Graphoidal Graphs

In this chapter we study the properties of the intersection graph $\Omega(\psi)$ where $\psi$ is a graphoidal cover or an acyclic graphoidal cover or a path partition of a unicyclic graph. We also obtain a characterization of complete multipartite graphoidal graphs.

Acharya and Sampathkumar [1] introduced the concept of graphoidal graphs. Since $E(G)$ is obviously a graphoidal cover of any graph $G$, it follows that line graphs are graphoidal. Further Acharya and Sampathkumar [1] verified that all the nine Beineke's forbidden subgraphs of line graphs are graphoidal and hence they conjectured that every graph is graphoidal. Arumugam and Pakkiam [4] have disproved this conjecture and they have obtained a characterization of all bipartite graphoidal graphs (Theorem 1.40). They also characterized $\Omega(\psi)$ where $\psi$ is a minimum graphoidal cover of a tree (Theorem 1.39). Recently Panda and Mohanty [18] considered the more general problem of characterizing intersection graph of a family of internally disjoint paths in a tree (called perfect-vertex graphs or $PV$-graphs). They have characterized $PV$-graphs in terms of finite set of forbidden subgraphs. In this chapter, we study the properties of $\Omega(\psi)$ where $\psi$ is a graphoidal cover or an acyclic graphoidal cover or a path partition of a unicyclic graph. We also characterize complete multipartite graphoidal graphs.
Definition 4.1 A graph $G$ is called a path cover graph if there exists a graph $H$ and an acyclic graphoidal cover $\psi$ of $H$ such that $G \cong \Omega(\psi)$.

Lemma 4.2 $G$ is a path cover graph if and only if $G$ is a graphoidal graph.

Proof Obviously any path cover graph is graphoidal. Conversely let $G$ be a graphoidal graph. Then there exists a graph $H$ and a graphoidal cover $\psi$ of $H$ such that $G \cong \Omega(\psi)$. If there exists a cycle $C$ in $\psi$, we choose an edge $e$ of $C$ and replace $H$ by $H - e$ and $C$ by $C - e$. Repeating this process, we obtain a graph $H_1$ and an acyclic graphoidal cover $\psi_1$ of $H_1$ such that $G \cong \Omega(\psi_1)$ so that $G$ is a path cover graph.

We now proceed to investigate the structure of the path cover graph of a unicyclic graph.

Let $G$ be a unicyclic graph and let $m$ denote the number of vertices of degree at least 3 on $C$. Let $\psi$ be a minimum acyclic graphoidal cover of $G$. Then it follows from Theorem 2.8 that every vertex of degree at least 2 except possibly one vertex is interior to $\psi$. Hence if $P_1$ and $P_2$ are any two paths in $\psi$ and $P_1 \cap P_2 \neq \phi$, then there exists a vertex $v$ such that $\deg v \geq 3$ and $v$ lies on both $P_1$ and $P_2$. Further there exists at most one pair of paths $P_1, P_2$ in $\psi$ such that $P_1$ and $P_2$ have two common vertices of degree at least 3. In fact such a pair of paths exists only when $m \geq 2$ and there exist two paths $P_1$ and $P_2$ in $\psi$ which together cover all the edges of $C$. 
**Theorem 4.3** Let $G$ be a unicyclic graph with unique cycle $C$. Let $m$ denote the number of vertices of degree greater than 2 on $C$ and let $n$ denote the number of vertices of degree 1. Let $\psi$ be a minimum acyclic graphoidal cover of $G$. Then

(i) Number of vertices in $\Omega(\psi) = \begin{cases} 
2 & \text{if } m = 0 \\
 n + 1 & \text{if } m = 1 \\
n & \text{otherwise.}
\end{cases}$

(ii) Number of edges in $\Omega(\psi) = \begin{cases} 
1 & \text{if } m = 0 \\
\sum \binom{\deg v - 1}{2} - 1 & \text{if there exist two paths in } \psi \\
\sum \binom{\deg v - 1}{2} & \text{with two vertices of degree at least 3 in common} \\
\sum \binom{\deg v - 1}{2} - 1 & \text{otherwise.}
\end{cases}$

where the summation is taken over all vertices of degree at least 3.

