CHAPTER 2
FUZZY FRAMES

2.1 Introduction

In this chapter we generalise the concept of Frame in to a Fuzzy Frame and some results related to that are obtained.

2.2 Fuzzy Frame

We give the following definition for fuzzy frame.

Definition 2.2.1. Let $F$ be a frame; then a fuzzy set $\mu: F \rightarrow [0,1]$ of $F$ is said to be a fuzzy frame if,

(F1) $\mu(\bigvee S) \geq \inf \{ \mu(a) \mid a \in S \}$ for every arbitrary $S \subseteq F$

(F2) $\mu( a \land b) \geq \min \{ \mu(a), \mu(b) \}$ for all $a, b \in F$

(F3) $\mu(e_F) = \mu(O_F) \geq \mu(a)$ for all $a \in F$, where $e_F$ and $O_F$ are respectively the unit and zero element of the frame $F$.

Example 2.2.2. Let $\mu^a$ be a fuzzy set of $I= [0,1]$ defined by,

$$\mu^a(x) = \begin{cases} 
a, & x = 0,1 \\
x, & 0 < x \leq \frac{1}{2} \quad \text{where a is some chosen element in} \ (\frac{1}{2},1] \\
1-x, & \frac{1}{2} < x < 1
\end{cases}$$

Then $\mu^a$ is a fuzzy frame of $I$.

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Example 2.2.3. Consider the set $\mathbb{R}$ of real numbers with usual topology $\tau$, which is a frame. Let $\mu$ be a fuzzy set in $\tau$ defined by,

$$\mu(u) = \begin{cases} 
1, & u = \mathbb{R} \phi \\
\frac{1}{2}, & u \neq \mathbb{R} \phi 
\end{cases}$$

where $u \in \tau$

Then $\mu$ is a fuzzy frame of $\tau$.

Example 2.2.4. Let $F$ be a frame with $n$ elements. Let $(F_i)_{i=1}^{2m}$ be a strictly increasing chain of subframes of $F$ where $F_1 = \{ e_F, O_F \}$ and $F_{2m} = F$. Define fuzzy sets $\mu$ and $\lambda$ on $F$ as follows,

$\mu: F \to [0,1]$ such that

$$\mu(e_F) = \mu(O_F) = 1, \mu(x) = \begin{cases} 
\frac{1}{2^{k+1}}, & \text{if } x \in F_{2k+1} - F_{2k-1} \text{ for } k = 1, 2, \ldots, m-1 \\
\frac{1}{2^{m+1}}, & \text{if } x \in F_{2m} - F_{2m-1} 
\end{cases}$$

$\lambda: F \to [0,1]$ such that

$$\lambda(e_F) = \lambda(O_F) = 1, \lambda(x) = \begin{cases} 
\frac{1}{2}, & \text{if } x \in F_2 - F_1 \\
\frac{1}{2k}, & \text{if } x \in F_{2k} - F_{2k-2} \text{ for } k = 2, \ldots, m 
\end{cases}$$

Then $\mu$ and $\lambda$ are fuzzy frames of $F$.

Proposition 2.2.5. If $\mu$ is a fuzzy frame of $F$ then $\mu_t$ is a sub frame of $F$ for any $t \in I$ with $t \leq \mu(e_F) = \mu(O_F)$.

Proof. For arbitrary $\{a_i\} \in \wedge \subseteq \mu_t$ we have $\mu(\lor a_i) \geq t$, since $\mu$ is a fuzzy frame and
\[ \mu(a_i) \geq t \text{ for all } i. \text{ Hence } \vee a_i \in \mu, \text{ Similarly for all } a, b \in \mu \text{ we have } a \wedge b \in \mu. \text{ Also clearly } e_F, O_F \in \mu \text{ Therefore } \mu \text{ is a subframe of } F. \]

**Remark 2.2.6.** If \( E \) is a subset of a frame \( F \) then \( E \) is a subframe of \( F \) if and only if \( \chi_E \) is a fuzzy frame of \( F \), where \( \chi_E \) is the characteristic function of \( E \).

**Definition 2.2.7.** Let \( \mu \) be a fuzzy frame and \( \mu_t \) be a level subset of the frame \( F \) for some \( t \in I \) with \( t \leq \mu(\varepsilon_F) \). Then \( \mu_t \) is called a level subframe of \( F \).

Denote \( \mu_t > \mu'_t \) if \( \mu_t \supset \mu'_t \). Now since \( t < t' \) if and only if \( \mu_t > \mu'_t \) for any \( t, t' \) in \( \mu(F) \) every fuzzy frame of a frame \( F \) gives a chain with level subframes of \( F \),

\[
\{ O_F, e_F \} = \mu_{t_0} < \mu_{t_1} < \ldots < \mu_{t_r} = F \text{ where } t_i \in \text{Im } \mu \text{ and } t_0 > t_1 > \ldots > t_r.
\]

Since all subframes of a frame \( F \) usually do not form a chain we have not all subframes are level subframes of the same fuzzy frame.

We shall denote the chain of level subframes of a frame \( F \) by \( \Gamma_{\mu} (F) \).

**Definition 2.2.8.** Let \( X \) be the set of all fuzzy frames of \( F \), the relation "~" in \( X \) defined by \( \mu \sim \mu' \) if and only if for all \( x, y \in F \), \( \mu(x) > \mu(y) \iff \mu'(x) > \mu'(y) \). Then "~" is an equivalence relation on \( X \).

**Proposition 2.2.9.** Let \( \mu \) and \( \mu' \) be two fuzzy frames of a frame \( F \) then \( \mu \sim \mu' \) if and only if \( \Gamma_{\mu} (F) = \Gamma_{\mu'} (F) \).
Proof. Let $\mu_t \in \Gamma_\mu (\mathbb{F})$ and take $t' = \inf \{ \mu_a \mid a \in \mu_t \}$ then $\mu_t = \mu'_{t'}$. Similarly if $\mu'_{t'} \in \Gamma_{\mu'} (\mathbb{F})$ and $t = \inf \{ \mu_a \mid a \in \mu'_{t'} \}$ then $\mu'_{t'} = \mu_t$. Hence $\Gamma_\mu (\mathbb{F}) = \Gamma_{\mu'} (\mathbb{F})$.

Conversely for any $x, y$ in $\mathbb{F}$ if $\mu(x) > \mu(y)$ then $y \not\in \mu(\mu(x)) = \mu'(t)$ and $\mu(y) < t \leq \mu(x)$ it follows that $\mu'(x) > \mu'(y)$. Similarly $\mu'(x) > \mu'(y)$ implies that $\mu(x) > \mu(y)$. Hence $\mu \sim \mu'$.

Note 2.2.10. Thus two fuzzy frames $\mu$ and $\eta$ of a frame $\mathbb{F}$ are said to be equivalent if they have the same family of level subframes otherwise $\mu$ and $\eta$ are non-equivalent.

We shall denote the equivalence class of $\mu$ by $[\mu]$.

Proposition 2.2.11. If two equivalent fuzzy frames $\mu$ and $\eta$ of a frame have the same image sets then $\mu = \eta$.

Proof. Obvious.

