CHAPTER-III
BAYESIAN ANALYSIS OF NORMAL SEQUENCE USING
NEW TYPE OF PRIORS

In this chapter Normal Sequence is considered for carrying out the analysis to illustrate the various newly developed novel priors for location and scale parameters through Bayesian methodology. The newly designed combinations of priors for location and scale parameters respectively are Double Exponential (DE) and Half-Normal priors (HN); Double Exponential with Taylor Series Approximation (DE-TSA) and Half-Normal priors; Double Exponential and Extended Inverted gamma priors (EIG); DETSA and Extended Inverted gamma priors; Double Exponential and Inverted gamma priors (IG); DETSA and Inverted gamma priors; Normal (N) and Extended Inverted gamma priors and Normal and Inverted gamma priors of these combinations are discussed elaborately in this Chapter. There are three cases; (i) unknown mean and known variance (ii) known mean and unknown variance and (iii) both mean and variance are unknown. These three cases of Normal sequence are considered for discussion in the Bayesian analysis.

In the first section we have considered the new combinations of Double Exponential and Half-Normal priors for location and scale parameters respectively instead of using usual type of Normal and gamma priors for mean parameter $\mu$ and scale parameter $\sigma$ of Normal Sequence. In case the prior of location parameter follows a distribution with long tailed nature then Double Exponential distribution is the only possibility to suitably describe the situation. In order to surmount these difficulties, we can select Double Exponential and Half-Normal priors for location and scale parameters of Normal sequence.
DOUBLE EXPONENTIAL AND HALF - NORMAL PRIORS

In this section Bayes Estimates are obtained for location parameter which follows Double Exponential prior and scale parameter which follows Half-Normal prior.

Case (i) Unknown mean $\mu$ and known variance $\sigma^2$

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a normal population with mean $\mu$ (unknown) and variance $\sigma^2$ (Known).

i.e., $f(x_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right),$

for $i=1,2,...,n$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$

The likelihood function of $P(X/\mu)$ is given by

$$P(X/\mu) \propto \frac{1}{\sigma^n} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right) \tag{3.1}$$

The prior distribution of $\mu$ is

$$P(\mu) \propto \frac{1}{\sqrt{v}} \exp \left( -\frac{\sqrt{v}}{\theta} |\mu - \theta| \right) \tag{3.2}$$

where $\theta > 0$ and $v > 0$, $-\infty < \mu < \infty$. By multiplying equation (3.1) and (3.2) we get the posterior distribution of $\mu$ and is given by

$$P(\mu/X) \propto \frac{1}{\sigma^n v^{n/2}} \exp \left( -\frac{n \mu^2}{2\sigma^2} \right) \exp \left( -\{c_1(\mu - \bar{x})^2 + c_2|\mu - \theta|\} \right) \tag{3.3}$$

where $c_1 = \frac{n}{2\sigma^2}$, $c_2 = \frac{\sqrt{v}}{\theta}$, $n\mu^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2$,

on simplification of the equation (3.3), we obtained $P(\mu/X)$ as

$$P(\mu/X) \propto \left\{ \begin{array}{ll}
\exp -\left( \frac{n}{2\sigma^2} \left[ \mu - \left( \bar{x} + \frac{c_2}{2c_1} \right) \right] ^2 + s_1 \right) & \text{if } -\infty < \mu < \theta, \\
\exp -\left( \frac{n}{2\sigma^2} \left[ \mu - \left( \bar{x} - \frac{c_2}{2c_1} \right) \right] ^2 + s_2 \right) & \text{if } 0 < \mu < \infty,
\end{array} \right. \tag{3.4}$$

where $s_1 = \bar{x}^2 - \left( \bar{x} + \frac{c_2}{2c_1} \right)^2 + c_2 \theta$,

$s_2 = \bar{x}^2 - \left( \bar{x} + \frac{c_2}{2c_1} \right)^2 - c_2 \theta$ and $c_1, c_2$ are mentioned in equation (3.3).
Integrate the equation (3.4) with respect to $\mu$ by taking origin we get $P(\mu/X)$ as

$$P(t/X) \propto \begin{cases} 
\exp \left\{ -\frac{n}{2\sigma^2} \left[ \left( t - \left( \bar{x} + \frac{c_2}{2c_1} - \theta \right) \right)^2 + s_1 \right] \right\} & \text{if } -\infty < t < 0, \\
\exp \left\{ -\frac{n}{2\sigma^2} \left[ \left( t - \left( \bar{x} - \frac{c_2}{2c_1} - \theta \right) \right)^2 + s_2 \right] \right\} & \text{if } 0 < t < \infty, 
\end{cases} \quad \text{... (3.5)}$$

where $t = \mu - \theta$ and $c_1, c_2, s_1$ and $s_2$ are mentioned in equation (3.3) and (3.4).

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The posterior distribution of $t$, where $t = \mu - \theta$ is obtained by normalizing the distribution mentioned in equation (3.5) and is given by

$$P(t/X) = \begin{cases} 
\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2}\frac{(t - \bar{x} + \frac{c_2}{2c_1} - \theta)}{\sigma^2} \right\} & \text{if } -\infty < t < 0, \\
\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2}\frac{(t - \bar{x} - \frac{c_2}{2c_1} - \theta)}{\sigma^2} \right\} & \text{if } 0 < t < \infty, 
\end{cases}$$

where $t = \mu - \theta, c_1, c_2, s_1$ and $s_2$ are mentioned in equation (3.3) and (3.4).

Posterior mean of $\mu$ is obtained by

$$E(\mu) = E(t + \theta) = \theta + E(t)$$

where $E(t) = \int_{-\infty}^{\infty} t p(t/x) \, dt. \quad \text{... (3.6)}$

**Case (ii) known mean $\mu$ and unknown variance $\sigma^2$**

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal population with mean $\mu$ (known) and variance $\sigma^2$ (unknown).

i.e., $f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right)$, for $i=1,2, ..., n, -\infty < \mu < \infty, 0 < \sigma < \infty \quad \text{... (3.7)}$

The likelihood function of $P(X/\sigma)$ is given by

$$P(X/\sigma) \propto \frac{1}{\sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \quad \text{... (3.8)}$$
The prior distribution of $\sigma$, by assuming Half- Normal distribution, is given as

$$P(\sigma) \propto \sqrt{\frac{2}{\pi a}} \exp \left( -\frac{\sigma^2}{2a^2} \right), \quad 0 < \sigma < \infty, \quad a > 0$$   ... (3.9)

By multiplying equation (3.8) and (3.9) we get the posterior distribution of $\sigma$ and is written as

$$P(\sigma/X) \propto \frac{1}{\sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\sigma^2}{2a^2} \right), \quad \cdots \quad (3.10)$$

where $w = \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\mu - \bar{x})^2$, and $a > 0$.

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The posterior distribution of $\sigma$ is obtained by normalizing the distribution mentioned in equation (3.10) and is given by

$$P(\sigma/X) = \frac{p(\sigma/X)}{\int_0^\infty p(\sigma/X)d\sigma} = \frac{\frac{1}{\sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\sigma^2}{2a^2} \right)}{\int_0^\infty \frac{1}{\sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\sigma^2}{2a^2} \right)d\sigma}, \quad 0 < \sigma < \infty, \quad a > 0$$   ... (3.11)

where $w$ is mentioned in equation (3.10).

The integral is evaluated in terms of the modified Bessel function of third kind $k_v(z)$ (cf. Erdelyi, 1953, P.5, formula(13)) by using its integral representation (Erdelyi, op. cit., P.82, formula(23)) and is given by

$$\int_0^\infty v^{-v-1} \exp \frac{-z}{2} \left( v + \frac{\mu^2}{\nu} \right) dv = \frac{2}{\mu^v} k_v(\mu z)$$   ... (3.12)

where $z > 0$ and $\mu z > 0$.

On simplification of equation (3.11) we obtained $P(\sigma/X)$ as

$$P(\sigma/X) = \frac{w^{n-1}}{k_{n-1}\sqrt{\frac{\pi a}{2}}} \frac{1}{\sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\sigma^2}{2a^2} \right), \quad 0 < \sigma < \infty, \quad a > 0$$

where $w$ is mentioned in equation (3.10).
Posterior mean of $\sigma$ is obtained by
\[
E(\sigma) = \int_0^{\infty} \sigma \ p(\sigma/X) \ d\sigma = \frac{n-1}{\frac{w}{2}} \int_0^{\infty} \sigma \ \frac{1}{\sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\sigma^2}{2a^2} \right) d\sigma.
\]
By using equation (3.12), on simplification, we get the posterior mean as
\[
E(\sigma) = \frac{1}{\frac{w}{2}} \frac{k_{n-2}}{\sqrt{\frac{w}{2}}} \quad \ldots \ (3.13)
\]
where $w$ is mentioned in equation (3.10), $a > 0$.

The Posterior variance of $\sigma$ is defined as
\[
V(\sigma) = E(\sigma^2) - [E(\sigma)]^2 \quad \ldots \ (3.14)
\]
where $E(\sigma^2) = \int_0^{\infty} \sigma^2 \ p(\sigma/X) \ d\sigma = \frac{n-1}{\frac{w}{2}} \int_0^{\infty} \sigma^2 \ \frac{1}{\sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\sigma^2}{2a^2} \right) d\sigma$
\[
V(\sigma) = \frac{\frac{w}{2} k_{n-2}}{\sqrt{\frac{w}{2}}} \quad \ldots \ (3.16)
\]
where $w$ is mentioned in equation (3.10).