**Proof** (i) follows from Theorem 2.8.

We now prove (ii). If $m = 0$, then $\Omega(\psi) = K_2$ and the result is trivial.

Suppose $m \geq 1$.

Case i No two paths in $\psi$ have two common vertices of degree at least 3.

Let $v$ be a vertex with $\deg v > 2$. Since every vertex of degree at least 2 except possibly one vertex is interior to $\psi$, $v$ is an internal vertex of
exactly one path in $\psi$ and hence there exist exactly $\deg v - 1$ paths in $\psi$ containing the vertex $v$ and any two of these paths determine an edge in $\Omega(\psi)$. Hence number of edges in $\Omega(\psi) \geq \sum \binom{\deg v - 1}{2}$. Further every edge in $\Omega(\psi)$ is of the form $P_iP_j$ where $P_i, P_j \in \psi$ and $P_i \cap P_j \neq \emptyset$. Hence there exists a vertex $v$ such that $\deg v \geq 3$ and $v$ lies on both $P_i$ and $P_j$ so that edge $P_iP_j$ is counted in $\sum \binom{\deg v - 1}{2}$ and hence the result follows.

Case ii There is one pair of paths $P_i, P_j$ in $\psi$ which have two common vertices of degree at least 3.

Then the edge $P_iP_j$ is counted twice in $\sum \binom{\deg v - 1}{2}$. Hence the result follows.

Theorem 4.4 Let $G$ be a graph having no isolated vertices and let $\psi$ be an acyclic graphoidal cover of $G$. Then $G$ is connected if and only if $\Omega(\psi)$ is connected.

Proof Since every acyclic graphoidal cover is a graphoidal cover, the result follows from Theorem 1.38.

Remark 4.5 If $\psi$ is any minimum acyclic graphoidal cover of a unicyclic graph $K_{1,n} + e$ $(n \geq 3)$, then $\Omega(\psi)$ is a complete graph on $n - 1$ vertices.

We now proceed to characterize $\Omega(\psi)$ where $\psi$ is an acyclic graphoidal cover of a unicyclic graph. Theorem 1.39 gives a characterization of $\Omega(\psi)$ where $\psi$ is a minimum graphoidal cover of a tree. We observe that
this theorem is true for any graphoidal cover of a tree, not necessarily minimum.

**Theorem 4.6** Let $G$ be a connected graph. Then there exists a unicyclic graph $H$ and an acyclic graphoidal cover $\psi$ of $H$ such that $G \cong \Omega(\psi)$ if and only if one of the following holds.

(i) $G$ is a block graph.

(ii) There exists exactly one block $B$ of $G$ which is not complete and $B$ is an edge disjoint union of complete graphs $G_1, G_2, \ldots, G_m$ such that $G_i$ and $G_{i+1}$ have exactly one common vertex for each $i = 1, 2, \ldots, m-1$ and $G_m$ and $G_1$ have exactly one common vertex.

(iii) There exists exactly one block $B$ of $G$ which is not complete and $B$ is the union of two complete graphs $G_1$ and $G_2$ such that $G_1$ and $G_2$ have exactly two common vertices.

**Proof** Suppose $G \cong \Omega(\psi)$ where $\psi$ is an acyclic graphoidal cover of a unicyclic graph $H$. Let $C$ be the unique cycle in $H$. Let $m$ denote the number of vertices of degree at least 3 on $C$.

**Case i** $m = 0$.

Then $G = C$. If $|\psi| = 2$, then $\Omega(\psi) = K_2$ which is a block graph. If $|\psi| = 3$, then $\Omega(\psi) = K_3$ which is also a block graph. If $|\psi| \geq 4$, then $\Omega(\psi) = C_n$ ($n \geq 4$) which is an edge disjoint union of $K_2$ satisfying (ii).