Proposition 2.2.12. If each non-empty level subset $\mu_t$, $t \in I$ of a fuzzy set $\mu$ is a subframe of $\mathbb{F}$, then $\mu$ is a fuzzy frame of $\mathbb{F}$.

Proof. Given $\mu_t = \{ x \in \mathbb{F} \mid \mu(x) \geq t \}$, $t \in I$ is a subframe of $\mathbb{F}$. $\mu_t$ being a subframe $\mathbb{O}_\mathbb{F}$, $\mathbb{E}_\mathbb{F} \in \mu_t$, $t \in I$. In particular we have $\mathbb{O}_\mathbb{F}$, $\mathbb{E}_\mathbb{F} \in \mu_t$ where $T$ the largest element of $I$ such that $\mu_T \neq \phi$. Hence $\mu(\mathbb{E}_\mathbb{F}) = \mu(\mathbb{O}_\mathbb{F}) = T \geq \mu(a)$ for all $a \in \mathbb{F}$. Now let $S$ an arbitrary subset of $\mathbb{F}$ and let $t = \inf \{ \mu(a) \mid a \in S \}$. Clearly we have $S \subseteq \mu_t$ hence
∧ S ∈ \( \mu \) and therefore \( \mu(\bigvee S) \geq \inf \{ \mu(a) \mid a \in S \} \). Similarly for all \( a, b \in F \) we have
\( \mu(a \land b) \geq \min \{ \mu(a), \mu(b) \} \). Hence \( \mu \) is a fuzzy frame of \( F \).

**Theorem 2.2.13.** Let \( \mu \) be a fuzzy subset of a frame \( F \). Then \( \mu \) is a fuzzy frame of \( F \) if and only if each non-empty level subset \( \mu_i \) of \( \mu \) is a subframe of \( F \).

**Proof.** Follows from Proposition 2.2.5 and Proposition 2.2.12.

**Theorem 2.2.14.** Let \( F \) be a frame of finite order then there exists a fuzzy frame \( \mu \) of \( F \) such that \( \Gamma_\mu(F) \) is a maximal chain of all subframes of \( F \).

**Proof.** Since \( F \) is frame of finite order, the number of subframes of \( F \) is finite. So there exists some maximal chain of subframes of \( F \).

Take \( F_0 = \{ O_F, \, e_F \} < F_1 < F_2 < \ldots < F_n = F \).

Now define \( \mu(F_0) = \{ 1 \} \) and \( \mu(F_{i+1} \setminus F_i) = \{ 1/\nu_{i+1} \} \) for any \( i, 0 \leq i < n \). Clearly \( \mu \) is a fuzzy frame of \( F \) and is given by the chain (1).

**Remark 2.2.15.** If \( F \) is a frame of finite order and \( \mu \) a fuzzy frame of it then \( \Gamma_\mu(F) \) is completely determined by \( \mu \) and conversely for any finite frame \( F \) and the subframe chain \( \{ O_F, \, e_F \} < F_1 < F_2 < \ldots < F_n = F \) there exists an equivalence class of fuzzy frames of \( F \) such that \( \Gamma_\mu(F) \) is the above chain.

**Remark 2.2.16.** If \( [\mu] \neq [0] \) then there exists a fuzzy frame \( \eta \) of \( F \) in \( [\mu] \) such that \( \eta(e_F) = \eta(O_F) = 1 \).
Theorem 2.2.17. If $H$ is a subframe of $F$, $\mu$ a fuzzy frame of $F$ and $\eta$ is restriction of $\mu$ to $H$ then $\eta$ is a fuzzy frame of $H$.

Proof. Obvious

Theorem 2.2.18. Let $\{I_\alpha \mid \alpha \in \Lambda \}$ be a collection of subframes of $F$ such that

i) $F = \bigcup_{\alpha \in \Lambda} I_\alpha$

ii) $s > t$ if and only if $I_s \subset I_t$ for all $s, t \in \Lambda$ where $\Lambda$ a collection of elements in $[0,1]$. Then a fuzzy set $\mu$ defined on $F$ by $\mu(x) = \sup \{ t \in \Lambda \mid x \in I_t \}$ is a fuzzy frame of $F$.

Proof. By Proposition 2.2.12 it is enough to show that non-empty level sets $\mu_t = \{x \in F \mid \mu(x) > t\}, t \in I$ are subframes of $F$. We have the following two cases,

Case-I. $t = \sup \{ s \in \Lambda \mid s < t \}$

$a \in \mu_t \Leftrightarrow a \in \{x \in F \mid \mu(x) > t\} \Leftrightarrow a \in I_s$ for all $s < t \Leftrightarrow a \in \bigcap_{s < t} I_s$

Therefore $\mu_t = \bigcap_{s < t} I_s$ is a subframe of $F$.

Case-II. $t \neq \sup \{ s \in \Lambda \mid s < t \}$

In this case $\mu_t = \bigcup_{s \geq t} I_s$. For if $a \in \bigcup_{s \geq t} I_s$ then $a \in I_s$ for some $s \geq t$.

Hence we have $\mu(x) \geq t$. Therefore $x \in \mu_t$ and hence $\bigcup_{s \geq t} I_s \subseteq \mu_t$.

Now suppose $x \notin \bigcup_{s \geq t} I_s$. Then $x \notin I_s$ for all $s \geq t$. 

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Since $t \neq \sup \{ s \in \Lambda \mid s < t \}$ there exist $\varepsilon > 0$ such that \((t - \varepsilon, t) \cap \Lambda = \emptyset\).

Hence $x \notin I_s$ for all $s \geq t - \varepsilon$. Thus $\mu(x) < t - \varepsilon < t$ and so $x \notin \mu_t$.

Therefore $\bigcup_{s \geq t} I_s \supseteq \mu_t$.

Thus $\mu_t = \bigcup_{s \geq t} I_s$, which is therefore a subframe of $F$.

Combining the two cases we have the required result.

Definition 2.2.19. Let $\mu$ be any fuzzy subset of the frame $F$ then the fuzzy frame generated by $\mu$ in $F$ is the least fuzzy frame of $F$ containing $\mu$ and is denoted by $\langle \mu \rangle$.

Theorem 2.2.20. Let $\mu$ be a fuzzy set of the frame $F$ then $\langle \mu \rangle(x) = \vee \{ t \mid x \in \langle \mu_t \rangle \}$ for all $x \in F$, where $\langle \mu_t \rangle$ is the subframe of $F$ generated by $\mu_t$.

Proof. Let $\eta$ be any fuzzy frame of the frame $F$ defined by $\eta(x) = \vee \{ t \mid x \in \langle \mu_t \rangle \}$ for all $x \in F$. Then for any arbitrary $S \subset F$ we have for all $x \in S$, $\eta(x) \geq \inf \{ \eta(y) \mid y \in S \}$. Now $S \subset \langle \mu_t \rangle \Rightarrow \vee S \in \langle \mu_t \rangle$, hence $\eta(\vee S) \geq \inf \{ \eta(y) \mid y \in S \}$. Also for $x, y \in F$ let $\eta(x) = t_1$ and $\eta(y) = t_2$. Suppose that $t_1 > t_2$. Then $y \in \langle \mu_{t_2} \rangle \Rightarrow x \in \langle \mu_{t_1} \rangle$ and so $x \wedge y \in \langle \mu_{t_1} \rangle$, hence $\eta(x \wedge y) \geq t_1 \wedge t_2$.