Case (iii) Unknown mean $\mu$ and unknown variance $\sigma^2$

Let $X = (x_1, x_2, \ldots, x_n)$ be a random sample from a Normal population with unknown mean $\mu$ and unknown variance $\sigma^2$. 

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i.e.,\( f(x_i/\mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right) \) for \( i = 1, 2, \ldots, n, \ -\infty < \mu < \infty, \ 0 < \sigma < \infty \)

The likelihood function is given by
\[
P(X/\mu, \sigma) \propto \frac{1}{\sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\}
\] ...
(3.17)

since \( \mu \) and \( \sigma \) are independent the joint prior distribution of \( \mu \) and \( \sigma \) can be obtained as
\[
P(\mu, \sigma / X) \propto \frac{1}{\sigma^n} \exp \left\{ -\frac{w}{2\sigma^2} \frac{\sigma^2}{\nu} |\mu - \theta| - \frac{\sigma^2}{2a^2} \right\}
\] ...
(3.18)

Posterior distribution of \( \mu \) can be obtained by integrating the equation (3.18) with respect to \( \sigma \) by using equation (3.12) and is given by
\[
P(\mu) \propto \exp \left\{ -\frac{\sqrt{w}}{\nu} |\mu - \theta| \right\} \frac{k_{n-1}}{w} \frac{\sigma^{2n}}{\nu^{n+1}}
\] ...
(3.19)

where \( w \) and \( a \) are mentioned in equation (3.13).

The posterior distribution of \( \sigma \) can be obtained by using conditional posterior
\[
(\mu = \mu_0 \text{ specified})
\]
(i.e.,) \( P(\sigma / \mu) = \frac{p(\mu, \sigma / X)}{p(\mu)} \)

By using equation (3.18) and (3.19) we get \( p(\sigma / \mu) \) as
\[
P(\sigma / \mu) = \frac{\sqrt{w}}{\nu} \frac{1}{\sigma^n} \exp \left\{ -\frac{w}{2\sigma^2} \frac{\sigma^2}{\nu} \right\}
\] ...
(3.20)

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The conditional posterior distribution of \( \sigma \) given \( \mu \), by using equation (3.20) and is obtained as \( P(\sigma / \mu) = \frac{p(\sigma / \mu)}{\int_{\sigma}^{\infty} p(\sigma / \mu) d\sigma} \)

(i.e) \( P(\sigma / \mu) = \frac{\frac{1}{\sigma^n} \exp \left\{ -\frac{w}{2\sigma^2} \frac{\sigma^2}{\nu} \right\} \frac{w^{n-1}}{\nu}}{\frac{k_{n-1}}{w^{n+1}} \frac{\sigma^{2n}}{\nu}} \) ...
(3.21)

Posterior mean of \( \sigma \) is defined as
\[
E(\sigma / \mu) = \int_{0}^{\infty} \sigma P(\sigma / \mu) d\sigma
\]
By using equation (3.21), on simplification we can obtain \( E(\sigma/\mu) \) as

\[
E(\sigma/\mu) = \frac{k_{n-2}}{2} \frac{\sqrt{w}}{a} \left( \frac{k_{n-1}}{2} \right)^{\frac{1}{2}} W^2
\]

… (3.22)

where \( w \) and \( a \) are mentioned in equation (3.13). For specified value of \( \mu (\mu = \mu_0) \) the variance of \( \sigma \) is defined as

\[
\text{Var}(\sigma/\mu) = E(\sigma^2/\mu) - [E(\sigma/\mu)]^2
\]

… (3.23)

\[
E(\sigma^2/\mu) = \int_0^{\infty} \sigma^2 P(\sigma/\mu) d\sigma.
\]

By using equation (3.21) we get \( E(\sigma^2/\mu) \) as

\[
E(\sigma^2/\mu) = \frac{k_{n-2}}{2} \frac{\sqrt{w}}{a} \left( \frac{k_{n-1}}{2} \right)^{\frac{1}{2}} W^2
\]

… (3.24)

\[
\text{Var}(\sigma/\mu) \text{ is evaluated by using equation (3.22) and (3.23) and is given by}
\]

\[
\text{Var}(\sigma/\mu) = \frac{k_{n-2}}{2} \frac{\sqrt{w}}{a} \left( \frac{k_{n-1}}{2} \right)^{\frac{1}{2}} W^2 - \left( \frac{k_{n-2}}{2} \frac{\sqrt{w}}{a} \left( \frac{k_{n-1}}{2} \right)^{\frac{1}{2}} W^2 \right)^2
\]

…(3.25)

where \( w \) and \( a \) are mentioned in equation (3.13).

In the next section, we have proposed the prior which is obtained through limiting distributions approach. Double Exponential with Taylor series approximation prior for location parameter and Half-Normal prior for scale parameter are assumed for Bayes estimates. This combination of priors influenced the studies to expose conceptual simplicity and easy derivation of the posterior mean with heavy tailed nature of observations.

**DOUBLE EXPONENTIAL WITH TAYLOR’S SERIES APPROXIMATION (DEUTA) AND HALF - NORMAL PRIORS**

In this section Bayes Estimates are obtained for location parameter which follows Double Exponential prior with Taylor series approximation (DEUTA) and scale parameter which follows Half-Normal prior.
Case (i) Unknown mean $\mu$ and known variance $\sigma^2$

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal population with mean $\mu$ (unknown) and variance $\sigma^2$ (known).

i.e., $f(x_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( - \frac{1}{2\sigma^2} (x_i - \mu)^2 \right), \ -\infty < \mu < \infty, \ 0 < \sigma < \infty,$

for $i = 1, 2, ..., n$.

The likelihood function of $P(X/\mu)$ is given by,

$$P(X/\mu) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\}, \quad \cdots \ (3.26)$$

The prior distribution of $\mu$ is given by,

$$P(\mu) = \frac{1}{v\sqrt{2}} \exp \left\{ -\frac{\sqrt{2}}{v} |\mu - \theta| \right\}, \quad \cdots \ (3.27)$$

where $\theta > 0$ and $v > 0, \ -\infty < \mu < \infty$. By multiplying equation (3.26) and (3.27) we get the posterior distribution of $\mu$ and is given by,

$$P(\mu/x) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \exp \left\{ -\frac{\sqrt{2}}{v} |\mu - \theta| \right\}, \quad \cdots \ (3.28)$$

By using Taylor’s series we have $P(\mu/x)$ as,

$$P(\mu/x) \sim \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{w}{2\sigma^2} \right\} \frac{1}{v\sqrt{2}} \left[ 1 - \frac{\sqrt{2}}{v} |\mu - \theta| + \frac{2}{v\sigma^2} (\mu - \theta)^2 \right]$$

$$P(\mu/x) \sim \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{ns^2}{2\sigma^2} \right\} \frac{1}{v\sqrt{2}} \left[ 1 - \frac{\sqrt{2}}{v} |\mu - \theta| + \frac{2}{v\sigma^2} (\mu - \theta)^2 \right] \left[ \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{n(\mu - \bar{x})^2}{2\sigma^2} \right\} - \frac{1}{v^2} \frac{1}{\sqrt{2\pi} \sigma^n} |\mu - \theta| \right]$$

$$\exp \left\{ -\frac{n(\mu - \bar{x})^2}{2\sigma^2} \right\} + \frac{1}{v^2} \frac{1}{\sqrt{2\pi} \sigma^n} (\mu - \theta)^2 \exp \left\{ -\frac{n(\mu - \bar{x})^2}{2\sigma^2} \right\}, \quad \cdots \ (3.28)$$

where $ns^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \ , \ v > 0$.

$$P(\mu/x) \sim \sum_{i=1}^{3} (-1)^{i+1} h_i (\mu/v, \theta) \frac{1}{\sqrt{2\pi} \sigma^2 n} \exp \left\{ -\frac{n}{2\sigma^2} \right\} \left[ s^2 + (\mu - \bar{x})^2 \right]$$

where $h_i(\mu/v, \theta) = \left\{ \begin{array}{ll} \frac{1}{2^i} \left( \frac{\sqrt{2}}{v} \right)^i |\mu - \theta|^{i-1}, & i = 1, 2, 3. \end{array} \right. \ \cdots \ (3.29)$
Bayes Estimates

The posterior distribution of $\mu$ is obtained by normalizing the distribution mentioned in equation (3.28) and is given by,

$$P(\mu/X) = \frac{p(\mu/X)}{\int_{-\infty}^{\infty} p(\mu/X) d\mu}$$

$$= \frac{1}{s} \left[ \frac{1}{v^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} - \frac{1}{\nu^2} \frac{1}{\sqrt{2\pi} \sigma^2} |\mu - \theta| \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right]$$

$$+ \frac{1}{v^3 \nu^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} (\mu - \theta)^2 \left\{ \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right\}$$

... (3.29)

where $v > 0$, $-\infty < \mu < \infty$ and

$$s = \left\{ \frac{1}{v^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} \left[ \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta) \right] + \frac{1}{\nu^2} \frac{1}{\sqrt{2\pi} \sigma^2} (\mu - \theta) \right\}$$