**Case ii** $m = 1$.
Let \( v \) be the unique vertex of degree greater than 2 on \( C \). Then there exists a path \( P_1 \) in \( \psi \) such that \( v \) is an end vertex of \( P_1 \) and \( E(P_1) \subseteq E(C) \).

If \( P_1 = (u, v) \), then let \( H_1 = H - e \) where \( e \) is the edge \( uv \). Otherwise let \( H_1 \) be the subgraph of \( H \) obtained by deleting all internal vertices of \( P_1 \). Then \( H_1 \) is a tree and \( \psi \setminus \{P_1\} \) is an acyclic graphoidal cover of \( H_1 \).

Hence by Theorem 1.39, \( \Omega(\psi \setminus \{P_1\}) \) is a block graph. Let \( B \) be the block in \( \Omega(\psi \setminus \{P_1\}) \) consisting of all paths in \( \psi \setminus \{P_1\} \) passing through the vertex \( v \). Since \( B \) is complete, \( B \cup \{P_1\} \) forms a block in \( \Omega(\psi) \) and since \( P_1 \) is adjacent to every vertex in \( B \), \( B \cup \{P_1\} \) is complete. Hence \( \Omega(\psi) \) is a block graph.

Case iii \( m \geq 2 \) and there exists a unique pair of paths \( P_1, P_2 \) in \( \psi \) having two common vertices \( v_1, v_2 \) of \( C \) such that \( \deg v_1, \deg v_2 > 2 \).

Let \( S_1 \) and \( S_2 \) denote the set of all paths in \( \psi \) passing through \( v_1 \) and \( v_2 \) respectively. Then the induced subgraph \( \langle S_1 \cup S_2 \rangle \) forms a block in \( \Omega(\psi) \) and this block is the union of two complete graphs \( \langle S_1 \rangle \) and \( \langle S_2 \rangle \) having exactly two common vertices namely \( P_1 \) and \( P_2 \). All the remaining blocks of \( \Omega(\psi) \) are complete.

Case iv \( m \geq 2 \) and any two paths in \( \psi \) have at most one common vertex of degree greater than 2.

Let \( C = (v_1, v_2, \ldots, v_n, v_1) \) and \( v_{i_1}, v_{i_2}, \ldots, v_{i_m} \) be the vertices of degree greater than 2 on \( C \) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \). Let \( G_j \) denote the complete subgraph of \( \Omega(\psi) \) induced by the set of all paths in \( \psi \) passing
through the vertex \( v_j \). Clearly \( G_j \) and \( G_{j+1} \) have exactly one common
vertex for each \( j = 1, 2, \ldots, m - 1 \) and \( G_m \) and \( G_1 \) have exactly one
common vertex so that \( B = G_1 \cup G_2 \cup \cdots \cup G_m \) forms a block in \( \Omega(\psi) \).
All the remaining blocks in \( \Omega(\psi) \) are complete.

We prove the converse by induction on the number of blocks of \( G \).
Suppose \( G \) has exactly one block. If \( G = K_n \) then by Remark 4.5, for any
minimum acyclic graphoidal cover \( \psi \) of the unicyclic graph \( H = K_{1,n+1} + e \),
we have \( G \cong \Omega(\psi) \). Suppose \( G = B \) where \( B \) is a block which is not
complete and \( B \) is an edge disjoint union of complete graphs \( G_1, G_2, \ldots, G_m \) such that \( G_i \) and \( G_{i+1} \) have exactly one common vertex for each
\( i = 1, 2, \ldots, m - 1 \) and \( G_m \) and \( G_1 \) have exactly one common vertex.
Let \( G_i = K_n, i = 1, 2, \ldots, m \). Let \( H \) be the unicyclic graph with cycle
\( C = (u_1, u_2, \ldots, u_m, u_1) \) and \( n_i - 1 \) pendant vertices \( u_{i1}, u_{i2}, \ldots, u_{i(n_i-1)} \)
adjacent to \( u_i \) for each \( i = 1, 2, \ldots, m \). For each \( i = 1, 2, \ldots, m \), let
\[
P_i = \begin{cases} 
(u_{i1}, u_i, u_{i+1}) & \text{if } n_i = 2 \\
(u_{i1}, u_i, u_{i2}) & \text{if } n_i \geq 3.
\end{cases}
\]
Let \( S \) denote the set of all edges of \( H \), not covered by the paths \( P_1, P_2, \ldots, P_m \). Then \( \psi = \{P_1, P_2, \ldots, P_m\} \cup S \) is an acyclic graphoidal cover of
\( H \) and \( \Omega(\psi) \cong B \).

