Chapter 4

THE GENERALIZED INACCURACY MEASURE

4.1 Introduction

In this chapter, we discuss a generalization for the concept of inaccuracy, discussed in Chapter three, analogous to the generalization of Shannon's entropy given in Rao (1965), Belzuance et al. (2004) and Nanda and Paul (2006).

Recently Nair and Gupta (2007) has extended the inaccuracy measure defined in equation (2.52) to the truncated situation in the form

\[ I(F, G; t) = \int_{-\infty}^{\infty} f(x) \log \left( \frac{g(x)}{F(t)} \right) dx, \]

and has provided characterization results for some well known life time distributions. Motivated by this, we also look into the problem of characterization of probability distributions, using the functional form of the truncated version of the generalized inaccuracy measure.

4.2 Definition and properties

For a non negative random variable \( X \), admitting an absolutely continuous distribution function, Khinchin (1957) has generalized the Shannon's entropy defined by equation (2.32) in the form,

\[ H_\phi(F) = \int_0^\infty f(x) \phi(f(x)) dx, \quad (4.1) \]

where \( \phi(.) \) is a convex function satisfying the condition \( \phi(1) = 0 \). Analogously inaccuracy measure defined in equation (2.52) can be modified as
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\[ I(F, G) = \int_{0}^{\infty} f(x) \left( \phi \left( g(x) \right) \right) dx. \]  \hspace{1cm} (4.2)

Taking \( \phi \left( g(x) \right) = \frac{1 - g'(x)}{r} \), equation (4.2) reduces to

\[ I_r(F, G) = \frac{1}{r} \int_{0}^{\infty} f(x) \left( 1 - g'(x) \right) dx; \quad r > -1, \quad r \neq 0. \]  \hspace{1cm} (4.3)

Our generalization of inaccuracy measure defined in equation (4.3), conforms to the spirit of the extension given in equation (4.1), provided that \( r \) is so chosen such that \( \frac{1 - g'(x)}{r} \) is a convex function. In equation (2.52), all the experimental events contributing to the evaluation of inaccuracy are assigned equal probabilities. This means that events with high and low probabilities have equal weight. It would be more reasonable to have higher probability for events, which impart more sensitivity to \( I(F, G) \) than those with lesser probabilities. This is achieved through our generalization given in equation (4.3). Note that as \( r \to 0 \), equation (4.3) reduces to equation (2.52).

In Section 4.2, we present some properties of \( I_r(F, G) \). Characterization results for probability distributions in the context of Cox’s proportional hazards model and proportional reversed hazards model are discussed in Section 4.3. In Section 4.4., we consider the generalized inaccuracy measure proposed by Nath (1968) and discuss characterization results for probability distributions using the functional form of this inaccuracy measure.

Properties

(i) The expression for \( I_r(F, G) \) given in equation (4.3) can be decomposed as the sum of two terms in the form

\[ I_r(F, G) = H_r(F) + \frac{1}{r} \int_{0}^{\infty} f(x) \left( f'(x) - g'(x) \right) dx. \]  \hspace{1cm} (4.4)
In the above equation, the first term is the generalization of $H(F)$, the Shannon’s entropy, given in equation (2.32) and the second term reduces to the Kullback-Leibler divergence measure defined by equation (2.47), as $r \to 0$. Hence equation (4.4) enables one to express the inaccuracy measure as the sum of a measure of uncertainty about $F$ and a measure of discrimination between the distributions.

(ii) $I_r(F, G)$ is minimum, when $f(x) = g(x)$. This is immediate since when $f(x) = g(x)$, $I_r(F, G)$ simplifies to $H_r(F)$, which is the minimum value. Further the error term in equation (4.4) is zero and $I_r(F, G)$ reduces to the generalization of Shannon's entropy measure given in Belzuance et al. (2004), namely

$$I_r(F) = \frac{1}{r} \int_0^\infty f(x) \left( 1 - f^r(x) \right) dx = H_r(F).$$

(iii) $I_r(F, G) = 0$ implies $E_f \left( g^r(x) \right) = 1$.

This result follows directly from the definition (4.3).