Posterior mean of $\mu$ is obtained by,

$$E(\mu) = \int_{-\infty}^{\infty} \mu \cdot P(\mu/X) d\mu,$$

$$= \int_{-\infty}^{\infty} \mu \left( \frac{1}{s} \left[ \frac{1}{v^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} - \frac{1}{\nu^2} \frac{1}{\sqrt{2\pi} \sigma^2} |\mu - \theta| \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right] \right.$$ \n
$$\exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} + \frac{1}{v^3 \nu^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} (\mu - \theta)^2 \left\{ \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right\} \right) d\mu$$

$$= \frac{1}{s} \left\{ \frac{x}{v^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} \left[ \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta) \right] + \right.$$

$$\frac{1}{\nu^2} \frac{1}{\sqrt{2\pi} \sigma^2} (\mu - \theta) \left\{ \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta) \right\} \right\}$$

... (3.30)

Posterior variance of $\mu$ is obtained by,

$$V(\mu) = E(\mu^2) - [E(\mu)]^2$$

... (3.31)

where $E(\mu^2) = \int_{-\infty}^{\infty} \mu^2 \cdot p(\mu/X) d\mu$

$$= \int_{-\infty}^{\infty} \mu^2 \left( \frac{1}{s} \left[ \frac{1}{v^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} - \frac{1}{\nu^2} \frac{1}{\sqrt{2\pi} \sigma^2} |\mu - \theta| \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right] \right.$$ \n
$$\exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} + \frac{1}{v^3 \nu^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} (\mu - \theta)^2 \left\{ \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right\} \right)$$

$$= \frac{1}{s} \left\{ \frac{1}{v^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} \left[ \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta) \right]^2 + \right.$$

$$\frac{1}{\nu^2} \frac{1}{\sqrt{2\pi} \sigma^2} (\mu - \theta)^2 \left\{ \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta) \right\} \right\}$$

$$+ \frac{1}{v^3 \nu^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} (\mu - \theta)^2 \left\{ \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right\}$$

$$+ \frac{1}{v^3 \nu^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi} \sigma^2} (\mu - \theta)^2 \left\{ \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right\}$$

$$\left\{ \exp \left\{ -\frac{n(\mu-x)^2}{2\sigma^2} \right\} \right\}$$

... (3.32)
V (µ) is evaluated by using equation (3.30) and (3.32)

$$V(\mu) = \left\{ \frac{1}{\sqrt{2\pi} s} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right) \right\} + \frac{1}{\sqrt{2\pi} s} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right)$$

where $s$ is mentioned in equation (3.29).

**Case (ii) known mean $\mu$ and unknown variance $\sigma^2$**

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal population with mean $\mu$ (known) and variance $\sigma^2$ (unknown).

i.e., $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right)$, for $i=1,2,...,n$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$

The likelihood function of $P(X|\sigma)$ is given by,

$$P(X|\sigma) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\}$$

The prior distribution of $\sigma$, by assuming Half-Normal distribution, is given as

$$P(\sigma) = \sqrt{\frac{2}{\pi}} \frac{1}{a} \exp \left( -\frac{\sigma^2}{2a^2} \right), \; 0 < \sigma < \infty, \; a > 0.$$

By multiplying equation (3.35) and (3.36) we get the posterior distribution of $\sigma$ and is given by,

$$P(\sigma|X) = \frac{1}{\pi a \sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\sigma^2}{2a^2} \right),$$

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Posterior mean of $\sigma$ is obtained by,

$$E(\sigma) = \int_{0}^{\infty} \sigma P(\sigma|X) d\sigma$$

$$= \int_{0}^{\infty} \sigma \frac{1}{\pi a \sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\sigma^2}{2a^2} \right) d\sigma.$$
The integral is evaluated in terms of the modified Bessel function of third kind $k_v(z)$ (cf. Erdelyi, 1953, P.5, formula (13)) by using its integral representation (Erdelyi, op. cit., P.82, formula (23)) and is given by

$$
\int_0^\infty v^{-\nu-1} \exp \left(-\frac{z^2}{2}\right) \frac{\nu^\nu}{\nu!} k_{\nu}(\mu z) \, dv = \frac{2^\nu}{\mu\nu} k_{\nu}(\mu z) \quad \ldots (3.39)
$$

By using equation (3.39), on simplification, we get the posterior mean as

$$
E(\sigma) = \frac{1}{2\pi a} \left[ \frac{n-2}{n-2} \frac{w}{\sigma^2} \right] k_{n-2} \left( \frac{w}{2a^2} \right) \quad \ldots (3.40)
$$

where $w$ and $a$ are mentioned in equation (3.37).

The posterior variance of $\sigma$ is obtained by,

$$
V(\sigma) = E(\sigma^2) - [E(\sigma)]^2, \quad \ldots (3.41)
$$

where $E(\sigma^2) = \int_0^\infty \sigma^2 \, p(\sigma/X) \, d\sigma$,

$$
= \int_0^\infty \sigma^2 \frac{1}{2\pi a} \left( -\frac{w}{2\sigma^2} + \frac{\sigma^2}{2a^2} \right) \, d\sigma.
$$

By using equation (3.39), on simplification we get $E(\sigma^2)$ as

$$
E(\sigma^2) = \frac{1}{2\pi a} \left[ \frac{n-2}{n-2} \frac{w}{\sigma^2} \right] k_{n-2} \left( \frac{w}{2a^2} \right) \quad \ldots (3.42)
$$

where $w$ and $a$ are mentioned in equation (3.37).

$V(\sigma)$ is evaluated by using equation (3.40) and (3.42)

$$
V(\sigma) = \frac{1}{2\pi a} \left[ \frac{n-2}{n-2} \frac{w}{\sigma^2} \right] k_{n-2} \left( \frac{w}{2a^2} \right) - \left( \frac{1}{2\pi a} \left[ \frac{n-2}{n-2} \frac{w}{\sigma^2} \right] k_{n-2} \left( \frac{w}{2a^2} \right) \right)^2 \quad \ldots (3.43)
$$

where $w$ and $a$ are mentioned in equation (3.37).

Case (iii) Unknown mean $\mu$ and unknown variance $\sigma^2$

Let $X = (x_1, x_2, \ldots, x_n)$ be a random sample from a Normal population with unknown mean $\mu$ and unknown variance $\sigma^2$.

i.e., $f(x_i/\mu, \sigma) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right)$, for $i=1,2,\ldots,n$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$,

The likelihood function is given by

$$
P(X/\mu, \sigma) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right) \quad \ldots (3.44)
$$
Since $\mu$ and $\sigma$ are independent, the joint prior distribution of $\mu$ and $\sigma$ can be obtained as

$$
P(\mu, \sigma / X) = \frac{1}{\sqrt{2\pi}\sigma^n} \exp \left\{ -\frac{1}{2a^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \frac{1}{\alpha_a} \exp \left( -\frac{\sqrt{\alpha}}{\sqrt{\sigma}} |\mu - \theta| \right) \frac{1}{\sqrt{\pi}} \exp \left( -\frac{\alpha^2}{2a^2} \right)$$

$$P(\mu, \sigma / X) \propto \frac{g}{\alpha_a \sqrt{\pi}} \frac{1}{\sigma^n} \exp \left( -\frac{w}{2\sigma^2} - \frac{\alpha^2}{2a^2} \right), \quad \ldots (3.45)$$

where $w$ and $\alpha$ are mentioned in equation (3.37), $g = \left\{ \frac{1}{\sqrt{\pi}} - \frac{\sqrt{\alpha}}{\sqrt{\sigma}} |\mu - \theta| + \frac{1}{\sqrt{\alpha}} (\mu - \theta)^2 \right\}$.

Posterior distribution of $\mu$ can be obtained by integrating the equation (3.45) with respect to $\sigma$ by using equation (3.37) and is given by,

$$P(\mu) \propto \frac{\alpha_{\mu}}{\sqrt{\pi}} \left[ \frac{1}{\sqrt{\alpha}} \frac{2}{a^2} \right]^{\alpha_{\mu}} \frac{k_{n-1}}{\alpha_{\sigma}} \frac{1}{\sqrt{\alpha_{\sigma}^2}} \left[ \frac{1}{\sqrt{\alpha}} \frac{w}{a^2} \right]$$

$$P(\mu) \propto \sum_{i=1}^{3} (-1)^{i+1} h_i(\mu/\nu, \theta) \frac{1}{\alpha_{\mu}} \left[ \frac{1}{\sqrt{\alpha}} \frac{2}{a^2} \right]^{\sum_{i=1}^{3} (-1)^{i+1} h_i(\mu/\nu, \theta)}$$

where $h_i(\mu/\nu, \theta) = \left( \frac{1}{\sqrt{\pi}} \frac{\sqrt{\alpha}}{\sqrt{\sigma}} |\mu - \theta|^{i-1}, \quad i = 1, 2, 3.$

Posterior distribution of $\sigma$ can be obtained by integrating the equation (3.45) with respect to $\mu$, we get $P(\sigma)$ as

$$P(\sigma) \propto m \left[ \frac{1}{\sqrt{2\pi\sigma^n}} - \frac{1}{\sqrt{\nu_{\sigma}^n}} (\bar{x} - \theta) + \frac{1}{\sqrt{2\nu_{\sigma}^n}} \left( \frac{\sigma^2}{n} + (\bar{x} - \theta)^2 \right) \right] \quad \ldots (3.47)$$

where $m = \frac{1}{\alpha_{\mu} \alpha_{\sigma}^n} \exp \left( -\frac{w}{2a^2} - \frac{\alpha^2}{2a^2} \right)$, $w = \sum_{i=1}^{n} (x_i - \bar{x})^2$, $a > 0, \nu > 0$.