Again since $e_F, O_F \in \langle \mu_t \rangle$ for all $t$ such that $\mu_t \neq \emptyset$ it follows that $\eta(e_F) = \eta(O_F) \geq \eta(x)$ for all $x \in F$. Thus $\eta$ is a subframe of $F$.

Let $\mu(x) = t$, then $x \in \mu_t \subseteq \langle \mu_t \rangle$ and thus $\eta(x) \geq \mu(x)$. Hence $\eta \geq \langle \mu \rangle$ since
\( \langle \mu \rangle \) is the smallest fuzzy frame of \( F \) which containing \( \mu \). Now let \( \gamma \) be any fuzzy frame of \( F \) such that \( \gamma \supseteq \mu \) then \( \gamma_t \supseteq \mu_t \) and so \( \gamma_t \supseteq \langle \mu_t \rangle \) for all \( t \). Hence \( \gamma \supseteq \eta \).

Thus \( \eta = \langle \mu \rangle \). Therefore the result follows.

### 2.3 Homomorphisms

**Theorem 2.3.1.** Let \( L \) and \( M \) be two frames, \( \Phi \) a frame homomorphism from \( L \) onto \( M \) and \( \mu \) a fuzzy frame of \( M \), then \( \lambda = \mu \circ \Phi \) is a fuzzy frame of \( L \).

**Proof.** Let \( S \) be an arbitrary subset of \( L \). Now \( \Phi(\vee S) \in M \) and equal to \( \vee \{ \Phi(a) \mid a \in S \} \).

Since \( \mu \) is a fuzzy frame by Definition 2.2.1,

\[
\mu \circ \Phi ( \vee S ) = \mu \{ \vee \{ \Phi(a) \mid a \in S \} \} \supseteq \inf \{ \mu(\Phi ( a )) \mid a \in S \}.
\]

Also for all \( a, b \in L \), \( \mu \circ \Phi ( a \land b ) = \mu \{ \Phi(a) \land \Phi(b) \} \supseteq \min \{ \mu(\Phi ( a )), \mu(\Phi ( b )) \} \).

Again \( \mu(\Phi ( \bot )) = \mu(\Phi ( \top )) \). Therefore \( \lambda \) is a fuzzy frame of \( L \).

**Definition 2.3.2.** Let \( \lambda, \mu \) be fuzzy frames of frames \( L \) and \( M \) respectively. If there is a frame homomorphism \( f \) from \( L \) onto \( M \) such that \( \lambda = \mu \circ f \) then we say \( \lambda \) is homomorphic to \( \mu \) and is denoted by \( f^{-1}(\mu) \).

If \( f \) is an isomorphism then we say that \( \mu \) and \( \lambda \) are isomorphic.

**Lemma 2.3.3.** Let \( f \) be a homomorphism from a frame \( L \) on to a frame \( M \) and let \( \mu \) be any fuzzy frame of \( M \) then \( (f^{-1}(\mu))_t = f^{-1}(\mu_t) \) for every \( t \in I \).

**Proof.** Let \( x \in L \)
Remark 2.3.4. Theorem 2.3.1 follows also from above lemma since the homomorphic preimage of subframe is a subframe and again by Theorem 2.2.13 if $\mu$ is any fuzzy frame of the frame $F$ then every non-empty level subset of $\mu$ is also a sub frame of $F$.

Theorem 2.3.5. Let $f : L \to M$ be a homomorphism between frames $L$ and $M$. Then for every fuzzy frame $\mu$ of $L$, $f(\mu)$ is a fuzzy frame of $M$.

Proof. Define for all $y \in M$,

$$f(\mu)(y) = \begin{cases} \sup \{\mu(x) | x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Now for any arbitrary $S \subset M$ we have,

$$f(\mu)(\lor S) = \sup \{\mu(x) | x \in f^{-1}(\lor S)\}$$

$$\geq \inf \{\sup \{\mu(x) | x \in f^{-1}(y)\} | y \in S\} = \inf \{f(\mu(y)) \}.$$ 

Again for all $a, b \in M$ we have,

$$f(\mu)(a \land b) = \sup \{\mu(x) | x \in f^{-1}(a \land b)\}$$

$$\geq \min \{\sup(\mu(x) | x \in f^{-1}(a)), \sup(\mu(x) | x \in f^{-1}(b))\}$$

$$= \min \{f(\mu(a)), f(\mu(b))\}.$$ 

Also $f(\mu)$ preserves the unit and the zero elements of $M$.

Hence $f(\mu)$ is a fuzzy frame of $M$. 

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Theorem 2.3.6. Let $F$ be a frame of finite order and $f: F \rightarrow F'$ be an onto homomorphism. Let $\mu$ be a fuzzy frame of $F$ with $\text{Im } \mu = \{t_0, t_1, \ldots, t_n\}$ and $t_0 > t_1 > \ldots > t_n$. If the chain of level subframes of $\mu$ is $\{O_F, e_F\} = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \ldots \subseteq \mu_{t_n} = F$. Then the chain of level subframe of $f(\mu)$ will be $\{O_{F'}, e_{F'}\} = f(\mu_{t_0}) \subseteq f(\mu_{t_1}) \subseteq \ldots \subseteq f(\mu_{t_n}) = F'$.

Proof. Given $F$ is a frame of finite order. We have $f(\mu)$ is a fuzzy frame of $F'$ by Theorem 2.3.5. Also clearly $\text{Im } f(\mu) \subseteq \text{Im } \mu$. Now $f(\mu)_{t_i} = f(\mu_{t_i})$ for each $t_i \in \text{Im } f(\mu)$.

For let $y \in f(\mu)_{t_i}$ then $f(\mu)(y) \geq t_i$ by definition of level subset. Hence $\sup \{\mu(x) | x \in f^{-1}(y)\} \geq t_i$ follows from the proof of Theorem 2.3.5. Now choose $x_0 \in F$ such that $f(x_0) = y \in f(\mu)_{t_i}$. It follows that $f(\mu)_{t_i} \subseteq f(\mu)_{t_i}$

(1)

Let $x \in f(\mu)_{t_i}$. Then $x \in \mu_{t_i}$ hence $\mu(x) \geq t_i$ which implies $\sup \{\mu(z) | z \in f^{-1}(f(x))\} \geq t_i$ which implies $f(\mu)(f(x)) \geq t_i$ by Theorem 2.3.5. Hence $f(x) \in f(\mu)_{t_i}$ by definition of the level subset.

It follows that $f(\mu)_{t_i} \subseteq f(\mu)_{t_i}$

(2)

From (1) and (2) we have $f(\mu)_{t_i} = f(\mu)_{t_i}$

(3)

Also if $\mu_{t_i} \subseteq \mu_{t_j}$ then $f(\mu)_{t_i} \subseteq f(\mu)_{t_j}$ for $t_i, t_j \in \text{Im } \mu$.

