma_s(G) \leq 2q - p$. If $\gamma_s(G) = 2q - p$, then $G$ is a tree.

**Proof.** Trivially $[p/(\Delta + 1)] \leq \gamma(G) \leq \gamma_s(G) \leq p - 2 = 2(p - 1) - p$. Since $G$ is connected, $q \geq p - 1$. Hence $\gamma_s(G) \leq 2q - p$. By the assumption $\gamma_s(G) = 2q - p$. From these we get that $2q - p \leq p - 2$ so that $q \leq p - 1$. Therefore $q = p - 1$ and so $G$ is a tree. \[\square\]

**Remark 4.2.26** Lower bound, lower strict inequality and upper strict inequality in Theorem 4.2.25 are attained. For the graph given in Figure 4.8, one can verify that $\gamma_s = 4$, $\Delta = 3$, $p = 12$, $[p/(\Delta + 1)] = 3$ and so $[p/(\Delta + 1)] < \gamma_s$.

For the graph given in Figure 4.6 one can verify that $\gamma_s = 2$, $p = 5$, $q = 6$, $\Delta = 3$, $[p/(\Delta + 1)] = 2$, $[p/(\Delta + 1)] = \gamma_s$ and $\gamma_s < 2q - p$. 

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4.3 Particular value for Split Domination Number

In this section we identify certain graphs for which $\gamma_s = p - 2$. For instance $\gamma_s(K_p - e) = p - 2$.

**Theorem 4.3.1** Let $G$ be a graph with $\delta = 1$. Then $\gamma_s(G) = p - 2$ if and only if $G$ is either $P_3$ or $P_4$.

**Proof.** Let $G$ be a graph with $\delta = 1$. Suppose that $\gamma_s(G) = p - 2$. Then $\gamma(G) = \gamma_s(G)$ [19]. So $\gamma(G) = p - 2 \leq p/2$. Therefore $p \leq 4$. Hence $G$ is $P_3$ or $P_4$. Converse part is obvious. $\square$

**Theorem 4.3.2** Let $G$ be a graph with $p \geq 4$, $\delta > 1$ and $\gamma_s(G) = p - 2$. Suppose $S$ is a $\gamma_s$-set of $G$, then every vertex in $V - S$ is adjacent to all the vertices in $S$.

**Proof.** By the assumption on $S$, $|S| = p - 2$ and hence $|V - S| = 2$. Let $V - S = \{x, y\}$. By the Proposition 4.2.13, $V - S$ is a dominating set of $G$. 80
Suppose \( x \) is not adjacent to a vertex \( u \) in \( S \), then \( u \) is adjacent to \( y \). Since the degree of \( u \) is greater than 1, \( u \) is also adjacent to at least one vertex in \( S \). Clearly \( S - \{u\} \) is a split dominating set of \( G \), which contradicts the minimality of \( |S| \). □

In view of Theorem 4.3.2, we have \( \gamma(G) \leq 2 \) and so the following corollaries.

**Corollary 4.3.3** Let \( G \) be a graph with \( p \geq 4, \delta > 1 \). If \( \gamma_s(G) = p - 2 \), then \( \gamma(G) + \gamma_s(G) = p \) or \( p - 1 \).

**Corollary 4.3.4** Let \( G \) be a graph with \( p \geq 4, \delta > 1 \) and \( \gamma_s(G) = p - 2 \), then \( K_{2,p-2} \) is a spanning subgraph of \( G \).

**Remark 4.3.5** Converse of Corollary 4.3.3 need not be true. For the graph \( G \), given in Figure 4.9, \( \gamma(G) = 2, \gamma_s(G) = 3 \) and \( \gamma(G) + \gamma_s(G) = p - 1 \), but \( \gamma_s(G) = 3 \neq p - 2 \).