In many practical situations, complete data may not be observable to the experimenter due to various reasons. For instance in lifetime studies, the interest may center around the life time of a unit after a specified period of time, say $t$. Observing that the probability density function of the random variable $X - t | X > t$ and $Y - t | Y > t$ respectively $\frac{f(t + x)}{F(t)}$ and $\frac{g(t + x)}{G(t)}$, the generalized inaccuracy measure in the truncated situation takes the form

$$I_r(F, G; t) = \frac{1}{r} \int_0^\infty f(x) \left( 1 - \left( \frac{g(x)}{G(t)} \right)^r \right) dx.$$ (4.5)
The generalized inaccuracy measure

For convenience in notation, we denote $I_r(F,G;t)$ by $I_r(t)$ in the sequel. Differentiating equation (4.5) with respect to $t$, and using the definition of hazard rate given in chapter two, one can have the representation

$$1 - r I_r(t) = \frac{h_1(t)h_2'(t) - r I_r'(t)}{rh_2(t) + h_1(t)},$$

(4.6)

where $h_1(t)$ and $h_2(t)$ are the hazard rates associated with $F(x)$ and $G(x)$ respectively. Equation (4.6) reveals that the functional form of $I_r(t)$ can mutually characterize $F(x)$ and $G(x)$.

In reliability studies it may happen that the life time may not be observable beyond a specified time point $t$. Hence the distributions of $X$ and $Y$ may be truncated to the right and one can look at the past life by taking $X$ and $Y$ in the interval $(0, t]$. Here the random variables under consideration are $X^*_t = t - X \mid X < t$ and $Y^*_t = t - Y \mid Y < t$. In this situation the generalized inaccuracy measure simplifies to

$$I_r^*(t) = \frac{1}{r} \int_0^t \frac{f(x)}{F(t)} \left(1 - \left(\frac{g(x)}{G(t)}\right)^r\right) dx; \quad 0 < x < t.$$  

(4.7)

Further, equation (4.7) can also be written as

$$1 - r I_r^*(t) = \frac{\lambda_1(t)\lambda_2'(t) + r I_r'^*(t)}{r \lambda_2(t) + \lambda_1(t)},$$

(4.8)

where $\lambda_1(t)$ and $\lambda_2(t)$ are the reversed hazard rates of $F(x)$ and $G(x)$ respectively, reviewed in Section 2.1.

When $(Y, \bar{G})$ is the proportional hazards model of $(X, F)$, equation (4.6) takes the form
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\[ 1 - rI_r(t) = \frac{\theta^r h_{r+1}^r(t) - rI_r(t)}{(1 + r\theta) h_r(t)}. \] (4.9)

In the context of proportional reversed hazards model considered in Section 2.3, equation (4.8) becomes

\[ 1 - rI_{r}^*(t) = \frac{\theta^r \lambda_{r+1}^r(t) + rI_r^*(t)}{(1 + r\theta) \lambda_r(t)}. \] (4.10)

\( I_r(t) \) defined in equation (4.5) provides a measure of discrimination between \( F(x) \) and \( G(x) \), where the observation made beyond time \( t \). In general, \( F(x) \) and \( G(x) \) need not be depend on each other. However when there is some dependence structure between two distributions one can arrive at certain characterization results for probability distributions. An extensively studied dependence structure is the proportional hazards model, reviewed in Section 2.2. In this situation we have

\[ \bar{G}(x) = \left( F(x) \right)^\theta, \theta > 0. \] (4.11)

When \( (Y, \bar{G}) \) is the proportional hazards model of \((X, \bar{F})\), the expression for \( I_r(F, G; t) \), given in equation (4.5), takes the form

\[ 1 - rI_r(F; t) = \frac{\theta^r}{\left( \bar{F}(t) \right)^{1+r\theta}} \int_t^\infty \left( f(x) \right)^{r+1} \left( \bar{F}(x) \right)^{(\theta-1)} dx. \] (4.12)

We first examine whether \( I_r(F; t) \) uniquely determine the distribution. The answer to the question is in the affirmative, which we given as Theorem 4.1.