(4)

Combining (3) and (4) we have the required result.
2.4 Intersection and union of fuzzy frames

Let $\mu$ and $\lambda$ be two fuzzy frames of $F$ then $\mu \subseteq \lambda$ means $\mu(x) \leq \lambda(x)$ for all $x \in F$. Let $F$ denote the set of all fuzzy frames of the frame $F$. We shall denote the supremum and infimum in $F$ by $\bigcup$ (union) and $\bigcap$ (intersection) respectively.

Thus $\bigcap_{i \in \Lambda} \mu_i(a) = \inf\{\mu_i(a) \mid i \in \Lambda\}$ and $\bigcup_{i \in \Lambda} \mu_i(a) = \sup\{\mu_i(a) \mid i \in \Lambda\}$ where $\mu_i \in F$. The greatest element of $F$ is $1_F$, which is the function $\chi_F$ and $F$ has no least element.

Proposition 2.4.1. The intersection of any set of fuzzy frames on the frame $F$ is a fuzzy frame.

Proof. We have $\bigcap_{i \in \Lambda} \mu_i(O_F) = \bigcap_{i \in \Lambda} \mu_i(E_F) \geq \bigcap_{i \in \Lambda} \mu_i(x)$ for all $x \in F$ clearly.

Also for all $x, y \in F$

$$\bigcap_{i \in \Lambda} \mu_i(x \wedge y) = \inf\{\mu_i(x \wedge y) \mid i \in \Lambda\} \geq \inf\{\min(\mu_i(x), \mu_i(y)) \mid i \in \Lambda\}$$

$$= \min(\inf\{\mu_i(x) \mid i \in \Lambda\}, \inf\{\mu_i(y) \mid i \in \Lambda\}) = \min(\bigcap_{i \in \Lambda} \mu_i(x), \bigcap_{i \in \Lambda} \mu_i(y))$$

Similarly for arbitrary $S \subseteq F$ we have,

$$\bigcap_{i \in \Lambda} \mu_i(\bigvee S) \geq \inf_{x \in S} \inf_{i \in \Lambda} (\mu_i(x)) = \inf_{x \in S} (\inf_{i \in \Lambda} (\mu_i(x)))$$

$$= \inf_{x \in S} (\bigcap_{i \in \Lambda} \mu_i(x))$$
Remark 2.4.2. The union of arbitrary family of fuzzy frames on a frame $F$ need not be a fuzzy frame.

For consider the frame $F = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ where $X = \{a, b, c\}$ and the order is set inclusion.

Consider the fuzzy sets $\mu$ and $\lambda$ defined on $F$ by,

\[
\mu(X) = \mu(\emptyset) = 1, \quad \mu(\{a\}) = \frac{1}{3}, \quad \mu(\{b\}) = \frac{1}{2}, \quad \mu(\{a, b\}) = \frac{1}{6}
\]

\[
\lambda(X) = \lambda(\emptyset) = 1, \quad \lambda(\{a\}) = \frac{2}{3}, \quad \lambda(\{b\}) = \frac{1}{3}, \quad \lambda(\{a, b\}) = \frac{1}{4}
\]

Clearly $\mu$ and $\lambda$ are fuzzy frames.

Here $(\mu \cup \lambda)(X) = (\mu \cup \lambda)(\emptyset) = 1, \quad (\mu \cup \lambda)(\{a\}) = \frac{2}{3}, \quad (\mu \cup \lambda)(\{b\}) = \frac{1}{2},

(\mu \cup \lambda)(\{a, b\}) = \frac{1}{3}$. Now $\mu \cup \lambda$ is not a fuzzy frame as,

\[
(\mu \cup \lambda)(\{a\} \vee \{b\}) = (\mu \cup \lambda)(\{a, b\}) = \frac{1}{3} < \inf\{(\mu \cup \lambda)(\{a\}), (\mu \cup \lambda)(\{b\})\}
\]

Remark 2.4.3. The union of any chain of fuzzy frames is clearly a fuzzy frame. We can also have two non-comparable fuzzy frames such that their union is a fuzzy frame. For consider Example 2.2.4 where we have $\mu$ and $\lambda$ are fuzzy frames of $F$ such that neither $\mu \leq \lambda$ nor $\lambda \leq \mu$. Also $\mu \cup \lambda$ is given by $(\mu \cup \lambda)(e_{F}) = (\mu \cup \lambda)(O_{F}) = 1,

(\mu \cup \lambda)(x) = \frac{1}{k}$ if $x \in F_{k} \setminus F_{k-1}$ for $k = 2, 3, \ldots, m$ Hence $\mu \cup \lambda$ is a fuzzy frame of $F$.

Theorem 2.4.4. Let $(\mu_{i})_{i=1, 2\ldots n}$ be a finite collection of fuzzy frames of a frame $F$. 

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Then $\bigcup_{i} \mu_i$ is a fuzzy frame if and only if for $t \in [0,1]$, $\mu_i(x) \geq t$ for all $x \in S$ an arbitrary subset of $F$ and $\mu_i(x) \geq t$, $\mu_i(y) \geq t$ for all $x, y \in F$ implies $\mu_k(\vee S) \geq t$ and $\mu_k(x \wedge y) \geq t$ for some $k, 1 \leq k \leq n$.

**Proof.** By Theorem 2.2.13 $\bigcup_{i} \mu_i$ is a fuzzy frame if and only if each nonempty level subset $(\bigcup_{i} \mu_i)_t$ is a subframe of $F$. Now $(\bigcup_{i} \mu_i)_t = \bigcup_{i} (\mu_i)_t$ for each $t \in [0,1]$.

But $\bigcup_{i} (\mu_i)_t$ is a subframe of $F$ if and only if for any arbitrary $S \subseteq \bigcup_{i} (\mu_i)_t$ and $x, y \in \bigcup_{i} (\mu_i)_t$ we have $\vee S \in \bigcup_{i} (\mu_i)_t$ and $x \wedge y \in \bigcup_{i} (\mu_i)_t$.

That is $\mu_i(x) \geq t$ for all $x \in S$ an arbitrary subset of $F$ and $\mu_i(x) \geq t$, $\mu_i(y) \geq t$ for all $x, y \in F$ implies $\mu_k(\vee S) \geq t$ and $\mu_k(x \wedge y) \geq t$ for some $k, 1 \leq k \leq n$.

**Proposition 2.4.5.** If the set of all fuzzy frames of $F$ under usual ordering of fuzzy set inclusion $\leq$ is not a complete lattice.

**Proof.** Since $F$ has no infimum the result follows.

**Theorem 2.4.6.** Let $S$ be the set of fuzzy frames of a frame $F$ such that $\mu_i(\emptyset_F) = \mu_i(\emptyset_F) = 1$ for all $\mu_i \in S$. Then $S$ forms a complete lattice under the usual ordering of fuzzy set inclusion $\leq$.