**BAYES ESTIMATES**

The posterior distribution of $\sigma$ is obtained by normalizing the distribution mentioned in equation (3.47) and is given by

$$P(\sigma/X) = \frac{P(\sigma/X)}{\int_{0}^{\infty} P(\sigma/X) \, d\sigma}$$

$$= \frac{1}{\nu_{\sigma}^n} \exp \left( -\frac{w}{2a^2} - \frac{\alpha^2}{2a^2} \right) \left[ \frac{1}{\sqrt{2\nu_{\sigma}^n}} (\frac{\sigma^2}{n} + (\bar{x} - \theta)^2) \right]$$

$$\ldots (3.48)$$
where \( p = \left[ \frac{2}{a^{n-1}} \right] k_{n-1} \sqrt{\frac{w}{a^2}} \left[ 1 - \frac{1}{v} \frac{1}{\sqrt{2n}} - \frac{(\bar{X} - \theta)}{v^2} \right] + \left[ \frac{2}{a^{n-2}} \right] k_{n-2} \sqrt{\frac{w}{a^2}} \left[ \frac{1}{\sqrt{2n} v^3 n} \right] \)

The Posterior mean of \( \sigma \) is obtained by

\[
E (\sigma) = \int_{0}^{\infty} \sigma P(\sigma) d\sigma
\]

\[
= \frac{1}{\pi a} \left( \left[ \frac{2}{a^{n-2}} \right] k_{n-2} \sqrt{\frac{w}{a^2}} \left( \frac{1}{v} \frac{1}{\sqrt{2n}} - \frac{(\bar{X} - \theta)}{v^2} \right) + \left[ \frac{2}{a^{n-3}} \right] k_{n-3} \sqrt{\frac{w}{a^2}} \left( \frac{1}{\sqrt{2n} v^3 n} \right) \right) \quad \ldots (3.49)
\]

where \( p \) is mentioned in equation (3.48)

The Posterior variance of \( \sigma \) is obtained by,

\[
V (\sigma) = E (\sigma^2) - [E (\sigma)]^2,
\]

\[
\ldots (3.50)
\]

where \( E (\sigma^2) = \int_{0}^{\infty} \sigma^2 P(\sigma) d\sigma \)

\[
= \frac{1}{\pi a} \left( \left[ \frac{2}{a^{n-2}} \right] k_{n-2} \sqrt{\frac{w}{a^2}} \left( \frac{1}{v} \frac{1}{\sqrt{2n}} - \frac{(\bar{X} - \theta)^2}{v^2} \right) + \left[ \frac{2}{a^{n-3}} \right] k_{n-3} \sqrt{\frac{w}{a^2}} \left( \frac{1}{\sqrt{2n} v^3 n} \right) \right) \quad \ldots (3.51)
\]

where \( p \) is mentioned in equation (3.48)

\[
V (\sigma) \text{ is evaluated by using equation (3.49) and (3.51)}
\]

\[
V (\sigma) = \frac{1}{\pi a} \left[ \left[ \frac{2}{a^{n-2}} \right] k_{n-2} \sqrt{\frac{w}{a^2}} \left( \frac{1}{v} \frac{1}{\sqrt{2n}} - \frac{(\bar{X} - \theta)^2}{v^2} \right) \right] + \left[ \frac{2}{a^{n-3}} \right] k_{n-3} \sqrt{\frac{w}{a^2}} \left( \frac{1}{\sqrt{2n} v^3 n} \right) - \left\{ \left[ \frac{2}{a^{n-2}} \right] k_{n-2} \sqrt{\frac{w}{a^2}} \left( \frac{1}{v} \frac{1}{\sqrt{2n}} \right) \right\}^2
\]

where \( p \) is mentioned in equation (3.48), \( v > 0 \) and \( a > 0 \).
Regarding Normal sequence, many authors used Normal or gamma prior for location as well as scale parameter. But in the next section, we deviated from this and assumed that Double Exponential will be a prior distribution of $\mu$ and Extended Inverted gamma will be a prior distribution of $\sigma$ for carrying out the Bayesian analysis.

**DOUBLE EXPONENTIAL AND EXTENDED INVERTED GAMMA PRIORS**

In this section, Bayes Estimates are obtained for location parameter which follows Double Exponential prior and scale parameter which follows Extended Inverted gamma prior.

**Case (i) Unknown mean $\mu$ and known variance $\sigma^2$**

This Case is Similar to the case (i) of Bayesian Analysis of Normal Sequence Using Double Exponential and Half – Normal Priors.

**Case (ii) known mean $\mu$ and unknown variance $\sigma^2$**

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal population with mean $\mu$ (known) and variance $\sigma^2$ (unknown).

i.e., $f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right)$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$, for $i = 1, 2, ..., n$.

The likelihood function of $P(x/\sigma)$ is given by

$$P(x/\sigma) = \frac{1}{\sqrt{2\pi}\sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \quad \text{... (3.52)}$$

The Prior distribution of $\sigma$, by assuming Extended Inverted gamma distribution is given by

$$P(\sigma) = 2^\alpha \frac{\sigma^{\alpha+k}}{\Gamma(\alpha,c)} \left( \frac{1}{\sigma^2} \right)^{\alpha+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-c}{\sigma^2} \right), \quad \text{... (3.53)}$$

where $\alpha, c > 0$, $k = 1, 2, ..., \infty$

By multiplying equation (3.52) and (3.53) we get the posterior distribution of $\sigma$ and is given by.
\[ P(\sigma /x) = \frac{1}{\sqrt{2\pi}\sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \cdot \frac{C^{a+k}}{k^*(a,c)} \left( \frac{1}{\sigma^2} \right)^{a+1/2} \times \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( -\frac{c}{\sigma^2} \right) \]

\[ P(\sigma /x) = \frac{2}{\sqrt{\pi}} \frac{C^{a+k}}{k^*(a,c)} \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{d}{\sigma^2} \right) \] \hspace{1cm} \ldots (3.54)

where \( d = \frac{w+2c}{2} \), \( w = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \)

**BAYES ESTIMATES**

Posterior mean of \( \sigma \) is obtained by,

\[ E(\sigma) = \int_0^\infty \sigma \, p(\sigma/X) \, d\sigma \]

\[ = \int_0^\infty \sigma \, \frac{2}{\sqrt{\pi}} \frac{C^{a+k}}{k^*(a,c)} \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{d}{\sigma^2} \right) d\sigma \]

\[ E(\sigma) = \frac{2}{\sqrt{\pi}} \frac{C^{a+k}}{k^*(a,c)} \int_0^\infty \sigma \, \left[ \frac{k^*(a+n/2-1, d)}{d^{a+n/2-1+k}} \right] \] \hspace{1cm} \ldots (3.55)

where \( d \) and \( w \) are mentioned in equation (3.54)

\[ E(\sigma^2) = \int_0^\infty \sigma^2 \, p(\sigma/X) \, d\sigma \]

\[ = \int_0^\infty \sigma^2 \, \frac{2}{\sqrt{\pi}} \frac{C^{a+k}}{k^*(a,c)} \int_0^\infty \sigma \, \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{d}{\sigma^2} \right) d\sigma \]

\[ E(\sigma^2) = \frac{2}{\sqrt{\pi}} \frac{C^{a+k}}{k^*(a,c)} \int_0^\infty \sigma \, \left[ \frac{k^*(a+n/2-1, d)}{d^{a+n/2-1+k}} \right] \] \hspace{1cm} \ldots (3.56)

Using equation (3.55) and (3.56) we get \( V(\sigma) \) as

\[ V(\sigma) = \frac{2}{\sqrt{\pi}} \frac{C^{a+k}}{k^*(a,c)} \left[ \frac{k^*(a+n/2-1, d)}{d^{a+n/2-1+k}} \right] \times \left( \frac{2}{\sqrt{\pi}} \frac{C^{a+k}}{k^*(a,c)} \frac{k^*(a+n/2-1, d)}{d^{a+n/2-1+k}} \right)^2 \]

where \( d \) and \( w \) are mentioned in equation (3.54)

**Case (iii) Unknown mean \( \mu \) and unknown variance \( \sigma^2 \)**

Let \( X = (x_1, x_2, ..., x_n) \) be a random sample from a Normal population with unknown mean \( \mu \) and unknown variance \( \sigma^2 \).

i.e., \( f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right), -\infty < \mu < \infty, 0 < \sigma < \infty, \) for \( i = 1, 2, ..., n \).
The likelihood function is given by
\[ P(x/\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \] \quad \ldots(3.57)