**Theorem: 4.1**

Let \( F(x) \) and \( G(x) \) be absolutely continuous distribution functions such that \((Y, \bar{G}) \) is the proportional hazards model of \((X, \bar{F})\). Assume that \( I_r(F; t) \) is increasing in \( t \). Then \( I_r(F; t) \) uniquely determines \( F(t) \).
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Proof

Using equation (4.11) in equation (4.5), we get

\[ 1 - r I_r (F; t) = \frac{\theta^r}{(F(t))^{1+r\theta}} \int_t^{\infty} \left( \frac{\theta}{F(x)} \right)^r f^{r+1}(x) \, dx. \]  \hspace{1cm} (4.13)

Suppose that \( F(x) \) and \( G(x) \) are distribution functions such that

\[ I_r (F; t) = I_r (G; t), \text{ for all } t \geq 0. \]  \hspace{1cm} (4.14)

From equations (4.13) and (4.14)

\[ \left( \frac{\theta}{F(t)} \right)^{r+1} \int_t^{\infty} \left( \frac{\theta}{F(x)} \right)^r f^{r+1}(x) \, dx = \left( \frac{\theta}{G(t)} \right)^{r+1} \int_t^{\infty} \left( \frac{\theta}{G(x)} \right)^r g^{r+1}(x) \, dx, \]

where \( f(x) \) and \( g(x) \) are the probability density functions corresponding to \( F(x) \) and \( G(x) \). Differentiating the above equation with respect to \( t \) and using the definition of hazard rate, we get

\[ \theta^r (h_1(t))^{r+1} - (1 - r I_r (f; t))(1 + r\theta) h_1(t) = \theta^r (h_2(t))^{r+1} - (1 - r I_r (g; t))(1 + r\theta) h_2(t). \]  \hspace{1cm} (4.15)

where \( h_1(t) \) and \( h_2(t) \) are the hazard rates corresponding to \( f(x) \) and \( g(x) \) respectively. To prove \( \overline{F}(t) = \overline{G}(t) \), it is enough to show that \( h_1(t) = h_2(t) \), for all \( t \geq 0 \).

Suppose

\[ h_1(t) > h_2(t) \text{ with } h_i(t) \neq 0; \ i = 1, 2. \]

From equation (4.15) we have

\[ \frac{h_1(t)}{h_2(t)} = \frac{\theta^r (h_1(t))^{r+1} - (1 - r I_r (F; t))(1 + r\theta)}{\theta^r (h_2(t))^{r+1} - (1 - r I_r (F; t))(1 + r\theta)} > 1, \]
or
\[
\theta' \left( h_2(t) \right) - (1-rI_r(G;t))(1+r\theta) > \theta' \left( h_1(t) \right) - (1-rI_r(F;t))(1+r\theta).
\]
Using equation (4.14), we get
\[
h_1(t) < h_2(t),
\]
which is a contradiction. Similarly, we can show that the inequality \( h_1(t) < h_2(t) \) also leads to a contradiction. This gives
\[
h_1(t) = h_2(t).
\]

4.3 Characterization results

In this section, we look into the problem of characterization of probability distributions using the functional form of \( I_r(t) \). First we examine the situation where \( I_r(t) \) is independent of \( t \).

**Theorem: 4.2**

Let \( F(x) \) and \( G(x) \) be absolutely continuous distribution functions and \( I_r(t) \) be as defined in equation (4.5). If \( I_r(t) \) is a positive constant \( \left( < \frac{1}{r} \right) \), then \( F(x) \) is exponential if and only if \( G(x) \) is exponential.

**Proof**

Let \( I_r(t) = c \), where \( c \) is a positive constant with \( c < \frac{1}{r} \) and that \( F(x) \) is the exponential distribution with survival function
\[
\bar{F}(x) = e^{-\lambda x}; \lambda > 0, x > 0.
\]
From equation (4.6) we get
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\[
1 - rc = \frac{\lambda h'_2(t)}{rh_2(t) + \lambda}
\]

or

\[
(1 - rc)(rh_2(t) + \lambda) = \lambda h'_2(t).
\]

Differentiating the above equation with respect to \( t \) we get,

\[
h'_2(t)(\lambda rh_2^{-1}(t) - r(1 - rc)) = 0.
\]

The solution to the above equation is \( h_2(t) = \beta \), where \( \beta \) is a constant.

Hence \( G(x) \) is exponential.