**Proof.** Let $\{ \mu_i \mid i \in \Lambda \}$ be a family of fuzzy frames of a frame $F$. Since $\bigcap_{i \in \Lambda} \mu_i$ is the
largest fuzzy frame of $F$ contained in each $\mu_i$ we set $\bigwedge_{i \in \Lambda} \mu_i = \bigcap_{i \in \Lambda} \mu_i$. Also since the fuzzy frame generated by the union $\bigcup_{i \in \Lambda} \mu_i$ is the largest fuzzy frame containing each $\mu_i$ we set $\bigvee_{i \in \Lambda} \mu_i = \left\langle \bigcup_{i \in \Lambda} \mu_i \right\rangle$, where $\left\langle \bigcup_{i \in \Lambda} \mu_i \right\rangle$ is the fuzzy frame generated by $\bigcup_{i \in \Lambda} \mu_i$. Also $X_{\{o_F, e_F\}}$ and $X_f$ are respectively the least and greatest element of $S$

Thus $S$ is a complete lattice.

Remark 2.4.7. $S$ is not atomic for if $\mu = X_{\{o_F, e_F\}} \vee a$, be an atom where $a \in F$ is a fuzzy singleton, then we can find a $t' < t$ such that $\mu' = X_{\{o_F, e_F\}} \vee a' < \mu$.

Theorem 2.4.8. Let $f$ be a homomorphism of a frame $F$ into a frame $F'$. Let $\{\mu_i \mid i \in \Lambda \}$ be a family of fuzzy frames of $F$.

i) If $\bigcup_{i \in \Lambda} \mu_i$ is a fuzzy frame of $F$, then $\bigcup_{i \in \Lambda} f(\mu_i)$ is a fuzzy frame of $F'$.

ii) If $\bigcup_{i \in \Lambda} f(\mu_i)$ is a fuzzy frame of $F'$, then $\bigcup_{i \in \Lambda} \mu_i$ is a fuzzy frame of $F$, provided $\mu_i$'s are $f$-invariant.

Proof. i) Suppose $\bigcup_{i \in \Lambda} \mu_i$ is a fuzzy frame of $F$. Then the homomorphic image $f(\bigcup_{i \in \Lambda} \mu_i)$ is a fuzzy frame of $F'$ by Theorem 2.3.5.

Now since $f(\bigcup_{i \in \Lambda} \mu_i) = \bigcup_{i \in \Lambda} f(\mu_i)$ by Proposition 1.5.19 we have $\bigcup_{i \in \Lambda} f(\mu_i)$ is a fuzzy frame of $F'$.
fuzzy frame of $F'$.

ii) Suppose $\bigcup_{i \in \Lambda} f(\mu_i)$ is a fuzzy frame of $F$. Then $f^{-1}(\bigcup_{i \in \Lambda} f(\mu_i))$ is a fuzzy frame of $F$ by theorem 4.2. Also since $f^{-1}(\bigcup_{i \in \Lambda} f(\mu_i)) = \bigcup_{i \in \Lambda} f^{-1}(f(\mu_i)) = \bigcup_{i \in \Lambda} \mu_i$ by Proposition 1.5.19 we have $\bigcup_{i \in \Lambda} \mu_i$ is a fuzzy frame of $F$.

Theorem 2.4.9. Let $f$ be a homomorphism of a frame $F$ onto a frame $F'$ and $\{\lambda_i \mid i \in \Lambda\}$ be a family of fuzzy frames of $F'$ then the following are equivalent,

i) $\bigcup_{i \in \Lambda} \lambda_i$ is a fuzzy frame of $F'$.

ii) $\bigcup_{i \in \Lambda} f^{-1}(\lambda_i)$ is a fuzzy frame of $F$.

Proof. Suppose $\bigcup_{i \in \Lambda} \lambda_i$ is a fuzzy frame of $F'$. Now by Theorem 2.3.1 $f^{-1}(\bigcup_{i \in \Lambda} \lambda_i)$ is a fuzzy frame of $F$. Also by Proposition 1.5.19 we have $f^{-1}(\bigcup_{i \in \Lambda} \lambda_i) = \bigcup_{i \in \Lambda} f^{-1}(\lambda_i)$. Therefore $\bigcup_{i \in \Lambda} f^{-1}(\lambda_i)$ is a fuzzy frame of $F$.

Conversely suppose $\bigcup_{i \in \Lambda} f^{-1}(\lambda_i)$ is a fuzzy frame of $F$. Now by Theorem 2.3.5 $f(\bigcup_{i \in \Lambda} f^{-1}(\lambda_i))$ is a fuzzy frame of $F'$. Also by Proposition 1.5.19 we have $f(\bigcup_{i \in \Lambda} f^{-1}(\lambda_i)) = \bigcup_{i \in \Lambda} \lambda_i$. Therefore $\bigcup_{i \in \Lambda} \lambda_i$ is a fuzzy frame of $F'$. 

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2.5 Product of Fuzzy frames

Definition 2.5.1. Let $(\mu, L)$ and $(\eta, M)$ be fuzzy frames where $L$ and $M$ underlying sets which are frame. A morphism $\tilde{f} : (\mu, L) \rightarrow (\eta, M)$ is a frame homomorphism $f : L \rightarrow M$ such that $\mu \leq \eta \circ f$. That is the degree of membership of $x$ in $L$ does not exceed that of $f(x)$ in $M$. The function $f : L \rightarrow M$ is called the underlying function of $\tilde{f}$.

Definition 2.5.2. Let $\tilde{f} : (\mu, L) \rightarrow (\eta, M)$ and $\tilde{g} : (\eta, M) \rightarrow (\gamma, N)$ be morphisms then $\tilde{g} \circ \tilde{f} : (\mu, L) \rightarrow (\gamma, N)$ is a frame homomorphism $g \circ f : L \rightarrow M$ such that $\mu \leq \gamma \circ g \circ f$.

Let FFrm denote a category whose objects are fuzzy frames and morphisms as defined above. We have the following theorem.

Theorem 2.5.3. The category FFrm of fuzzy frames has equalizers.

Proof. Let $(\mu, L)$ and $(\eta, M)$ be fuzzy frames.

Let $\tilde{f} : (\mu, L) \rightarrow (\eta, M)$ and $\tilde{g} : (\mu, L) \rightarrow (\eta, M)$ be two morphisms.

Consider $\xymatrix{ L \ar[r]^f & M \ar[l]_g}$

Let $K = \{ x \in L \mid f(x) = g(x) \}$ which is a subframe of $L$ and let $i : K \rightarrow L$ be the inclusion map. Then clearly $f \circ i = g \circ i$.

Define a fuzzy set $\lambda$ on $K$ as follows, for $a \in K$ let $\lambda(a) = \mu(a)$.

Then $\tilde{i}$ is morphism from $(\lambda, K)$ to $(\mu, L)$. 

