Chapter 1

Introduction

Ever since the advent of science, nonlinear systems have been a part of it. Nature has bestowed almost all dynamical systems with nonlinearity. But progress in this field was very slow, owing to the analytic inaccessibility of almost all these systems. Analysis was possible only for very few special cases and parameter sets. Recently, this problem has been overcome with the developments in the field of numerical computations, aided by fast computers. Chaotic and orderly behaviour are exhibited by these dynamical systems. It would not be an exaggeration to say, ‘chaos is the order of the day, and order is only an exception’. Earliest known work in this field is due to Henri Poincaré. Subsequently, noteworthy, early mathematical treatments of chaotic dynamics have been provided by G. Birkhoff, M. L. Cartwright and J. E. Littlewood, S. Smale and A. N. Kolmogrov.

The study of chaos in nonlinear dynamical systems has been a field of intense scientific research for the last three decades. The complexities of chaotic behaviour are exhibited by systems in diverse disciplines such as chemical reactions [1], electronics [2, 3], economics [4, 5], astronomy and astrophysics [6], population biology [7, 8], lasers [9], neural networks [10], physiology [11] and plasmas [12, 13],
to list a few. These interdisciplinary studies seem to break the barriers between the different disciplines, suggesting new synthesis and unification in science.

1.1 Chaos and Nonlinear Dynamics

Chaotic behaviour is characteristic of nonlinear dynamical systems. In many cases of physical interest we deal with linear systems, which have been helpful in unraveling many secrets of nature and formulating laws of science. In reality, almost all these systems are nonlinear and we have analysed only the linear part of these for the sake of analytic viability. In many situations this kind of linearization gives results comparable to the actual behaviour of the systems. But there are many phenomena like the 'butterfly effect' which can be explained only by analysing the nonlinear part of the system. This makes the study of nonlinear systems very important.

In nonlinear dynamics, the time evolution of the system is studied using nonlinear equations of motion, which may be described by ordinary differential equations, partial differential equations, iteration of maps, integral equations, etc. The importance of nonlinear nature of the equations of motion has been identified properly only during the last quarter of the twentieth century. Nonlinear equations can produce a wide range of interesting, complex phenomena which the linear equations are totally incapable of generating.

Chaos is the apparently chaotic state generated by the nonlinear equations of motion. Chaos is not the chaotic state of disorderly behaviour in the literal sense of the word, where occurrence of events are totally unconnected and unpredictable. What we describe is ‘deterministic chaos’, where the state is complex and apparently irregular, but is totally different from random behaviour.
irregular and disordered behaviour of a dynamical system has traditionally been attributed to the large number of degrees of freedom. But the chaos exhibited by nonlinear systems are deterministic in nature, for the occurrence of which a very few degrees of freedom are sufficient. Deterministic, it is referred to as, for the fact that given the present state and the equations of motion, the future states can be predicted. The prediction will be accurate if the present state also is accurately prescribed. Even an infinitesimal error in the prescription of the initial condition will make the future prediction erratic. Thus there is a sensitive dependence on the initial conditions, which is a hallmark of chaos. Chaos is thus totally different from randomness which is the outcome of probabilistic or stochastic processes, where there is no such dependence on initial conditions. Chaotic behaviour is qualified as unpredictable only in the sense that, of the multitudes of final states available for any given initial condition, which state is going to correspond to a particular time.

In the vast field of dynamical systems, the theory of chaos is a relatively new development. As mentioned earlier, Henry Poincaré the French mathematician has pioneered the modern approach to the study of dynamical systems. Poincaré introduced modern qualitative techniques in exploring dynamical behaviour. Two important hallmarks of his method are (i) a global geometric point of view and (ii) replacement of analytical techniques by qualitative methods. In the geometrical point of view a dynamical system is visualized as a field of vectors in phase space where a solution is a smooth curve, at every point of which the tangent is along the vector, based at that point. Though the qualitative or the geometrical properties of the flow could be deduced from qualitative solutions, Poincaré tried to obtain the qualitative information by geometric methods. We describe in the following sections, a systematic approach to the study of dynamical systems, that
has evolved through this century, since Poincaré.

1.2 Dynamical Systems

A dynamical system is essentially a prescription for the time evolution of all points in a given state space. If the system is n-th order, the Euclidean n-space \( \mathbb{R}^n \) represents the state space. For an initial state \( x_0 \) at \( t = t_0 \) in \( \mathbb{R}^n \), the dynamical system tells us what the state vector \( x_t \) would be at a later time \( t \). To every dynamical system there corresponds a unique vector field \( f(x) \) in \( \mathbb{R}^n \). This vector field is the mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), which is of class \( C^1 \). The vector field satisfy Lipschitz\(^1\) condition in every compact domain\(^2\) of \( \mathbb{R}^n \).