Conversely assume

\[
\overline{G}(x) = \exp(-\beta x), \text{ with } \beta > (1 - rc)^{\frac{1}{r}} > 0.
\]

From equation (4.6) we get

\[
h_1(t) = \frac{r\beta(1 - rc)}{rc + \beta - 1}.
\]

Using the relationship

\[
\overline{F}(x) = \exp \left( - \int h_1(t) \, dt \right),
\]

we get

\[
\overline{F}(x) = \exp \left( - \frac{r\beta(1 - rc)}{rc + \beta - 1} x \right).
\]

From the above expression, \( F(x) \) is exponential.

The following theorem provides a characterization result for the generalized Pareto model in the context of proportional hazards model.
Theorem: 4.3

Let \( F(x) \) and \( G(x) \) be two absolutely continuous distribution functions, \( f(x) \) and \( g(x) \) be the corresponding probability density functions and \( h(t) \) be the hazard rate of \( X \). Assume \( \{Y, G\} \) is the proportional hazards model of \( (X, F) \) then the relationship

\[
1 - rI_r(t) = \beta \left( h(t) \right)^r,
\]

(4.16)

where \( \beta \) is a constant holds if and only if \( F(x) \) is the generalized Pareto distribution with survival function specified by equation (3.4).

Proof

Under the assumptions of the theorem, when \( X \) has generalized Pareto distribution, straight forward computations using equation (4.9), gives

\[
1 - rI_r(t) = \frac{\theta (a+1) \left( b + at \right)^{-r}}{((a+1)(1+r\theta)+ar)}
\]

\[
= \beta \left( \frac{b + at}{a+1} \right)^{-r}, \text{where } \beta = \frac{\theta (a+1)}{((a+1)(1+r\theta)+ar)} \text{ is a constant.}
\]

Observing that for the generalized Pareto model, the hazard rate is \( h(t) = \frac{a+1}{b + at} \), the if part follows.

To prove the converse, differentiating equation (4.16) with respect to \( t \) we get

\[
-rI'_r(t) = r \beta \left( h(t) \right)^{-1} h(t).
\]

(4.17)

Using equations (4.9) and (4.16), equation (4.17) can be reads as

\[
h(t) \left( (1 + r\theta) - \theta' \right) = r \beta \left( h(t) \right)^{-1} h(t)
\]

or

\[
\frac{h(t)}{\left( h(t) \right)^2} = \frac{(1 + r\theta) \beta - \theta'}{r \beta}.
\]
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That is,

\[- \frac{d}{dt} \left( \frac{1}{h_1(t)} \right) = \frac{(1+r\theta) \beta - \theta'}{r \beta}.\]

The above equation gives

\[\frac{1}{h_1(t)} = \left( \frac{\theta' - (1+r\theta) \beta}{r \beta} \right) t + c_2\]

or

\[h_1(t) = \frac{1}{c_1 t + c_2},\]

where, \(c_1 = \frac{\theta' - (1+r\theta) \beta}{r \beta}\) and \(c_2^{-1} = h_1(0)\).

Hall and Wellner (1981) have shown that (4.18) is a characteristic property of the generalized Pareto distribution, specified by equation (3.4). The necessary part follows from this result.

Remarks

(i) As \(r \to 0\), the result reduces to the Theorem 3.1 in Nair and Gupta (2007), reviewed in Section 2.8.

(ii) The theorem provides a characterization result for the re-scaled beta distribution specified by

\[\overline{F}(x) = \left( 1 - \frac{x}{R} \right)^c ; 0 < x < R, c, R > 0 ,\]

when \(c = -\left(1 + \frac{1}{a}\right)\) and \(R = -\frac{b}{a}\) in equation (3.4).

Further it may be noted that the uniform distribution arises as a special case when \(c = 1\).

(iii) A characterization result for the Lomax distribution specified by

\[\overline{F}(x) = \beta^c (x + \beta)^{-c} ; x > 0 , \beta > 0 , c > 0 ,\]
The generalized inaccuracy measure can be obtained when $\beta = \frac{a+1}{a}$ and $c = \frac{b}{a}$ in equation (3.4).

A parallel result exists for $I_r'(t)$ which we state as follows.