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If for arbitrary fuzzy frame \((\xi, N)\), \( \tilde{h} \) is a morphism from \((\xi, N)\) to \((\mu, L)\) such that 
\[ f \circ h = g \circ h \]
then there exist \( \theta: N \rightarrow K \) such that 
\[ i \circ \theta = h. \]

Also \( \xi \leq \lambda \circ \theta \) as for \( z \in N \), 
\[ \xi(z) \leq \mu \circ h(z) = \mu(h(z)) = \mu(i(\theta(z))) = \mu(i(\theta(z))) = \lambda(\theta(z)) = (\lambda \circ \theta)(z) \]
Thus \( \tilde{\theta} \) is a morphism from \((\xi, N)\) to \((\lambda, K)\)

Now for \( z \in N \)
\[ (\mu \circ i \circ \theta)(z) = (\mu \circ i)(\theta(z)) = \mu(i(\theta(z))) = \mu(\theta(z)) = (\lambda \circ \theta)(z) \leq \xi(z) \]
Hence \( \mu \circ i \circ \theta \geq \xi \). Therefore the result follows.

**Definition 2.5.4.** Let \( F_{\alpha} \) be fuzzy frame of the frame \( F_{\alpha} \) for \( \alpha \in \Lambda \). The product of \( \mu_{\alpha} \)'s is the function 
\[ \mu = \prod_{\alpha \in \Lambda} \mu_{\alpha} \]
defined on the product 
\[ F = \prod_{\alpha \in \Lambda} F_{\alpha} \]
with usual order by 
\[ \mu((x_{\alpha})_{\alpha \in \Lambda}) = \inf_{\alpha \in \Lambda} \{ \mu_{\alpha}(x_{\alpha}) \} \]

**Proposition 2.5.5.** \( \mu = \prod_{\alpha \in \Lambda} \mu_{\alpha} \) is a fuzzy frame of \( F = \prod_{\alpha \in \Lambda} F_{\alpha} \)

**Proof.** We have \( F = \{ (a_{\alpha})_{\alpha \in \Lambda} | a_{\alpha} \in F_{\alpha} \text{ for } \alpha \in \Lambda \} \)
\( e_{F} = (e_{F_{\alpha}})_{\alpha \in \Lambda} \) and \( o_{F} = (o_{F_{\alpha}})_{\alpha \in \Lambda} \) are respectively the unit and zero element of \( F \).

i) For arbitrary \( S \subseteq F \) we have,
\[ \mu(\lor S) = \mu(\lor \{ x_{\alpha} | \alpha \in \Lambda \}) = \mu((\lor x_{\alpha})_{\alpha \in \Lambda}) \]
\[= \inf_{a \in \Lambda} \{ \mu_a(x_a) \} \]
\[\geq \inf_{a \in \Lambda} \{ \inf_{x_a} \{ \mu_a(x) \} \} \]
\[= \inf_{a \in \Lambda} \{ \inf_{x_a} \{ \mu_a(x) \} \} \]
\[= \inf_{x \in S} \mu(x) \]

ii) For all \( x = (x_a)_{a \in \Lambda}, y = (y_a)_{a \in \Lambda} \in \mathcal{F} \)
\[\mu((x_a)_{a \in \Lambda} \land (y_a)_{a \in \Lambda}) = \mu((x_a \land y_a)_{a \in \Lambda}) \]
\[= \inf_{a \in \Lambda} \{ \mu_a(x_a \land y_a) \} \]
\[\geq \inf_{a \in \Lambda} \{ \min \{ \mu_a(x_a), \mu_a(y_a) \} \} \]
\[= \min \{ \inf_{a \in \Lambda} \{ \mu_a(x_a) \}, \inf_{a \in \Lambda} \{ \mu_a(y_a) \} \} \]
\[= \min \{ \mu(x), \mu(y) \} \]

iii) \( \mu(\mathcal{F}) = \prod_{a \in \Lambda} \mu_a(e_{\mathcal{F}}) \)
\[= \inf_{a \in \Lambda} \{ \mu_a(e_{\mathcal{F}_a}) \} \]
\[= \inf_{a \in \Lambda} \{ \mu_a(O_{\mathcal{F}_a}) \} \]
\[= \prod_{a \in \Lambda} \mu_a(O_{\mathcal{F}}) = \mu(O_{\mathcal{F}}) \]

also \( \mu(\mathcal{F}) = \prod_{a \in \Lambda} \mu_a(e_{\mathcal{F}}) = \inf_{a \in \Lambda} \{ \mu_a(e_{\mathcal{F}_a}) \} \)
\[\geq \inf_{a \in \Lambda} \{ \mu_a(a_{\mathcal{F}}) \} \]
\[ \prod_{a \in \Lambda} \mu_a(a) \text{ for all } a = (a_a)_{a \in \Lambda} \in F = \mu(a) \]

Hence we have the required result.

Theorem 2.5.6. The category FFnm of fuzzy frames has products.

Proof. Consider a family of fuzzy frames \( \{(\mu_a, F_a) \mid a \in \Lambda\} \). Corresponding to the product \( F = \prod_{a \in \Lambda} F_a \) we have the fuzzy frame \((\mu, F)\) where \( \mu = \prod_{a \in \Lambda} \mu_a \). Now consider the projection (homomorphism) \( p_a : F \to F_a \). We have \( \mu((x_a)_{a \in \Lambda}) = \inf \{ \mu_a(x_a) \} \). Hence \( \mu \leq \mu_a \circ p_a \) for \( a \in \Lambda \).

Therefore \( p_a \) is morphism from \((\mu, F)\) to \((\mu_a, F_a)\) for \( a \in \Lambda \).

Now for arbitrary fuzzy frame \((\xi, M)\) if \( \tilde{u}_a \) is a morphism from \((\xi, M)\) to \((\mu_a, F_a)\). Then define \( \theta : M \to F \) as \( (\theta(z))_a = p_a(\theta(z)) = (p_a \circ \theta)(z) = u_a(z) \) for all \( a \in \Lambda \) and \( z \in M \). Now \( \theta(z) = (u_a(z)) \) is a frame map as \( u_a \) for \( a \in \Lambda \) is a frame map.

\[
\begin{array}{ccc}
M & \xrightarrow{\theta} & F \\
\downarrow{u_a} & & \downarrow{p_a} \\
F_a & &
\end{array}
\]

Also for \( z \in M \) we have \( \xi(z) \leq \mu_a \circ u_a(z) \) for all \( a \in \Lambda \) and hence,

\[ \xi(z) \leq \inf_{a \in \Lambda} \mu_a(u_a(z)) = \inf_{a \in \Lambda} \{ \mu_a(\theta(z)) \} = \mu(\theta(z)) = \mu \circ \theta(z). \]

Hence \( \xi \leq \mu \circ \theta \). Thus \( \tilde{\theta} \) a morphism from \((\xi, M)\) to \((\mu, F)\).

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Clearly \( P_{a} \circ \theta = u_{a} \) for all \( \alpha \in \Lambda \).

Also \( (\mu_{a} \circ P_{a} \circ \theta)(z) = (\mu_{a} \circ P_{a})(\theta(z)) \)

\[ = \mu_{a}(P_{a}(\theta(z))) \]

\[ = \mu_{a}(u_{a}(z)) \]

\[ = (\mu_{a} \circ u_{a})(z) \geq \xi(z) \]

Hence \( \xi \leq \mu_{a} \circ P_{a} \circ \theta \).