Suppose \( G = B \) where \( B \) is a block which is not complete and \( B \) is the
union of two complete graphs \( G_1 = K_{n_1} \) and \( G_2 = K_{n_2} \) such that \( G_1 \) and
\( G_2 \) have exactly two common vertices. Let \( H \) be the unicyclic graph with
cycle $C = (v_1, v_2, v_3, v_1)$, $n_1 - 1$ pendant vertices $u_1, u_2, \ldots, u_{n_1 - 1}$ adjacent to $v_1$ and $n_2 - 1$ pendant vertices $w_1, w_2, \ldots, w_{n_2 - 1}$ adjacent to $v_2$. Let $P_1 = (u_1, v_1, v_2, w_1)$ and $P_2 = (v_2, v_3, v_1)$. Let $S$ denote the set of all edges of $H$ not covered by $P_1$ and $P_2$. Then $\psi = \{P_1, P_2\} \cup S$ is an acyclic graphoidal cover of $H$ and $\Omega(\psi) \cong B$.

We now assume that the result is true for all graphs with $k$ blocks satisfying the conditions stated in the theorem where $k \geq 1$. Let $G$ be a graph with $k + 1$ blocks satisfying the conditions stated in the theorem. Let $B = K_m$ be a block of $G$ which has exactly one cut vertex $v$ of $G$. Removal of all vertices of $B$ other than $v$ gives a graph $G$ with $k$ blocks. Hence there exists a unicyclic graph $H_1$ with unique cycle $C$ and an acyclic graphoidal cover $\psi_1$ of $H_1$ such that $G_1 \cong \Omega(\psi_1)$. Let $P_1$ be a path in $\psi_1$ corresponding to $v$ and let $P_1 = (v_1, v_2, \ldots, v_n)$. Subdivide the edge $v_{n-1}v_n$ by introducing a new vertex $w$. Let $P = (v_1, v_2, \ldots, v_{n-1}, w, v_n)$. Let $H$ be the unicyclic graph obtained by adjoining the edges $wu_1, wu_2, \ldots, wu_{m-1}$ to $H_1$. Let $P'_i = (w, u_i)$, $i = 1, 2, \ldots, m - 1$.

Now $\psi = (\psi_1 \setminus \{P_1\}) \cup \{P\} \cup \{P'_i \mid i = 1, 2, \ldots, m - 1\}$ is an acyclic graphoidal cover of $H$ and $G \cong \Omega(\psi)$. This completes the induction and the proof.

Remark 4.7 The proof of Theorem 4.6 is constructive. Hence given a graph $G$ satisfying the conditions stated in Theorem 4.6, the proof of the theorem suggests in a straight forward way an algorithm for constructing
Figure 4.1

a unicyclic graph $H$ and an acyclic graphoidal cover $\psi$ of $H$ such that $G \cong \Omega(\psi)$. We illustrate this algorithm with an example.