Since the prior distributions of \( \mu \) and \( \sigma \) respectively are defined in the equations (3.2) and (3.53), the joint prior distribution of \( \mu \) and \( \sigma \) can be obtained as
\[ P(\mu,\sigma/x) = \frac{1}{v\sqrt{\pi}} \frac{c^{\alpha+k}}{|k^*(a,c)|} \exp \left( -\frac{\sqrt{2}}{v} |\mu - \theta| \right) \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{d}{\sigma^2} \right) \] \quad \ldots(3.58)

where \( d \) and \( w \) are mentioned in equation (3.54)

The posterior distribution of \( \mu \) can be obtained by integrating equation (3.58) with respect to \( \sigma \)
\[ P(\mu) = \frac{1}{v\sqrt{\pi}} \frac{c^{\alpha+k}}{|k^*(a,c)|} \exp \left( -\frac{\sqrt{2}}{v} |\mu - \theta| \right) \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{d}{\sigma^2} \right) \frac{d\sigma}{d^*\sigma^{\alpha+n/2+k}} \] \quad \ldots(3.59)

where \( d \) and \( w \) are mentioned in equation (3.54)

The posterior distribution of \( \sigma \) can be obtained by using conditional posterior

\( \mu = \mu_0 \) Specified

(i.e.,) \[ P(\sigma/\mu) = \frac{p(\mu,\sigma/x)}{p(\mu)} \]

By using equation (3.58) and (3.59) we get \( p(\sigma/\mu) \) as
\[ P(\sigma/\mu) = \frac{d^{\alpha+n/2+k}}{|k^*(a+n/2, d)|} \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{d}{\sigma^2} \right) \] \quad \ldots(3.60)

**BAYES ESTIMATES**

Posterior mean of \( \sigma \) is obtained by
\[ E(\sigma) = \int_0^\infty \sigma P(\sigma/\mu) d\sigma \]
\[ E(\sigma) = \frac{d^{\alpha+n/2+k}}{|k^*(a+n/2, d)|} \left[ k^*(a+n/2+1/2, d) \right] \] \quad \ldots(3.61)

\[ E(\sigma^2) = \int_0^\infty \sigma^2 p(\sigma/x) d\sigma \]
Using equation (3.61) and (3.62) we get $V(\sigma)$ as

$$V(\sigma) = d^{1/2-k} \frac{|k^*(\alpha+n/2,d)|}{|k^*(\alpha+n/2,d)|} \frac{|k^*(\alpha+n/2,1,d)|}{|k^*(\alpha+n/2,1,d)|} - \left( \frac{d^{1/2-k}}{|k^*(\alpha+n/2,d)|} \frac{|k^*(\alpha+n/2,1,d)|}{|k^*(\alpha+n/2,1,d)|} \right)^2,$$

where $d$ and $w$ are mentioned in equation (3.54).

In the next section, we have developed some other new set of priors in which DETSA and Extended Inverted gamma prior respectively will be assumed for mean and scale parameters for obtaining Bayes estimates of Normal sequence.

**DETSA AND EXTENDED INVERTED GAMMA PRIORS**

In this section, Bayes Estimates are obtained for location parameter which follows DETSA and scale parameter which follows Extended Inverted gamma prior.

**Case (i) Unknown mean $\mu$ and known variance $\sigma^2$**

This Case is Similar to the case (i) of Bayesian Analysis of Normal Sequence Using Double Exponential prior as Taylor’s series form of prior and Half-Normal Prior.

**Case (ii) known mean $\mu$ and unknown variance $\sigma^2$**

This Case is Similar to the case (ii) of Bayesian Analysis of Normal Sequence Using Double Exponential and Extended Inverted Gamma Prior.

**Case (iii) Unknown mean $\mu$ and unknown variance $\sigma^2$**

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal population with unknown mean $\mu$ and unknown variance $\sigma^2$. i.e., $f(x_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right),$ $-\infty < \mu < \infty,$ $0 < \sigma < \infty,$ for $i = 1, 2, ..., n.$

The likelihood function is given by

$$P(x/\mu,\sigma) = \frac{1}{\sqrt{2\pi \sigma^n}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\}$$

... (3.63)
Since the prior distributions of $\mu$ and $\sigma$ are respectively defined in the equations (3.2) and (3.53), the joint prior distribution of $\mu$ and $\sigma$ can be obtained as

$$P(\mu, \sigma|x) \propto \frac{c^{\alpha+k}}{\sqrt{\pi} |k^*(\alpha, c)|} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{d}{\sigma^2} \right) \quad \ldots (3.64)$$

where $d$ and $w$ are mentioned in equation (3.54), $g$ is mentioned in equation (3.45).

The posterior distribution of $\mu$ can be obtained by integrating equation (3.64) with respect to $\sigma$

$$P(\mu) \propto \frac{\sqrt{\pi} |k^*(\alpha, c)|}{C^{\alpha+k}} \int_0^\infty \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{d}{\sigma^2} \right) d\sigma$$

$$P(\mu) \propto \frac{\sqrt{\pi} |k^*(\alpha, c)|}{C^{\alpha+k}} \left[ k^*(\alpha+n/2, d) \right] \quad \ldots (3.65)$$

The posterior distribution of $\sigma$ can be obtained by integrating equation (3.64) with respect to $\mu$

$$P(\sigma) \propto \frac{1}{\sqrt{\pi} |k^*(\alpha, c)|} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{c}{\sigma^2} \right) \int_0^\infty \exp \left( -\frac{w}{\sigma^2} \right) d\mu$$

$$P(\sigma) \propto \frac{1}{\sqrt{\pi} |k^*(\alpha, c)|} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2} (1 + \frac{1}{\sigma^2})^k \exp \left( -\frac{c}{\sigma^2} - \frac{nz^2}{2\sigma^2} \right) \sqrt{\frac{2\pi}{v\sqrt{n}}} \quad \ldots (3.66)$$

where $j = \left[ 1 - \frac{\sqrt{2}}{v} (\bar{x} - \theta) + \frac{1}{v^2} \left( \frac{\sigma^2}{n} + (\bar{x} - \theta)^2 \right) \right]$

**BAYES ESTIMATES**

The posterior distribution of $\sigma$ is obtained by normalizing the distribution mentioned in equation (3.66) and is given by

$$P(\sigma) = \frac{P(\sigma)}{\int_0^\infty P(\sigma) d\sigma}$$

$$P(\sigma) = \int_0^\infty \frac{1}{\sigma^2} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2} (1 + \frac{1}{\sigma^2})^k \exp\left( -\frac{d}{\sigma^2} \right) \frac{\sqrt{2\pi}}{v\sqrt{n}} f \quad \ldots (3.67)$$

Where $t = \left[ k^*(\alpha+n/2, d) \right] \sqrt{\frac{2\pi}{v^2n}} f + \sqrt{\frac{2\pi}{v^2n}} \left[ k^*(\alpha+n/2, d) \right]$ and

$$f = 1 - \frac{\sqrt{2}}{v} (\bar{x} - \theta) + \frac{1}{v^2} (\bar{x} - \theta)^2$$
Posterior mean of $\sigma$ is obtained by

$$
E(\sigma) = \int_0^\infty \sigma \ p(\sigma) \ d\sigma
$$

$$
= \frac{1}{t} \int_0^\infty \sigma \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \ \exp \left( -\frac{d}{\sigma^2} \right) \ \frac{\sqrt{2\pi}}{v\sqrt{n}} \ f \ d\sigma
$$

$$
E(\sigma) = \frac{1}{t} \left[ \frac{k^* (\alpha + \frac{n}{2} - 1, d)}{\alpha^{n/2-1+k}} \right] \ \frac{\sqrt{2\pi}}{v\sqrt{n}} \ f + \frac{\sqrt{2\pi}}{v^3 n \sqrt{n}} \ \left[ \frac{k^* (\alpha + \frac{n}{2} - 2, d)}{\alpha^{n/2-2+k}} \right] \quad \ldots(3.68)
$$

$$
E(\sigma^2) = \frac{1}{t} \left[ \frac{k^* (\alpha + \frac{n}{2} - 3, d)}{\alpha^{n/2-1+k}} \right] \ \frac{\sqrt{2\pi}}{v\sqrt{n}} \ f + \frac{\sqrt{2\pi}}{v^3 n \sqrt{n}} \ \left[ \frac{k^* (\alpha + \frac{n}{2} - 2, d)}{\alpha^{n/2-2+k}} \right] \quad \ldots(3.69)
$$

Using equation (3.68) and (3.69) we get $v(\sigma)$ as

$$
v(\sigma) = \frac{1}{t} \left[ \frac{k^* (\alpha + \frac{n}{2} - 3, d)}{\alpha^{n/2-1+k}} \right] \ \frac{\sqrt{2\pi}}{v\sqrt{n}} \ f + \frac{\sqrt{2\pi}}{v^3 n \sqrt{n}} \ \left[ \frac{k^* (\alpha + \frac{n}{2} - 2, d)}{\alpha^{n/2-2+k}} \right] - \\
\frac{1}{t} \left[ \frac{k^* (\alpha + \frac{n}{2} - 1, d)}{\alpha^{n/2-1+k}} \right] \ \frac{\sqrt{2\pi}}{v\sqrt{n}} \ f + \frac{\sqrt{2\pi}}{v^3 n \sqrt{n}} \ \left[ \frac{k^* (\alpha + \frac{n}{2} - 2, d)}{\alpha^{n/2-2+k}} \right] \right]^2
$$

where $f$ is mentioned in equation (3.67)

Bayesian analysis of Normal sequence for location parameter which follows Double Exponential prior and scale parameter which follows inverted gamma prior is discussed in the next section.