Thus for each family \((\mu_{a}, F_{a})_{a \in \Lambda}\) of fuzzy frames there is a fuzzy frame \((\mu, F)\) and morphisms \( \tilde{P}_{a}: (\mu, F) \to (\mu_{a}, F_{a}) \) such that for any fuzzy frame \((\xi, M)\) and family of morphisms \( \tilde{u}_{a}: (\xi, M) \to (\mu_{a}, F_{a}) \) there is a unique morphism \( \tilde{\theta}: (\xi, M) \to (\mu, F) \) such that \( P_{a} \circ \theta = u_{a} \) and \( \xi \leq \mu_{a} \circ P_{a} \circ \theta \) for all \( \alpha \in \Lambda \).

Therefore the result follows.

Theorem 2.5.7. The category FFrm of fuzzy frames is complete.

Proof. Follows from Theorem 2.5.3 and Theorem 2.5.6. \( \Box \)

Theorem 2.5.8. Let \( \mu_{1} \) and \( \mu_{2} \) be fuzzy sets of frames \( F_{1} \) and \( F_{2} \) respectively such that \( \mu_{1} \times \mu_{2} \) is a fuzzy frame of \( F_{1} \times F_{2} \). Then \( \mu_{1} \) or \( \mu_{2} \) is a fuzzy frame of \( F_{1} \) or \( F_{2} \) respectively.
Proof. We have \( \mu_1 \times \mu_2(e_{F_1}, e_{F_2}) \geq \mu_1 \times \mu_2(x, y) \) for all \( (x, y) \in F_1 \times F_2 \) also

\[
\mu_1 \times \mu_2(o_{F_1}, o_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2}) \text{ where } (e_{F_1}, e_{F_2}) \text{ and } (o_{F_1}, o_{F_2}) \text{ are}
\]

respectively the unit and zero elements of the frame \( F_1 \times F_2 \).

Now \( \mu_1 \times \mu_2(x, y) = \inf \{\mu_1(x), \mu_2(y)\} \) for all \( (x, y) \in F_1 \times F_2 \) by Definition 2.5.3.

Then \( \mu_1(x) \leq \mu_1(e_{F_1}) \) or \( \mu_2(y) \leq \mu_2(e_{F_2}) \) also \( \mu_1(e_{F_1}) \) and \( \mu_2(e_{F_2}) \) are equal to either \( \mu_1(o_{F_1}) \) or \( \mu_2(o_{F_2}) \).

If \( \mu_1(x) \leq \mu_1(e_{F_1}) \) then \( \mu_1(x) \leq \mu_2(e_{F_2}) \) or \( \mu_2(y) \leq \mu_2(e_{F_2}) \).

Let \( \mu_1(x) \leq \mu_2(e_{F_2}) \) for all \( x \in F_1 \)

Then for all \( x \in F_1 \), \( \mu_1 \times \mu_2(x, e_{F_2}) = \inf \{\mu_1(x), \mu_2(e_{F_2})\} = \mu_1(x) \)

Now for arbitrary \( S \subseteq F_1 \) we have

\[
\mu_1(\bigvee S) = \mu_1 \times \mu_2(\bigvee S, e_{F_2}) = \mu_1 \times \mu_2(\bigvee_{x \in S} x, e_{F_2}) = \mu_1 \times \mu_2(\bigvee_{x \in S} (x, e_{F_2})) \geq \inf_{x \in S} \{\mu_1(x) \times \mu_2(x, e_{F_2})\} = \inf_{x \in S} \{\mu_1(x)\}
\]

For all \( x, y \in F_1 \) we have,
\[
\mu_1(x \land y) = \mu_1 \times \mu_2(x \land y, e_{F_2})
\]
\[
= \mu_1 \times \mu_2((x, e_{F_2}) \land (y, e_{F_2}))
\]
\[
\geq \min\{\mu_1 \times \mu_2(x, e_{F_2}), \mu_1 \times \mu_2(y, e_{F_2})\}
\]
\[
= \min\{\mu_1(x), \mu_1(y)\}
\]

Now \(\mu_1(e_{F_1}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2}) \geq \mu_1 \times \mu_2(x, e_{F_2}) = \mu_1(x)\) for all \(x \in F_1\)

Also \(\mu_1(O_{F_1}) = \mu_1 \times \mu_2(O_{F_1}, e_{F_2}) = \inf\{\mu_1(O_{F_1}), \mu_2(e_{F_2})\}\)

If \(\mu_2(e_{F_2}) = \mu_1(O_{F_1})\) then

\(\mu_1(O_{F_1}) = \inf\{\mu_2(e_{F_2}), \mu_2(e_{F_2})\} = \mu_2(e_{F_2}) \geq \mu_1(x)\) for all \(x \in F_1\)

If \(\mu_2(e_{F_2}) = \mu_2(O_{F_2})\) then

\(\mu_1(O_{F_1}) = \inf\{\mu_1(e_{F_1}), \mu_2(O_{F_2})\} = \mu_1 \times \mu_2(O_{F_1}, O_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2})\)

\(= \inf\{\mu_1(e_{F_1}), \mu_2(e_{F_2})\} = \mu_1(e_{F_1})\)

Thus \(\mu_1(e_{F_1}) = \mu_1(O_{F_1}) \geq \mu_1(x)\) for all \(x \in F_1\)

Therefore \(\mu_1\) is a fuzzy frame of \(F_1\). \hfill (1)

Now let \(\mu_1(x) \leq \mu_2(e_{F_2})\) is not true for all \(x \in F_1\). That is if \(\mu_1(x) > \mu_2(e_{F_2})\) for all \(x \in F_1\) then \(\mu_2(y) \leq \mu_2(e_{F_2})\) for all \(y \in F_2\).

Therefore for all \(y \in F_2\), \(\mu_1 \times \mu_2(e_{F_1}, y) = \inf\{\mu_1(e_{F_1}), \mu_2(y)\} = \mu_2(y)\)
Now for arbitrary $S \subseteq F_1$ we have

$$\mu_2(\vee S) = \mu_1 \times \mu_2(e_{F_1}, \vee S)$$

$$= \mu_1 \times \mu_2(e_{F_1}, \vee x)_{x \in S}$$

$$= \mu_1 \times \mu_2(\vee_{x \in S} e_{F_1}, x)$$

$$\geq \inf_{x \in S} \{ \mu_1 \times \mu_2(e_{F_1}, x) \} = \inf_{x \in S} \{ \mu_2(x) \}$$

Similarly for all $x, y \in F_2$ we have $\mu_2(x \wedge y) \geq \min \{ \mu_2(x), \mu_2(y) \}$

Now $\mu_2(e_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2}) \geq \mu_1 \times \mu_2(e_{F_1}, x) = \mu_2(x)$ for all $x \in F_2$.

Also $\mu_2(O_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, O_{F_2}) = \inf \{ \mu_1(e_{F_1}), \mu_2(O_{F_2}) \}$

If $\mu_1(e_{F_1}) = \mu_1(O_{F_1})$ then,

$$\mu_2(O_{F_2}) = \inf \{ \mu_1(O_{F_1}), \mu_2(O_{F_2}) \} = \mu_1 \times \mu_2(O_{F_1}, O_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2})$$

$$= \inf \{ \mu_1(e_{F_1}), \mu_2(e_{F_2}) \} = \mu_2(e_{F_2})$$

If $\mu_1(e_{F_1}) = \mu_2(O_{F_2})$ then,

$$\mu_2(O_{F_2}) = \inf \{ \mu_1(e_{F_1}), \mu_1(e_{F_1}) \} = \mu_1(e_{F_1}) \geq \mu_2(e_{F_2}) \geq \mu_2(x)$$

for all $x \in F_2$.