Example 4.8 Consider the graph $G$ given in Figure 4.1. The block $B_1$ is not complete whereas the other blocks are complete. Also $B_1$ is an edge disjoint union of three complete graphs $G_1 = K_3$, $G_2 = K_2$ and $G_3 = K_4$ such that $G_i$ and $G_{i+1}$ have exactly one common vertex for each $i = 1, 2$ and $G_3$ and $G_1$ have exactly one common vertex. The construction of unicyclic graph $H$ and an acyclic graphoidal cover $\psi$ of $H$ such that $G \cong \Omega(\psi)$ is shown in Figure 4.2.

Theorem 4.9 For a connected graph $G$, there exists a unicyclic graph $H$ with unique cycle $C$ and a graphoidal cover $\psi$ of $H$ with $C$ as a member of $\psi$ such that $G \cong \Omega(\psi)$ if and only if $G$ is a block graph.

Proof If one vertex $v$ on $C$ is internal in some member $P \neq C$ of $\psi$, let $e = uv$ be an edge of $C$ incident with $v$. Otherwise choose $e^*$ to be
any edge of $C$. Now $H - e$ is a tree and $\psi_1 = (\psi - \{C\}) \cup (C - e)$ is a graphoidal cover of $H - e$ and hence $\Omega(\psi_1)$ is a block graph. Also $\Omega(\psi) = \Omega(\psi_1)$ and hence $\Omega(\psi)$ is a block graph. Converse can be easily proved by using induction on the number of blocks of $G$.

**Definition 4.10** A graph $G$ is called a path partition graph if there exists a graph $H$ and a path partition $\psi$ of $H$ such that $G \cong \Omega(\psi)$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure42.png}
\caption{Figure 4.2}
\end{figure}
Remark 4.11 Since $E(G)$ is trivially a path partition of $G$, any line graph is a path partition graph.

We now proceed to investigate the structure of the path partition graph of a tree.

Theorem 4.12 Let $T$ be a tree with $k$ odd vertices and let $\psi$ be a minimum path partition of $T$. Then

(i) Number of vertices in $\Omega(\psi) = \frac{k}{2}$

(ii) Number of edges in $\Omega(\psi) = \sum \left( \left\lfloor \frac{\deg v}{2} \right\rfloor \right)$

where summation is taken over all vertices of degree greater than 2.

Proof (i) follows from Theorem 1.24. Now let $\deg v > 2$. Since $\psi$ is a minimum path partition of $T$, there are exactly $\left\lfloor \frac{\deg v}{2} \right\rfloor$ paths containing $v$ and any two of these paths determine a line in $\Omega(\psi)$. Hence (ii) follows.

Theorem 4.13 Let $G$ be a graph. Then there exists a tree $T$ and a path partition $\psi$ of $T$ such that $G \cong \Omega(\psi)$ if and only if $G$ is a block graph.

Proof Similar to the proof of Theorem 1.39.
Theorem 4.14 Let $G$ be a connected graph. Then there exists a unicyclic graph $H$ and a path partition $\psi$ of $H$ such that $G \cong \Omega(\psi)$ if and only if one of the following holds.

(i) $G$ is a block graph.

(ii) There exists exactly one block $B$ of $G$ which is not complete and $B$ is an edge disjoint union of complete graphs $G_1, G_2, \ldots, G_m$ such that $G_i$ and $G_{i+1}$ have exactly one common vertex for each $i = 1, 2, \ldots, m-1$ and $G_m$ and $G_1$ have exactly one common vertex.

(iii) There exists exactly one block $B$ of $G$ which is not complete and $B$ is the union of two complete graphs $G_1$ and $G_2$ such that $G_1$ and $G_2$ have exactly two common vertices.

**Proof** Similar to that of Theorem 4.6.

Theorem 4.15 Let $G$ be a connected graph with $n$ vertices. Then there exists a complete bipartite graph $K_{2,n+1}$ and a minimum acyclic graphoidal cover $\psi$ of $K_{2,n+1}$ such that $G \cong \Omega(\psi)$ if and only if $G$ is complete.