**DOUBLE EXPONENTIAL AND INVERTED GAMMA PRIORS**

In this section, Bayes Estimates are obtained for location parameter which follows Double Exponential prior and scale parameter which follows Inverted gamma prior.

**Case (i) Unknown mean $\mu$ and known variance $\sigma^2$**

This Case is Similar to the case (i) of Bayesian Analysis of Normal Sequence Using Double Exponential and Half – Normal Prior
Case (ii) known mean μ and unknown variance σ²

Let \( X = (x_1, x_2, ..., x_n) \) be a random sample from a Normal population with mean \( \mu \) (known) and variance \( \sigma^2 \) (unknown).

i.e., \( f(x_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right) \), \( -\infty < \mu < \infty, \ 0 < \sigma < \infty \), for \( i = 1, 2, ..., n \).

The likelihood function of \( P(x/\sigma) \) is given by

\[
P(x/\sigma) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \quad \text{... (3.70)}
\]

The prior distribution of \( \sigma \), by assuming Inverted gamma distribution, is given as

\[
P(\sigma) = \frac{2^\alpha \beta^\alpha}{\sigma^{2\alpha+1}} \exp \left( -\frac{\beta}{\sigma^2} \right), \quad \text{... (3.71)}
\]

where \( \alpha, \beta > 0, \ 0 < \sigma < \infty \).

By multiplying equation (3.70) and (3.71) We get the Posterior distribution of \( \sigma \) and is given by,

\[
P(\sigma/x) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \frac{2^\alpha \beta^\alpha}{\sigma^{2\alpha+1}} \exp \left( -\frac{\beta}{\sigma^2} \right)
\]

\[
P(\sigma/x) = \frac{2^\alpha \beta^\alpha}{\sqrt{\pi} |\alpha|} \sigma^{2\alpha+n+1} \exp \left( -\frac{\beta}{\sigma^2} \right) \quad \text{... (3.72)}
\]

where \( \beta' = \frac{w+2c}{2}, \ w = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \)

BAYES ESTIMATES

Posterior mean of \( \sigma \) is obtained by

\[
E(\sigma) = \int_{0}^{\infty} \sigma p(\sigma/x) \ d\sigma
\]

\[
= \int_{0}^{\infty} \sigma \sqrt{\frac{2}{\pi}} \frac{\beta^\alpha}{|\alpha|} \sigma^{2\alpha+n+1} \exp \left( -\frac{\beta}{\sigma^2} \right) \ d\sigma
\]

\[
E(\sigma) = \frac{1}{2\pi} \frac{\beta^\alpha}{|\alpha|} \left[ \frac{2^{2\alpha+n-1}}{\beta' \Gamma(\alpha)} \right], \quad \text{... (3.73)}
\]

where \( \beta' \) and \( w \) are mentioned in equation (3.72)

\[
E(\sigma^2) = \int_{0}^{\infty} \sigma^2 p(\sigma/x) \ d\sigma
\]

\[
= \frac{2}{\sqrt{\pi}} \frac{\beta^\alpha}{|\alpha|} \int_{0}^{\infty} \sigma^{2\alpha+n+1} \exp \left( -\frac{\beta}{\sigma^2} \right) \ d\sigma
\]
\[ E(\sigma^2) = \frac{1}{2\pi} \left( \frac{1}{\beta} \right) \left[ \frac{2\alpha + n - 2}{2} \right] \] \quad \text{...(3.74)}

\[ V(\sigma) = \frac{1}{2\pi} \left( \frac{1}{\beta} \right) \left[ \frac{2\alpha + n - 2}{2} \right] - \left( \frac{1}{2\pi} \left( \frac{1}{\beta} \right) \left[ \frac{2\alpha + n - 2}{2} \right] \right)^2 \]

where \( \beta \) and \( w \) are mentioned in equation (3.72)

**Case (iii) Unknown mean \( \mu \) and unknown variance \( \sigma^2 \)**

Let \( X = (x_1, x_2, \ldots, x_n) \) be a random sample from a Normal population with unknown mean \( \mu \) and unknown variance \( \sigma^2 \).

i.e., \( f(x_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right), \) \( -\infty < \mu < \infty, \) \( 0 < \sigma < \infty, \) for \( i = 1, 2, \ldots, n. \)

The likelihood function is given by

\[ P(x/\mu, \sigma) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \quad \text{...(3.75)} \]

Since the prior distributions of \( \mu \) and \( \sigma \) are respectively defined in the equations (3.2) and (3.70), the joint prior distribution of \( \mu \) and \( \sigma \) can be obtained as

\[ P(\mu, \sigma/x) = \frac{1}{\sqrt{2\pi} \alpha^{\frac{n}{2}}} (\sigma)^{-(2\alpha+n+1)} \exp \left( -\frac{w}{2\sigma^2} - \frac{v}{\sigma^2} |\mu - \theta| \right), \quad \text{...(3.76)} \]

where \( w \) is mentioned in equation (3.72)

The posterior distribution of \( \mu \) can be obtained by integrating equation (3.76) with respect to \( \sigma \)

\[ P(\mu) = \frac{1}{\sqrt{2\pi} \alpha^{\frac{n}{2}}} \exp \left( -\frac{\sqrt{v}}{v} |\mu - \theta| \right) \left\{ \int_{0}^{\infty} \sigma^{-(2\alpha+n+1)} \exp \left( -\frac{w}{2\sigma^2} - \frac{\beta}{\sigma^2} \right) d\sigma \right\} \]

\[ P(\mu) = \frac{1}{\sqrt{2\pi} \alpha^{\frac{n}{2}}} \exp \left( -\frac{\sqrt{v}}{v} |\mu - \theta| \right) \left[ \frac{2\alpha + n}{2} \right] \quad \text{...(3.77)} \]

where \( c = \frac{w + 2\beta}{2} \), \( w \) is mentioned in equation (3.72)

The posterior distribution of \( \sigma \) can be obtained by using conditional posterior (\( \mu = \mu_0 \) specified)

\[ \text{(i.e.,)} \quad P(\sigma/\mu) = \frac{P(\mu, \sigma/x)}{P(\mu)} \]
By using equation (3.76) and (3.77) we get \( p(\sigma/\mu) \) as

\[
P(\sigma/\mu) = 2 \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \sigma^{(2\alpha+n-1)} \exp \left( \frac{-w}{2\sigma^2} - \frac{\beta}{\sigma^2} \right),
\]

...(3.78)

where \( c \) and \( w \) are mentioned in equation (3.77).

**BAYES ESTIMATES**

Posterior mean of \( \sigma \) is obtained by

\[
E(\sigma) = \int_0^\infty \sigma P(\sigma/\mu) d\sigma
\]

\[
E(\sigma) = \int_0^\infty \sigma \left( \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \sigma^{(2\alpha+n-1)} \exp \left( \frac{-w}{2\sigma^2} - \frac{\beta}{\sigma^2} \right) \right) d\sigma
\]

E \( (\sigma) = 2 \left( \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \right) \right)^{\frac{1}{2}} \left[ \frac{\sigma^{(2\alpha+n-1)}}{\sigma^{(2\alpha+n-1)}} \right] \right)^{\frac{1}{2}} \right], \]

...(3.79)

E \( (\sigma^2) = \int_0^\infty \sigma^2 p(\sigma|x) d\sigma
\]

E \( (\sigma^2) = 2 \left( \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \right) \right)^{\frac{1}{2}} \left[ \frac{\sigma^{(2\alpha+n-2)}}{\sigma^{(2\alpha+n-2)}} \right] \right)^{\frac{1}{2}} \right], \]

...(3.80)

Using equation (3.79) and (3.80) we get \( V(\sigma) \) as

\[
V(\sigma) = 2 \left( \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \right) \right)^{\frac{1}{2}} \left[ \frac{\sigma^{(2\alpha+n-2)}}{\sigma^{(2\alpha+n-2)}} \right] \right)^{\frac{1}{2}} \right] - \left( \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \right) \right)^{\frac{1}{2}} \left[ \frac{\sigma^{(2\alpha+n-1)}}{\sigma^{(2\alpha+n-1)}} \right] \right)^{\frac{1}{2}} \right] \right] - \left( \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \right) \right)^{\frac{1}{2}} \left[ \frac{\sigma^{(2\alpha+n-2)}}{\sigma^{(2\alpha+n-2)}} \right] \right)^{\frac{1}{2}} \right], \]

where \( c \) and \( w \) are mentioned in equation (3.77).

In the next section, we have developed a different combination of priors as DETSA and the basic prior of Inverted gamma respectively for mean and scale parameters for obtaining Bayes estimates of Normal sequence.

**DETSA AND INVERTED GAMMA PRIORS**

In this section, Bayes Estimates are obtained for location parameter which follows DETSA and scale parameter which follows Inverted gamma prior.
Case (i) Unknown mean $\mu$ and known variance $\sigma^2$

This Case is Similar to the case (i) of Bayesian Analysis of Normal Sequence Using Double Exponential prior and Half-Normal Prior.