Thus $\mu_2(e_{F_2}) = \mu_2(O_{F_2}) \geq \mu_2(x)$ for all $x \in F_2$.

Therefore $\mu_2$ is a fuzzy frame of $F_2$. (2)

Hence from (1) and (2) either $\mu_1$ or $\mu_2$ is a fuzzy frame of $F_1$ or $F_2$ respectively.
Theorem 2.5.9. Let $\mu_{\alpha}$ be fuzzy set of the frame $F_{\alpha}$ for $\alpha \in \Lambda$ such that $\prod_{\alpha \in \Lambda} \mu_{\alpha}$ is a fuzzy frame of $F = \prod_{\alpha \in \Lambda} F_{\alpha}$. Now for $x_\alpha \in F_{\alpha}$ ($\alpha \in \Lambda$) if $\mu_{\alpha}(e_{F_{\alpha}}) = \mu_{\alpha}(O_{F_{\alpha}}) \geq \mu_{\alpha}(x_\alpha)$ and $\mu_{\alpha}(e_{F_{\alpha}}) = \mu_{\alpha}(e_{F_\beta})$ for all $\alpha, \beta \in \Lambda$ where $e_{F_{\alpha}}$, $O_{F_{\alpha}}$ are respectively the unit and zero element of the frame $F_{\alpha}$ then $\mu_{\alpha}$ is a fuzzy frame of $F_{\alpha}$ for all $\alpha \in \Lambda$.

Proof. We have $\prod_{\alpha \in \Lambda} \mu_{\alpha}((e_{F_{\alpha}})_{\alpha \in \Lambda}) = \prod_{\alpha \in \Lambda} \mu_{\alpha}((O_{F_{\alpha}})_{\alpha \in \Lambda}) \geq \prod_{\alpha \in \Lambda} \mu_{\alpha}((x_\alpha)_{\alpha \in \Lambda})$ for all $(x_\alpha)_{\alpha \in \Lambda} \in F$ where $(e_{F_{\alpha}})_{\alpha \in \Lambda}$ and $(O_{F_{\alpha}})_{\alpha \in \Lambda}$ are respectively the unit and zero elements of the frame $F$.

Now for $y \in F_{\alpha}$ consider $(y_\beta)_{\beta \in \Lambda} \in F$ where $y_\beta = \begin{cases} y & \text{if } \beta = \alpha \\ e_{F_\beta} & \text{otherwise} \end{cases}$

Then for all $y \in F_{\alpha}$, $\prod_{\beta \in \Lambda} \mu_{\beta}((y_\beta)_{\beta \in \Lambda}) = \inf_{\beta \in \Lambda} \{ \mu_{\beta}(y_\beta) \} = \mu_{\alpha}(y)$

Consider $\alpha \in \Lambda$

Now for arbitrary $S \subseteq F_{\alpha}$ we have,

$$\mu_{\alpha}(\bigvee S) = \prod_{\beta \in \Lambda} \mu_{\beta}((y_\beta)_{\beta \in \Lambda}) \text{ where } y_\beta = \begin{cases} \bigvee S & \text{if } \beta = \alpha \\ e_{F_\beta} & \text{otherwise} \end{cases}$$

$$= \prod_{\beta \in \Lambda} \mu_{\beta}(\bigvee_{x \in S} (x_\beta)_{\beta \in \Lambda}) \text{ where } x_\beta = \begin{cases} x & \text{if } \beta = \alpha \\ e_{F_\beta} & \text{otherwise} \end{cases}$$

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Similarly it can be shown that for all \( x, y \in F_\alpha \)

\[
\mu_\alpha(x \land y) = \prod_{\beta \in \Lambda} \mu_\beta((y_\beta)_\beta) \quad \text{where} \quad y_\beta = \begin{cases} 
x \land y & \text{if } \beta = \alpha \\
e_{F_\beta} & \text{otherwise} 
\end{cases}
\]

\[
\geq \min\{\mu_\alpha(x), \mu_\beta(y)\}
\]

Hence the result follows.

\[
\text{Theorem 2.5.10. Let } F \text{ be a frame and } f \text{ a homomorphism on } F. \text{ Let } \mu \text{ and } \sigma \text{ are fuzzy frames of the frame } f(F) \text{ then } \mu \times \sigma \text{ is a fuzzy frame of } f(F) \times f(F). \text{ The pre image } \mu \circ f \text{ and } \sigma \circ f \text{ are fuzzy frames of } F \text{ and } (\mu \times \sigma) \circ (f,f) \text{ a fuzzy frame of } F \times F. \text{ We study this relation.}
\]

\[
\text{Proof. For all } (x_1, x_2) \in F \times F \text{ we have,}
\]

\[
(\mu \times \sigma) \circ (f,f)(x_1, x_2) = (\mu \times \sigma)(f(x_1), f(x_2))
\]

\[
= \inf \{ \mu(f(x_1)), \sigma(f(x_2)) \}
\]

\[
= \inf \{ \mu \circ f(x_1), \sigma \circ f(x_2) \}
\]

\[
= (\mu \circ f \times \sigma \circ f)(x_1, x_2) \quad \square
\]
The relation between images of product of fuzzy frames of a frame $F$ is given as follows.

**Theorem 2.5.11.** Let $\mu$ and $\sigma$ be fuzzy frames of the frame $F$. If $f$ is a homomorphism on $F$, the product $f(\mu) \times f(\sigma)$ and $(f, f)(\mu \times \sigma)$ satisfies

$$(f, f)(\mu \times \sigma) \subseteq f(\mu) \times f(\sigma).$$

**Proof.** $f(\mu)$ and $f(\sigma)$ are fuzzy frames of $f(F)$ and $f(\mu) \times f(\sigma)$ is a fuzzy frame of $(f, f)(F \times F) = f(F) \times f(F)$.

Now for each $y = (y_1, y_2) \in f(F) \times f(F)$ we have,

$$[(f, f)(\mu \times \sigma)](y) = \sup \{(\mu \times \sigma)(x) | x \in F^{-1}(y)\}$$

where $F = (f, f)$ and $x = (x_1, x_2)$

$$= \sup \{\inf(\mu(x_1), \sigma(x_2)) | (x_1, x_2) \in F^{-1}(y)\}$$

$$\leq \inf(\sup\{\mu(x_1) | x_1 \in f^{-1}(y_1)\}, \sup\{\sigma(x_2) | x_2 \in f^{-1}(y_2)\})$$

$$= \inf\{f(\mu(y_1)), f(\sigma(y_2))\}$$

$$= (f(\mu) \times f(\sigma))(y)$$

That is $[(f, f)(F \times F)](y) \leq (f(\mu) \times f(\sigma))(y)$ for all $y \in f(F) \times f(F)$

Therefore the result follows. \qed