**Proof** Suppose there is a complete bipartite graph $K_{2,n+1}$ with bipartition $(X, Y)$ where $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, \ldots, y_{n+1}\}$ and a minimum acyclic graphoidal cover $\psi$ of $K_{2,n+1}$ such that $G \cong \Omega(\psi)$. Since $\eta_n(K_{2,n+1}) = n$ and every path in $\psi$ contains both $x_1$ and $x_2$, $\Omega(\psi)$ is complete.

Converse is trivial.
We now proceed to characterize complete multipartite graphs which are graphoidal.

Lemma 4.16 Let $X$ be an independent set of a graph $G$ and let
\[
\bar{q} = \sum_{v \in X} \deg v.
\]
If $\bar{q} > 2p$ then $G$ is not graphoidal.

Proof Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = V \setminus X = \{y_1, y_2, \ldots, y_m\}$. Let $n_j$ denote the number of vertices in $X$ to which $y_j$ is adjacent. Let
\[
k_0 = |\{j \mid 1 \leq j \leq m \text{ and } n_j = 0\}|,
\]
\[
k_1 = |\{j \mid 1 \leq j \leq m \text{ and } n_j = 1\}| \text{ and }
\]
\[
k_2 = |\{j \mid 1 \leq j \leq m \text{ and } n_j \geq 2\}|
\]
so that $k_0 + k_1 + k_2 = m$. Suppose $G$ is graphoidal. Then there exists a graph $H$ and a graphoidal cover $\psi$ of $H$ such that $G \cong \Omega(\psi)$. Let $P_1, P_2, \ldots, P_n$ be the paths corresponding to $x_1, x_2, \ldots, x_n$ and $Q_1, Q_2, \ldots, Q_m$ be the paths corresponding to $y_1, y_2, \ldots, y_m$. Since each $Q_j$ intersects $n_j$ of the mutually vertex disjoint paths $P_1, P_2, \ldots, P_n$, $Q_j$ has $n_j - 2$ internal vertices which are external vertices of $P_1, P_2, \ldots, P_n$ so that $\sum_{n_j \geq 2} (n_j - 2) \leq 2n$. Hence $\sum_{n_j \geq 2} n_j \leq 2n + 2k_2$. Now
\[
\bar{q} = \sum_{n_j} n_j = \sum_{n_j = 1} n_j + \sum_{n_j \geq 2} n_j \leq k_1 + 2n + 2k_2 \leq 2n + 2m = 2p.
\]
Thus $\bar{q} \leq 2p$ which is a contradiction. \[\]
Theorem 4.17 Let \( m_1 \geq 3, n \geq 3 \) and \( m_1 \geq m_2 \geq \cdots \geq m_n \). Then the complete multipartite graph \( G = K_{m_1,m_2,\ldots,m_n} \) is graphoidal if and only if
\[
m_1(m_2 + m_3 + \cdots + m_n) \leq 2(m_1 + m_2 + \cdots + m_n).
\]

Proof Suppose \( K_{m_1,m_2,\ldots,m_n} \) is graphoidal. Let \( (X_1, X_2, \ldots, X_n) \) be an \( n \)-partition of \( G \). Since \( X_1 \) is an independent set of \( G \), by Lemma 4.16
\[
\sum_{v \in X_1} \deg v \leq 2p \quad \text{and hence} \quad m_1(m_2 + m_3 + \cdots + m_n) \leq 2(m_1 + m_2 + \cdots + m_n).
\]

Conversely suppose \( m_1(m_2 + m_3 + \cdots + m_n) \leq 2(m_1 + m_2 + \cdots + m_n) \). Then \((m_1 - 2)k \leq 2m_1\) where \( k = m_2 + m_3 + \cdots + m_n \).

Case i \( m_1 \geq 7 \).