**Theorem: 4.4**

Let $X$ and $Y$ be non-negative non-degenerate random variables admitting absolutely continuous distribution functions $F(x)$ and $G(x)$ respectively. Further assume that $(Y, G)$ is the proportional reversed hazards model of $(X, F)$. Then the relationship

$$1 - r I_r^*(t) = k\left(\lambda_i(t)\right)^\gamma,$$

(4.19)

where $\lambda_i(t)$ is the reversed hazard rate holds for all $t \geq 0$ if and only if $X$ follow the power distribution with distribution function,

$$F(x) = \left(\frac{x}{b}\right)^c; \quad b \neq 0.$$

(4.20)

**Proof**

By direct calculation using equation (4.10), we get

$$1 - r I_r^*(t) = \frac{c \theta' \left(c/t\right)^\gamma}{(c + (c\theta - 1)r)},$$

which is of the form given in equation (4.19) with

$$\lambda_i(t) = \frac{c}{t} \quad \text{and} \quad k = \frac{c \theta'}{(c + (c\theta - 1)r)},$$

and the if part follows.

To prove the converse, differentiating equation (4.19) with respect to $t$, we get

$$-r I_r''(t) = k r \left(\lambda_i(t)\right)^{-1} \lambda_i(t).$$

(4.21)

Using equations (4.19) and (4.21) in equation (4.10), we have

$$\left(\lambda_i(t)\right)^{-1} \left(\theta' - \left(1 + \theta r\right)k^2 \lambda_i^2(t) - k r \lambda_i(t)\right) = 0.$$
This gives either $\lambda_1(t) = 0$

or

$$\frac{\lambda_1(t)}{(\lambda_1(t))^2} = \frac{\theta^\prime - (1 + r\theta)\beta}{kr}.$$  \hspace{1cm} (4.22)

The former solution is inadmissible, in this situation the distribution function $F$ becomes degenerate.

From equation (4.22) we get

$$\lambda_1(t) = \frac{1}{pt + q}, \hspace{1cm} (4.23)$$

where $p = \frac{(1 + \theta r)k - \theta^\prime}{kr}$ and $q$ is the constant of integration.

As $t \to 0$, $q = 0$. Using the result (2.12), we obtain the random variable $X$ follows power function distribution.

In the context of equilibrium distribution, reviewed in Section 2.4, $I_r(F,G)$ becomes

$$I^E_r = \frac{1}{r} \left(1 - \frac{\mu^r}{r+1}\right), \hspace{1cm} (4.24)$$

where $\mu = E(X)$. The following relationship exists between the generalized inaccuracy measure defined in equation (4.3) and the mean residual life function $m(t) = E(X - t | X > t)$.

That is,

$$I^E_r(t) = \frac{1}{r} \left(1 - \frac{(m(t))^r}{r+1}\right). \hspace{1cm} (4.25)$$

It may be observed that in the context of equilibrium distributions, the knowledge of the mean residual life function of the original distribution
determines the inaccuracy measure uniquely. Hence, the characterization results using the form of the mean residual life function can be suitably translated to the inaccuracy measure.

4.4 An alternative measure of inaccuracy

Nath (1968) defines inaccuracy of order \( r \) as

\[
H_r(P,Q) = \frac{1}{1-r} \log \left( \sum_{x=0}^{\infty} p(x)q^{-r}(x) \right); r \neq 1, r > 0. \tag{4.26}
\]

For the applications of the measure defined in equation (4.26), we refer to Nath (1968).

If \( X \) and \( Y \) are two non-negative random variables admitting absolutely continuous distribution functions \( F(x) \) and \( G(x) \) respectively, then the continuous analogue of equation (4.26) can be taken as

\[
H_r(F,G) = \frac{1}{1-r} \log \left( \int_0^\infty f(x)g^{-r}(x)dx \right); r \neq 1, r > 0. \tag{4.27}
\]

Note that as \( r \to 1 \), equation (4.27) reduces to the inaccuracy measure defined by equation (2.52). In this sense \( H_r(F,G) \) provides a generalization for the inaccuracy measure given in Nath (1968). In the left truncated context, equation (4.27) becomes

\[
H_r(F,G; t) = \frac{1}{1-r} \log \left( \int_t^\infty \frac{f(x)}{F(t)} \left( \frac{g(x)}{G(t)} \right)^{-r} dx \right); r \neq 1, r > 0, t > 0. \tag{4.28}
\]

Further in the right truncated situation, equation (4.27) can be written as

\[
\overline{H}_r(F,G; t) = \frac{1}{1-r} \log \left( \int_0^t \frac{f(x)}{F(t)} \left( \frac{g(x)}{G(t)} \right)^{-r} dx \right); r \neq 1, r > 0, t > 0. \tag{4.29}
\]