![Figure 4.9](image)

**Theorem 4.3.6** Let \( S \) be a \( \gamma_s \)-set of a graph \( G \) with \( p \geq 4, \delta > 1 \) and \( \gamma_s(G) = p - 2 \), then every vertex in \( S \) is not adjacent to at most one vertex in \( S \).
Proof. If not, let $u$ in $S$ be not adjacent to two vertices $x$ and $y$ in $S$. Then $V - \{u, x, y\}$ is a split dominating set of $G$ and hence $\gamma_s(G) \leq p - 3$ a contradiction to $\gamma_s(G) = p - 2$.

In view of Theorem 4.3.2 and Theorem 4.3.6 we have the following corollary.

**Corollary 4.3.7** Let $G$ be a graph with $p \geq 4$ and $\delta > 1$. If $\gamma_s(G) = p - 2$, then $\text{diam}(G) = 2$.

**Remark 4.3.8** The converse of Corollary 4.3.7 need not be true. For consider the graph $G$ given in Figure 4.10.

![Figure 4.10](image)

In this graph $G$, $S = \{1\}$ is a $\gamma_s(G)$-set and $\gamma_s(G) = 1 \neq p - 2$ but $\text{diam}(G) = 2$.

**Theorem 4.3.9** Let $S$ be a $\gamma_s$-set of a graph $G$ with $p \geq 4$, and $\delta > 1$. If $\gamma_s(G) = p - 2$, then $\beta_0(G) = 2$.

**Proof.** Let $S$ be a $\beta_0$-set of a graph $G$. If $\beta_0(G) > 2$, then $V - S$ is a split dominating set of $G$ which implies that $\gamma_s(G) < p - 2$, a contradiction. □
Remark 4.3.10 Converse of Theorem 4.3.9 need not be true. For the graph $G$ given in Figure 4.11, $p = 6$, $\delta > 1$ and $\beta_{\delta}(G) = 2$ but $\gamma_{\delta}(G) = 2 \neq p - 2$.

![Figure 4.11:](image)

**Proposition 4.3.11** Let $G$ be a graph with $p \geq 4$ and $\delta > 1$. If $\gamma_{\delta}(G) = p - 2$, then $\bar{G}$ is disconnected.

**Proof.** Let $S$ be a $\gamma_{\delta}$-set of a graph $G$. By the Theorem 4.3.2, every vertex in $V - S$ is adjacent to all the vertices in $S$. Hence $\bar{G}$ is disconnected. \qed

**Remark 4.3.12** Proposition 4.3.11 fails if $\delta = 1$. For example $\gamma_{\delta}(P_{4}) = 2 = p - 2$ but $\bar{P}_{4} = P_{4}$ is connected.

**Theorem 4.3.13** Let $G$ be a graph with $p > 4$ and $\delta > 1$. If $G$ and $\bar{G}$ are connected then $\gamma_{\delta}(G) + \gamma_{\delta}(|\bar{G}|) \leq 2(p - 3)$ and $\gamma_{\delta}(G) \cdot \gamma_{\delta}(|\bar{G}|) \leq (p - 3)^{2}$.

**Proof.** Since $G$ and $\bar{G}$ are connected, in view of Proposition 4.2.2 and Proposition 4.3.11, both $\gamma_{\delta}(G)$ and $\gamma_{\delta}(\bar{G})$ are less than $p - 2$. Therefore $\gamma_{\delta}(G) \leq p - 3$ and $\gamma_{\delta}(|\bar{G}|) \leq p - 3$ and so $\gamma_{\delta}(G) + \gamma_{\delta}(\bar{G}) \leq 2(p - 3)$ and $\gamma_{\delta}(G) \cdot \gamma_{\delta}(|\bar{G}|) \leq (p - 3)^{2}$.

\qed
Remark 4.3.14 The above result obtains a better upper bound for $\gamma_s(G) + \gamma_c(G)$ than the bound obtained in Theorem 12 [19].

Theorem 4.3.15 Let $G$ be a graph with $\delta > 1$ and $p \geq 4$. If $\gamma_s(G) = p - 2$, then $\gamma_s(G) \geq \gamma_c(G)$.