Then \( k \leq 2 \) so that \( m_2 = m_3 = 1 \) and \( n = 3 \). Now
\[
P_{1j} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, \ldots, m_1
\]
\[
P_{21} = (x_{2m_1-3}, x_1, x_3, x_5, \ldots, x_{2m_1-5}, x_{2m_1-1})
\]
\[
P_{31} = (x_{2m_1-3}, x_2, x_4, x_6, \ldots, x_{2m_1-4}, x_{2m_1})
\]
gives a collection \( \psi \) of internally disjoint and edge disjoint paths such that
\[
\Omega(\psi) \cong K_{m_1,1,1}.
\]

Case ii \( m_1 = 5 \) or \( 6 \).

Then \( k \leq 3 \) so that \( n \leq 4 \). Since any induced subgraph of a graphoidal graph is graphoidal we can assume that \( k = 3 \) and \( m_1 = 6 \) so that
(a) \( m_2 = m_3 = m_4 = 1 \) and \( n = 4 \) or
(b) \( m_2 = 2, m_3 = 1 \) and \( n = 3 \).

If (a) holds, then the required collection of paths is given by
\[
P_{1j} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, \ldots, 6, \quad P_{21} = (y_5, x_2, x_4, x_6, x_8, x_{12}),
\]
\( P_{31} = (x_6, x_{10}, x_{12}, x_1, x_3, x_7) \) and \( P_{41} = (x_1, x_5, x_7, x_9, x_{11}, x_4) \).

If (b) holds, then the required collection of paths is given by

\[
P_{11} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, \ldots, 6, \quad P_{21} = (y_5, x_2, x_4, x_6, x_8, x_{12}),
\]
\[
P_{22} = (y_3, x_{10}, x_{12}, x_1, x_3, x_7) \quad \text{and} \quad P_{31} = (x_2, x_5, x_7, x_9, x_{11}, x_3).
\]

Case iii \( m_1 = 4 \).

Then \( k \leq 4 \). The possible values of \( m_i \) \( (i \geq 2) \) and \( n \) are

(a) \( m_2 = m_3 = m_4 = m_5 = 1 \) and \( n = 5 \)

(b) \( m_2 = 2, m_3 = 2 \) and \( n = 3 \)

(c) \( m_2 = 2, m_3 = 1, m_4 = 1 \) and \( n = 4 \)

(d) \( m_2 = 3, m_3 = 1 \) and \( n = 3 \).

If (a) holds, then the required collection of paths is given by

\[
P_{ij} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, 3, 4, \quad P_{21} = (y_3, x_2, x_4, x_8),
\]
\[
P_{31} = (x_2, x_6, x_8, x_3), \quad P_{41} = (y_3, x_1, x_3, x_7) \quad \text{and} \quad P_{51} = (x_2, x_5, x_7, x_4).
\]

If (b) holds, then the required collection of paths is given by

\[
P_{ij} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, 3, 4, \quad P_{21} = (y_3, x_2, x_4, x_7),
\]
\[
P_{22} = (y_1, x_6, x_8, x_3), \quad P_{31} = (y_3, x_1, x_3, y_4) \quad \text{and} \quad P_{32} = (y_1, x_5, x_7, y_2).
\]

If (c) holds, then the required collection of paths is given by

\[
P_{ij} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, 3, 4, \quad P_{21} = (y_3, x_2, x_4, x_7),
\]
\[
P_{22} = (y_1, x_6, x_8, x_3), \quad P_{31} = (y_3, x_1, x_3, y_4) \quad \text{and} \quad P_{41} = (x_3, x_5, x_7, x_2).
\]

If (d) holds, then the required collection of paths is given by

\[
P_{ij} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, 3, 4, \quad P_{21} = (y_3, x_2, x_4, x_7),
\]
\[
P_{22} = (y_1, x_6, x_8, y_2), \quad P_{23} = (y_4, x_1, x_3, x_5) \quad \text{and} \quad P_{31} = (x_1, x_5, x_7, y_2).
\]
Case iv \( m_1 = 3 \).