For the convenience in the sequel, we denote \( H_r(F,G; t) \) and \( \overline{H}_r(F,G; t) \) by \( H_r(t) \) and \( \overline{H}_r(t) \) respectively. Differentiating equation (4.28) with respect
to \( t \), and rearranging the terms we get a relationship between \( H_r(t) \) and hazard rates namely

\[
(1-r)H_r(t) = h_1(t) + (r-1)h_2(t) - h_1(t)(h_2(t))^{-1}\exp(-(1-r)H_r(t)),
\]

(4.30)

where \( h_1(t) \) and \( h_2(t) \) are the hazard rates associated with the distribution functions \( F(x) \) and \( G(x) \) respectively and \( H_r(t) \) denotes the derivative of \( H_r(t) \). Similarly, from equation (4.29) one can have the representation

\[
(1-r)\bar{H}_r(t) = \lambda_1(t)(\lambda_2(t))^{-1}\exp(-(1-r)\bar{H}_r(t)) - \lambda_1(t) + (1-r)\lambda_2(t),
\]

(4.31)

where \( \lambda_1(t) \) and \( \lambda_2(t) \) are the reversed hazard rates.

When \( \left(Y, \bar{G}\right) \) is the proportional hazards model of \( \left(X, F\right) \), equation (4.30) takes the form

\[
(1-r)\bar{H}_r(t) = (1+(r-1)\theta)h_1(t) - \sigma^{-1}(h_2(t))^{-1}\exp(-(1-r)\bar{H}_r(t)).
\]

(4.32)

Proceeding as similar lines, when \( G \) is the proportional reversed hazards model of \( F \), equation (4.31) can be written as

\[
(1-r)\bar{H}_r(t) = \phi^{-1}(\lambda_1(t))^{-1}\exp(-(1-r)\bar{H}_r(t)) - (1-(1-r)\phi)\lambda_1(t).
\]

(4.33)

**Theorem: 4.5**

For the random variables \( X \) and \( Y \) considered in Theorem 4.4, assume that \( \left(Y, \bar{G}\right) \) is the proportional hazards model of \( \left(X, F\right) \) and \( H_r(t) \) is increasing in \( t \). Then \( H_r(t) \) uniquely determines \( F(t) \).

**Proof**

The proof of the theorem is similar to that of Theorem 4.1 and hence omitted.
Characterization results

The following theorem focuses attention on the situation when $H_r(t)$ is independent of $t$.

**Theorem: 4.6**

Let $X$ and $Y$ be non-negative non-degenerate and absolutely continuous random variables with survival functions $\overline{F}(x)$ and $\overline{G}(x)$ respectively. Assume $(Y, \overline{G})$ is the proportional hazards model of $(X, \overline{F})$. Then the distribution of $X$ is exponential if and only if $H_r(t) = k$, where $k$ is a constant, for every $t > 0$.

**Proof**

Under the assumptions of the theorem, when $X$ follows exponential distribution with survival function

$$\overline{F}(x) = \exp(-\lambda x); \quad x > 0, \lambda > 0,$$

by direct calculation, we get

$$H_r(t) = \frac{1}{r-1} \log \left( 1 + (r-1)\theta \right) - \log(\lambda \theta)$$

$$= k, \text{ independent of } t.$$

Conversely assume that

$$H_r(t) = k$$

From equation (4.32), we have

$$h_i(t) \left[ 1 + (r-1)\theta - \theta^{-1} \exp \left( - (1-r)K \right) h_i^{-1}(t) \right] = 0.$$

This gives either $h_i(t) = 0$ or $h_i(t) = c$, where $c$ is a constant. But the former solution is inadmissible since in this situation $X$ becomes degenerate. From the later solution we conclude that $X$ follows exponential distribution.
In the following theorem, we give a characterization result for a family of distributions using a relationship between $H_r(t)$ and the hazard rate.

**Theorem: 4.7**

Let $X$ be a non-negative random variable with absolutely continuous distribution function $F(x)$ and let $(Y, G)$ is the proportional hazards model of $(X, \overline{F})$. The relationship

\[ H_r(t) = k(\theta) - \log h_i(t), \quad (4.34) \]

where $k(\theta)$ is a real valued function independent of $t$ and $h_i(t)$ is the hazard rate of $X$ holds for every $t > 0$ if and only if $X$ follows any one of the following three distributions.