Proof. When $p = 4$, the graph is nothing but $C_4$. Hence $\gamma_s(G) = 2$ and $\gamma_c(G) = 2$. When $p > 4$. Let $S$ be a $\gamma_s$-set of $G$. Then $|S| = p - 2$. Suppose $\langle S \rangle$ is not connected. Since $V - S$ is a dominating set, it is also a split dominating set with $|V - S| < |S|$, which is a contradiction. Hence $\langle S \rangle$ is connected and $S$ is a connected domination set of $G$. Therefore $|S| = \gamma_s(G) \geq \gamma_c(G)$.

Remark 4.3.16 Converse of Theorem 4.3.15 need not be true. Since, for any complete bipartite graph $K_{m,n}$ with $2 < m \leq n$, $\gamma_s(K_{m,n}) = m$ and $\gamma_c(K_{m,n}) = 2$. Hence $\gamma_c(K_{m,n}) < \gamma_s(K_{m,n})$. But $\gamma_s(K_{m,n}) = m \neq p - 2$ where $p = m + n$.

Example 4.3.17 Equality and strict inequality hold good in Theorem 4.3.15.

For the graph given in Figure 4.12, $\gamma_s = 4, \gamma_c = 2$ and $\gamma_s > \gamma_c$.

For the graph given in Figure 4.7, $\gamma_s = \gamma_c = 2$.

Theorem 4.3.18 Let $G$ be a graph with $\delta > 1$ and $p > 4$. If $\gamma_s(G) = p - 2$, then $\gamma_{ns}(G) \leq 2$.

Proof. Let $S$ be a $\gamma_s$-set of a graph $G$. Clearly $|S| = p - 2$ and $|V - S| = 2$. By Theorem 4.2.13, $V - S$ is a dominating set of $G$ and by Theorem 4.3.6 $\langle S \rangle$ is
connected. Thus $V - S$ is a non split dominating set of $G$ and $|V - S| \geq \gamma_{ns}(G)$.

That is $\gamma_{ns}(G) \leq 2$. \hfill \Box

**Example 4.3.19** The upper and lower bounds given in Theorem 4.3.18 are reachable. For the graph $G$ given in Figure 4.12, $\gamma_s = 4 = p - 2$ but $\gamma_{ns} = 2$.

For the graphs given in Figure 4.13, $\gamma_s = 3 = p - 2$ but $\gamma_{ns} = 1$.

**Theorem 4.3.20** Let $G$ be a graph with $\delta > 1$. Then the following equivalent:

(i) $\gamma_s(G) + \gamma(G) = p$

(ii) $\gamma_s(G) = p - 2$ and $\gamma(G) = 2$
(iii) $G$ is $p - 2$ regular graph.

Proof. (i) $\Leftrightarrow$ (ii) Follows from Lemma 4.2.16.

(ii) $\Rightarrow$ (iii) Let $S$ be a $\gamma_s$-set of a graph $G$. Then $|S| = p - 2$ and $|V - S| = 2$. Let $V - S = \{x, y\}$. Since $x$ and $y$ are non-adjacent and by Theorem 4.3.2, $d(x) = d(y) = p - 2$. Now we claim that each vertex in $S$ is of degree $p - 2$. If not there exists a vertex $v \in S$ of degree less than $p - 2$. Therefore there exists at least two vertices say $u$ and $w$ which are not adjacent to $v$. Now clearly $V - \{u, v, w\}$ is a dominating set containing $V - S$ and $\langle \{u, v, w\} \rangle$ is disconnected. Hence $V - \{u, v, w\}$ is a split dominating set of $G$ with cardinality $p - 3$, which is a contradiction to $\gamma_s(G) = p - 2$. Hence $\delta(G) = p - 2$. Since $G$ is non-complete, $\Delta(G) = p - 2$ and so $G$ is $p - 2$ regular.

(iii) $\Rightarrow$ (ii) Since any induced subgraph induced by three or more vertices in $G$ is connected, $\gamma_s(G) \geq p - 2$. Also $\beta_o(G) = 2$. Hence the complement of $\beta_o(G)$-set is a split dominating set with cardinality $p - 2$ and so $\gamma_s(G) \leq p - 2$. From these facts $\gamma_s(G) = p - 2$. Hence the result follows. \qed

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