Then \( k \leq 6 \). The possible values of \( m_i \) (\( i \geq 2 \)) and \( n \) are

(a) \( m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = 1 \) and \( n = 7 \)

(b) \( m_2 = m_3 = m_4 = 2 \) and \( n = 4 \)

(c) \( m_2 = m_3 = 2, m_4 = m_5 = 1 \) and \( n = 5 \)

(d) \( m_2 = 2, m_3 = m_4 = m_5 = m_6 = 1 \) and \( n = 6 \)

(e) \( m_2 = 3, m_3 = m_4 = m_5 = 1 \) and \( n = 5 \)

(f) \( m_2 = 3, m_3 = 2, m_4 = 1 \) and \( n = 4 \)

(g) \( m_2 = m_3 = 3 \) and \( n = 3 \).

If (a) holds, then the required collection of paths is given by

\[
P_{1j} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, 3, \quad P_{21} = (y_2, x_2, x_6), \\
P_{31} = (x_6, x_4, y_1), \quad P_{41} = (y_2, x_6, x_1), \quad P_{51} = (x_6, x_1, y_2), \\
P_{61} = (x_6, x_3, y_1) \text{ and } P_{71} = (y_2, x_5, y_1).
\]

If (b) holds, then the required collection of paths is given by

\[
P_{1j} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, 3, \quad P_{21} = (y_2, x_2, x_6), \\
P_{22} = (x_5, x_4, x_1), \quad P_{31} = (y_1, x_6, x_4), \quad P_{32} = (y_2, x_1, x_5), \\
P_{41} = (x_6, x_3, x_1) \text{ and } P_{42} = (y_2, x_5, y_1).
\]

If (c) holds, then the required collection of paths is given by

\[
P_{1j} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, 3, \quad P_{21} = (x_4, x_6, x_2), \\
P_{22} = (y_3, x_1, x_3), \quad P_{31} = (y_3, x_3, x_2), \quad P_{32} = (x_4, x_5, x_1), \\
P_{41} = (x_4, x_2, y_3) \text{ and } P_{51} = (y_3, x_4, y_1).
\]

If (d) holds then the required collection of paths is given by

\[
P_{1j} = (x_{2j-1}, y_j, x_{2j}) \quad j = 1, 2, 3, \quad P_{21} = (x_6, x_3, x_1),
\]
$P_{22} = (x_4, x_5, y_1), P_{31} = (x_4, x_2, x_6), P_{41} = (y_1, x_4, x_6), P_{51} = (y_1, x_6, y_2)$ and $P_{61} = (x_6, x_1, x_4)$.

If (e) holds then the required collection of paths is given by

$P_{1j} = (x_{2j-1}, y_j, x_{2j})$ \(j = 1, 2, 3\), $P_{21} = (y_2, x_2, x_5)$,

$P_{22} = (y_3, x_4, x_1), P_{23} = (y_1, x_6, x_3), P_{31} = (x_3, x_1, x_5)$,

$P_{41} = (x_2, x_3, y_3)$ and $P_{51} = (x_1, x_7, x_5, x_3)$.

If (f) holds then the required collection of paths is given by

$P_{1j} = (x_{2j-1}, y_j, x_{2j})$ \(j = 1, 2, 3\), $P_{21} = (y_2, x_2, x_5)$,

$P_{22} = (y_3, x_4, x_1), P_{23} = (y_1, x_6, x_3), P_{31} = (y_2, x_1, x_6)$,

$P_{32} = (x_2, x_3, y_3)$ and $P_{41} = (x_1, x_5, x_3)$.

If (g) holds, then the required collection of paths is given by

$P_{1j} = (x_{2j-1}, y_j, x_{2j})$ \(j = 1, 2, 3\), $P_{21} = (y_2, x_2, x_5)$,

$P_{22} = (y_3, x_4, x_1), P_{23} = (y_1, x_6, x_3), P_{31} = (y_2, x_1, x_6)$,

$P_{32} = (y_3, x_3, x_2)$ and $P_{33} = (y_1, x_5, x_4)$. 

\[\]