(i) the exponential distribution with survival function

\[ \overline{F}(x) = \exp(-\lambda x); \quad x \geq 0, \lambda > 0 \quad (4.35) \]

(ii) the Pareto distribution with survival function

\[ \overline{F}(x) = \left( \frac{a}{a+x} \right)^b; \quad x \geq 0, b > 1, 0 < a < \infty \quad (4.36) \]

(iii) the Beta distribution with survival function

\[ \overline{F}(x) = \left( 1 - \frac{x}{R} \right)^c; \quad 0 < x < R, c > 1. \quad (4.37) \]

**Proof**

Assume that equation (4.34) holds and differentiating with respect to $t$, we get

\[ H'_r(t) = -\frac{h_i(t)}{h_i(t)}. \quad (4.38) \]
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Substituting equation (4.38) in (4.32) we get

\[-(1-r) \frac{h'_t(t)}{h_t(t)} = (1+(r-1)\theta) - \theta^{r-1} \exp(-(1-r)k)\]

or

\[\frac{d}{dt} \left(\frac{1}{h_t(t)}\right) = \frac{1}{1-r} \left(1+(r-1)\theta - \theta^{r-1} \exp(-(1-r)k)\right).\]

This gives

\[\frac{1}{h_t(t)} = \left(\frac{1+(r-1)\theta - \theta^{r-1} \exp(-(1-r)k)}{1-r}\right) t + c, \quad (4.39)\]

where \(c\) is the constant of integration. Equation (4.39) takes the form

\[h_t(t) = (pt + c)^{-1}, \quad (4.40)\]

where

\[p = \frac{1+(r-1)\theta - \theta^{r-1} \exp(-(1-r)k)}{1-r}.\]

From Mukharjee and Roy (1986), equation (4.40) characterizes the exponential distribution for \(p = 0\), the Pareto distribution for \(p > 0\) and the beta distribution for \(p < 0\).

The if part of the theorem follows by direct calculations using the expression for \(H_r(t)\) and \(h_t(t)\) given in the table given below.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>(H_r(t))</th>
<th>(h_t(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>(\frac{1}{r-1} \log (1+(r-1)\theta) - \log (\lambda \theta))</td>
<td>(\lambda)</td>
</tr>
<tr>
<td>Pareto</td>
<td>(\frac{1}{1-r} \log \left(\frac{\theta^{r-1}b' \left(t + \alpha\right)^{1-r}}{r + b(r\theta - \theta + 1) - 1}\right))</td>
<td>(\frac{b}{t + \alpha})</td>
</tr>
<tr>
<td>Beta</td>
<td>(\frac{1}{1-r} \log \left(\frac{\theta^{r-1}c' \left(R-t\right)^{1-r}}{c(r\theta - \theta + 1) - r + 1}\right))</td>
<td>(\frac{c}{R-t})</td>
</tr>
</tbody>
</table>
In the following theorem, we give a characterization for generalized Pareto distribution based on relationship between $H_r(t)$ and mean residual life function.

**Theorem: 4.8**

For the random variables considered in Theorem 4.7, let $m_t(t)$ be the mean residual life function of $X$. Then the relation

$$H_r(t) - \log m_t(t) = k,$$  \hspace{1cm} (4.41)

where $k$ is a constant holds for all $t \geq 0$ if and only if $X$ follows generalized Pareto distribution.

**Proof**

Assume that equation (4.41) holds. Differentiating equation (4.41) with respect to $t$, we get

$$H_r'(t) = \frac{m_t(t)}{m_t(t)}.$$

Using equation (4.42), equation (4.32) can be written as

$$(1-r)m_t'(t) = (1+(r-1)\theta)m_t(t)h_t(t) - \theta^{-1}\exp(-(1-r)k)(m_t(t)h_t(t))'.$$  \hspace{1cm} (4.43)

Take $\exp(-(1-r)k) = c$ and using the relation $m_t(t)h_t(t) = 1 + m_t(t)$, we can see that $m(t)$ is a linear function. That is, $F$ is generalized Pareto distribution.

The if part of the theorem follows by direct calculation using $H_r(t)$ and $m_t(t)$. 