Case (ii) known mean $\mu$ and unknown variance $\sigma^2$

This Case is Similar to the case (ii) of Bayesian Analysis of Normal Sequence Using Double Exponential and Inverted Gamma Prior.

Case (iii) Unknown mean $\mu$ and unknown variance $\sigma^2$

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal population with unknown mean $\mu$ and unknown variance $\sigma^2$.

i.e., $f(x_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left(-\frac{1}{2\sigma^2} (x_i - \mu)^2\right), -\infty < \mu < \infty, 0 < \sigma < \infty$, for $i = 1, 2, ..., n$.

The likelihood function is given by

$$P(x/\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma^n} \exp \left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2\right\} \quad \text{...(3.81)}$$

Since the prior distributions of $\mu$ and $\sigma$ are respectively defined in the equations (3.2) and (3.70), the joint prior distribution of $\mu$ and $\sigma$ can be obtained as

$$P(\mu, \sigma / X) \propto \frac{1}{\sigma^{\alpha \frac{\beta}{\alpha} + 1}} \exp \left(-\frac{c}{\sigma^2}\right), \quad \text{...(3.82)}$$

where $c$ and $w$ are mentioned in equation (3.77), $g$ is mentioned in equation (3.45)

The posterior distribution of $\mu$ can be obtained by integrating equation (3.82) with respect to $\sigma$

$$P(\mu) \propto \frac{1}{\sigma^{\alpha \frac{\beta}{\alpha} + 1}} \int_{0}^{\infty} \sigma^{\alpha \frac{\beta}{\alpha} + 1} \exp \left(-\frac{c}{\sigma^2}\right) d\sigma$$

$$P(\mu) \propto \frac{1}{\sigma^{\alpha \frac{\beta}{\alpha} + 1}} \left[1 + \frac{\left(2\alpha + n\right)}{c^{\frac{\beta}{\alpha}}} \right] \quad \text{...(3.83)}$$

The posterior distribution of $\sigma$ can be obtained by integrating equation (3.82) with respect to $\mu$
\[ P(\sigma) \propto \frac{1}{\sqrt{\pi}} \left| \alpha \right|^\alpha \sigma^{-(2\alpha+n+1)} \exp\left(\frac{-\beta}{\sigma^2}\right) \int_0^\infty \exp\left(\frac{-w}{\sigma^2}\right) \left[ \frac{1}{v} \frac{\mu - \theta}{v^2} \right] d\mu, \]

\[ P(\sigma) \propto \frac{1}{\sqrt{\pi}} \left| \alpha \right|^\alpha \frac{\sqrt{2\pi}}{v\sqrt{n}} \sigma^{-(2\alpha+n)} \exp\left(\frac{-\beta}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right) \] \hspace{1cm} \text{(3.84)}

where \( j \) is mentioned in equation (3.66)

**BAYES ESTIMATES**

The posterior distribution of \( \sigma \) is obtained by normalizing the distribution mentioned in equation (3.84) and is given by

\[ P(\sigma) = \frac{P(\sigma)}{\int_0^\infty P(\sigma) d\sigma} \]

\[ P(\sigma) = \frac{j}{\sqrt{2\pi}} \sigma^{-(2\alpha+n)} \exp\left(\frac{-\beta}{\sigma^2}\right) \frac{\sqrt{2\pi}}{v\sqrt{n}} \] \hspace{1cm} \text{(3.85)}

where \( p = \left[ \frac{\alpha + n - 1}{\alpha + n - 2} \right] \frac{f}{v\sqrt{2n}} + \frac{1}{v^2n\sqrt{2n}} \frac{2\alpha + n - 3}{(c)} \)

\[ f = 1 - \frac{\sqrt{v}}{v} (\bar{x} - \theta) + \frac{1}{v^2} (\bar{x} - \theta)^2 \]

Posterior mean of \( \sigma \) is obtained by

\[ E(\sigma) = \int_0^\infty \sigma P(\sigma) d\sigma \]

\[ E(\sigma) = \frac{1}{p} \left[ \frac{2\alpha + n - 2}{2} \frac{f}{v\sqrt{2n}} + \frac{1}{v^2n\sqrt{2n}} \frac{2\alpha + n - 4}{(c)} \right] \] \hspace{1cm} \text{(3.86)}

\[ E(\sigma^2) = \frac{1}{p} \left[ \frac{2\alpha + n - 3}{2} \frac{f}{v\sqrt{2n}} + \frac{1}{v^2n\sqrt{2n}} \frac{2\alpha + n - 5}{(c)} \right] \] \hspace{1cm} \text{(3.87)}

\[ v(\sigma) = \frac{1}{p} \left[ \frac{2\alpha + n - 2}{2} \frac{f}{v\sqrt{2n}} + \frac{1}{v^2n\sqrt{2n}} \frac{2\alpha + n - 4}{(c)} \right]^2 \] \hspace{1cm} \text{(3.88)}

Where \( p \) is mentioned in equation (3.85), \( f \) is mentioned in equation (3.85)
In the next section, we have proposed a new combination of priors which are Normal and Extended Inverted gamma for mean and scale parameter respectively for obtaining Bayes estimates of Normal sequence.

NORMAL AND EXTENDED INVERTED GAMMA PRIORS

In this section, Bayes Estimates are obtained for location parameter which follows Normal and scale parameters which follows Extended Inverted gamma prior.

Case (i) Unknown mean $\mu$ and known variance $\sigma^2$

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal Population with mean $\mu$ (unknown) and variance $\sigma^2$ (known). i.e., the p.d.f of $x_i$ is given by

$$f(x_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right), \ -\infty < \mu < \infty, 0 < \sigma < \infty,$$

for $i = 1, 2, ..., n$.

The Likelihood function of $P(x/\mu)$ is given by,

$$P(x/\mu) = \frac{1}{\sqrt{2\pi} \sigma^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \ \ ...(3.89)$$

The prior distribution of $\mu$ is given by

$$P(\mu) = \frac{\sqrt{\tau}}{\sqrt{2\pi} \sigma} \exp \left\{ -\frac{\tau}{2\sigma^2} (\mu - \theta)^2 \right\}, \ \text{where} \ 0, \tau > 0. \ \ ...(3.90)$$

By multiplying equation (3.89) and (3.90) we get the Posterior distribution of $\mu$ and is given by,

$$P(\mu|x) = \frac{1}{\sqrt{2\pi} \sigma^{n+1}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left( x_i - \bar{x} \right)^2 + n(\mu - \bar{x})^2 \right\} \sqrt{\frac{\tau}{2\pi} \sigma}$$

$$\exp \left\{ -\frac{\tau}{2\sigma^2} (\mu - \theta)^2 \right\}$$

$$P(\mu|x) = \frac{1}{2\pi \sigma^{n+1}} \exp \left( -\frac{1}{2\sigma^2} (\tau+n)(\mu - \mu')^2 \right) \exp \left( \frac{-\beta'}{\sigma^2} \right) \ \ ...(3.91)$$

where $\beta' = \frac{1}{2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + \frac{\tau n}{2(\tau+n)} (\bar{x} - \theta)^2$, $\mu' = \frac{n\bar{x} + \tau \theta}{\tau + n}$.
Case (ii) known mean $\mu$ and unknown variance $\sigma^2$

This Case is Similar to case (ii) of Bayesian Analysis of Normal Sequence Using Double Exponential and Extended Inverted Gamma Prior

Case (iii) Unknown mean $\mu$ and unknown variance $\sigma^2$

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal population with mean $\mu$ and variance $\sigma^2$, where both mean and variance are unknown.

i.e., $f(x_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left(-\frac{1}{2\sigma^2} (x_i - \mu)^2\right)$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$, for $i = 1, 2, ..., n$.

The likelihood function is given by

$$P(x/\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^n}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\}.$$ ... (3.92)

Since the prior distributions of $\mu$ and $\sigma$ are respectively defined in the equations (3.90) and (3.53), the joint prior distribution of $\mu$ and $\sigma$ can be obtained as

$$P(\mu, \sigma/x) = \frac{1}{\sqrt{2\pi\sigma^n}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \frac{1}{\sqrt{2\pi}} 
\exp \left[ -\frac{n}{2\sigma^2} (\mu - \theta)^2 \right] 2 \frac{\gamma^{a+k}}{\Gamma(a, c)} \left( \frac{1}{\sigma^2} \right)^{a+1/2} \exp \left( -\frac{c}{\sigma^2} \right)$$

$$P(\mu, \sigma /x) = \frac{1}{\sqrt{2\pi}} \frac{\gamma^{a+k}}{\Gamma(a, c)} \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} \exp \left( -\frac{1}{2\sigma^2} (\tau + \mu) \right) \exp \left( -\frac{c}{\sigma^2} \right)$$ ... (3.93)

where $\beta = c + \frac{1}{2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + \frac{\tau n}{2(\tau + n)} (\bar{x} - \theta)^2$, $\mu' = \frac{n\bar{x} + \tau \theta}{\tau + n}$

The posterior distribution of $\mu$ can be obtained by integrating equation (3.93) with respect to $\sigma$

$$P(\mu) = \frac{\sqrt{\pi}}{\Gamma(a, c)} \int_{0}^{\infty} \left( \frac{1}{\sigma^2} \right)^{a+n/2+1/2} \exp \left( -\frac{c}{\sigma^2} \right) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma^2} (\beta + \frac{\tau+n}{\tau} (\mu - \mu')^2) \right) d\sigma$$


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Since Reference from Article – Finite Mixture of Certain Distributions – pg.no.2128

\[ \int_0^\infty \left( \frac{1}{\sigma^2} \right)^{\alpha+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-\sigma}{\sigma^2} \right) d\sigma = \frac{|k^*(\alpha,c)|}{c^{a+k}} \]

\[ P(\mu) = \frac{\sqrt{\pi}}{\Gamma(\alpha,c)} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-\beta^2}{\sigma^2} \right) \]

where \( d = \beta + \frac{\tau}{2} (\mu - \mu)^2, \beta' \) and \( \mu' \) are mentioned in equation (3.93)

The posterior distribution of \( \sigma \) can be obtained by integrating equation (3.93) with respect to \( \mu \)

\[ P(\sigma) = \frac{\sqrt{\pi}}{\Gamma(\alpha,c)} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-\beta^2}{\sigma^2} \right) \]

\[ \int_{-\infty}^{\infty} \exp \left( \frac{-1}{2\sigma^2} (\tau + n)(\mu - \mu)^2 \right) d\mu \]

\[ P(\sigma) = \frac{2\pi}{\Gamma(\alpha,c)} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-\beta^2}{\sigma^2} \right) \]

where \( \beta' \) is mentioned in equation (3.93)

**BAYES ESTIMATES**

Posterior mean of \( \sigma \) is obtained by,

\[ E(\sigma) = \int_0^\infty \sigma P(\sigma) d\sigma \]

\[ = \int_0^\infty \sigma \frac{2\pi}{\Gamma(\alpha,c)} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-\beta^2}{\sigma^2} \right) d\sigma \]

\[ E(\sigma) = \frac{2\pi}{\Gamma(\alpha,c)} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-\beta^2}{\sigma^2} \right) \]

**E(\sigma^2) = \int_0^\infty \sigma^2 P(\sigma) d\sigma**

\[ = \sqrt{\pi(\alpha+c)} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-\beta^2}{\sigma^2} \right) \]

\[ E(\sigma^2) = \frac{2\pi}{\Gamma(\alpha,c)} \left( \frac{1}{\sigma^2} \right)^{\alpha+n/2+1/2} \left( 1 + \frac{1}{\sigma^2} \right)^k \exp \left( \frac{-\beta^2}{\sigma^2} \right) \]

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Using equation (3.95) and (3.96) we get $V(\sigma)$ as

$$V(\sigma) = \sqrt{\frac{2\pi}{\Gamma(\tau+n)}} \frac{c^{\alpha+k}}{|k^*(\alpha,c)|^{\alpha+n+2}} \left[ k^*(\alpha+n-1,\beta) \right] - \sqrt{\frac{2\pi}{\Gamma(\tau+n)}} \frac{c^{\alpha+k}}{|k^*(\alpha,c)|^{\alpha+n+2}} \left[ k^*(\alpha+n-1,\beta) \right]$$

where $\beta' = c + \frac{1}{2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + \frac{\tau n}{2(\tau+n)} (\bar{x} - \theta)^2$.

Finally, in the next section Bayesian analysis of Normal sequence using the usual combinations of priors such as Normal and Inverted gamma priors for mean and scale parameters are discussed for the purpose of comparative studies.

**NORMAL AND INVERTED GAMMA PRIORS**

In this section, Bayes Estimates are obtained for location parameter which follows Normal and scale parameter which follows Inverted gamma prior.

**Case (i) Unknown mean $\mu$ and known variance $\sigma^2$**

This Case is Similar to the case (i) of Bayesian Analysis of Normal Sequence Using Normal and Extended inverted gamma Prior.

**Case (ii) known mean $\mu$ and unknown variance $\sigma^2$**

This Case is Similar to the case (ii) of Bayesian Analysis of Normal Sequence Using Double Exponential and Inverted Gamma Prior

**Case (iii) Unknown mean $\mu$ and unknown variance $\sigma^2$**

Let $X = (x_1, x_2, ..., x_n)$ be a random sample from a Normal population with unknown mean $\mu$ and unknown variance $\sigma^2$.

i.e., $f(x_i) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right)$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$, for $i = 1, 2, ..., n$.

The likelihood function is given by

$$P(x/\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^n}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\}$$  \(\text{...(3.98)}\)
Since the prior distributions of $\mu$ and $\sigma$ are respectively defined in the equations (3.90) and (3.70), the joint prior distribution of $\mu$ and $\sigma$ can be obtained as

$$P(\mu, \sigma/x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\mu - \bar{x})^2 \right\} \sqrt{\frac{\tau}{2\pi}}$$

$$\exp \left( -\frac{\tau}{2\sigma^2} (\mu - \theta)^2 \right) \frac{2^\beta}{\Gamma(\alpha)} (\sigma)^{(2\alpha+1)} \exp \left( -\frac{\beta}{\sigma^2} \right)$$

$$P(\mu, \sigma/x) = \frac{\sqrt{\tau}}{\pi} \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma)^{(2\alpha+n+2)} \exp \left( -\frac{1}{2\sigma^2} (\tau+n)(\mu - \mu)^2 \right) \exp \left( -\frac{\beta}{\sigma^2} \right)$$

... (3.99)

where $\beta = \beta + \frac{1}{2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + \frac{\tau n}{2(\tau+n)} (\bar{x} - \theta)^2$, $\mu' = \frac{n\bar{x} + n\theta}{\tau+n}$.

The posterior distribution of $\mu$ can be obtained by integrating equation (3.98) with respect to $\sigma$

$$P(\mu) = \frac{\sqrt{\tau}}{\pi} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{0}^{\infty} (\sigma)^{(2\alpha+n+1+1)} \exp \left( -\frac{1}{2\sigma^2} (\beta + \frac{\tau+n}{2}(\mu - \mu)^2) \right) \, d\sigma$$

Since by Reference from Box and Tiao pg.no.145

$$\int_{0}^{\infty} x^{-(p+1)} e^{-x^2/2} \, dx = \frac{1}{2} \int_{0}^{\infty} x^{-(p+1)} \, dx$$

$$P(\mu) = \frac{\sqrt{\tau}}{2\pi} \frac{\beta^\alpha}{\Gamma(\alpha)} \left( \frac{2\alpha+n+1}{2\beta + \frac{(\mu - \mu)^2}{\tau+n}} \right)$$

... (3.100)

It implies that the posterior distribution of $\mu$ follows ’t’ distribution with $2\alpha+n$ degrees of freedom. $\mu'$ is the estimate of the parameter $\mu$.

$$E(\mu) = \mu' = \frac{n\bar{x} + n\theta}{\tau+n}$$

The Posterior distribution of $\sigma$ can be obtained by integrating equation (3.98) with respect to $\mu$.

$$P(\sigma) = \frac{\sqrt{\tau}}{\pi} \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma)^{(2\alpha+n+1)} \exp \left( -\frac{\beta}{\sigma^2} \right) \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2\sigma^2} (\tau + n)(\mu - \mu)^2 \right) \, d\mu$$

$$P(\sigma) = \frac{\sqrt{2\pi\tau}}{\pi} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{\sqrt{\tau+n}} (\sigma)^{(2\alpha+n+1)} \exp \left( -\frac{\beta}{\sigma^2} \right)$$

... (3.101)

where $\beta = \beta + \frac{1}{2} \sum_{i=1}^{n} (x_i - \bar{x})^2 + \frac{\tau n}{2(\tau+n)} (\bar{x} - \theta)^2$. 

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Posterior mean of $\sigma$ is obtained by

$$\mathbb{E}(\sigma) = \int_0^\infty \sigma p(\sigma|x) \, d\sigma$$

$$= \int_0^\infty \sigma \frac{\sqrt{2\pi} \beta^\alpha}{\pi^{|\alpha|}} \frac{1}{\sqrt{\tau+n}} (\sigma)^{2\alpha+n+1} \exp\left(\frac{-\sigma^2}{\sigma^2}\right) \, d\sigma$$

$$\mathbb{E}(\sigma) = \frac{\sqrt{2\pi} \beta^\alpha}{\pi^{|\alpha|}} \frac{1}{\sqrt{\tau+n}} \frac{\alpha^{2\alpha+n-1}}{2^{2\alpha+n-1}}$$

$$\mathbb{E}(\sigma^2) = \frac{\sqrt{2\pi} \beta^\alpha}{\pi^{|\alpha|}} \frac{1}{\sqrt{\tau+n}} \frac{\alpha^{2\alpha+n-2}}{2^{2\alpha+n-2}}$$

Using equation (3.101) and (3.102) we get $V(\sigma)$ as

$$V(\sigma) = \frac{\sqrt{2\pi} \beta^\alpha}{\pi^{|\alpha|}} \frac{1}{\sqrt{\tau+n}} \frac{\alpha^{2\alpha+n-2}}{2^{2\alpha+n-2}} \left[ \frac{\sqrt{2\pi} \beta^\alpha}{\pi^{|\alpha|}} \frac{1}{\sqrt{\tau+n}} \frac{\alpha^{2\alpha+n-1}}{2^{2\alpha+n-1}} \right]^2$$

where $\beta'$ is mentioned in equation 3.101.