Thus a dynamical system can formally be defined as a \( C^1 \) map\(^3\) \( \mathbb{R}^n \times \mathbb{R}^n \), where the map \( \phi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called the flow of the system. Based on the time dependence this can be distinguished as autonomous and nonautonomous.

1.2.1 Autonomous Dynamical Systems

An autonomous dynamical system of \( n \) -th order is defined by

\( ^1 \)Lipschitz condition: Suppose \( f \) is defined in a domain \( D \) of the \((t, z)\) plane. If there exists a constant \( k > 0 \) such that for every \((t, z_1)\) and \((t, z_2)\) in \( D \)

\[ |f(t, z_1) - f(t, z_2)| \leq k |z_1 - z_2| \]

then \( f \) is said to satisfy a Lipschitz condition.

\( ^2 \)A set is termed compact if it is bounded and closed.

\( ^3 \)\( C^1 \)class: Let \( I \) denote an open interval on the real line \(-\infty < t < \infty \). The set of all complex valued functions having \( k \) continuous derivatives on \( I \) is denoted by \( C^k \) on \( I \). If \( f : \in C^k \), then \( f \) is said to be of class \( C^k \).
\[ \dot{x} = f(x) ; \quad x(t_0) = x_0, \]  

(1.1)

where \( x = (x_1, x_2, \ldots, x_n) \), \( \dot{x} = \frac{dx}{dt} \), \( x(t) \in \mathbb{R}^n \) gives the state at time \( t \). Since the vector field \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) does not depend on time, the initial time can always be fixed as \( t_0 = 0 \). The trajectory \( \phi_t(x_0) \) is the solution of (1.1) with initial condition \( x = x_0 \) at time \( t = t_0 \). The dynamical system 1.1 is linear or nonlinear according as \( f(x) \) is linear or not in \( x \).

For any finite \( t \), \( \phi_t \) can be taken as a \( C^1 \) diffeomorphism\(^4\). This implies that \( \phi_t(x) = \phi_t(y) \) if and only if \( x = y \). Hence trajectories of autonomous systems are uniquely decided by specified initial conditions.

### 1.2.2 Non autonomous Dynamical Systems

If the differential equation (1.1) has an explicit time dependence

\[ \dot{x} = f(x, t) ; \quad x(t_0) = x_0, \]  

(1.2)

it defines a nonautonomous dynamical system. Since the vector field \( f \) depends on time, the initial time cannot arbitrarily be set to zero. The solution to Eq. (1.2), with the initial condition \( x = x_0 \) at \( t = t_0 \), is the trajectory. Eq. (1.2) defines a nonlinear system if \( f(x) \) is nonlinear in \( x \).

If there exists a \( T > 0 \) such that \( f(x, t) = f(x, t + T) \) for all \( x \) and \( t \), the system is said to be time periodic with period \( T \). The smallest such \( T \) is called the minimal period.

An \( n \)-th order time periodic, nonautonomous dynamical system can always be converted to an \((n+1)\) th order autonomous dynamical system by introducing

\(^4\)A function \( f \) is a diffeomorphism if \( f^{-1} \) exists and both \( Df \) and \( Df^{-1} \) exist and are continuous.
an extra state $\theta = 2\pi t/T$. The resulting autonomous system is given by

$$
\begin{align*}
\dot{x} &= f(x, \theta T/2\pi) \quad ; \quad x(0) = x_0 \\
\dot{\theta} &= 2\pi/T \quad ; \quad \theta_0 = 2\pi t_0/T
\end{align*}
$$

(1.3)

Just as $f$ is periodic in $T$, the system (1.3) is periodic in $\theta$ with period $2\pi$. Here, the plane $\theta = 0$ can be identified with the plane $\theta = 2\pi$. This enables us to transform from the Euclidean space $\mathbb{R}^{n+1}$ to the cylindrical state space $\mathbb{R}^{n+1} \times S$, where $S = [0, 2\pi)$ is the circle. This copes with the $T-$ periodicity of the vector field. Restricting $0 \leq \theta < 2\pi$, the solution in the cylindrical state space is

$$
\begin{bmatrix}
x(t) \\
\theta(t)
\end{bmatrix} =
\begin{bmatrix}
\phi_t(x_0, t_0) \\
(2\pi t/T) \mod 2\pi
\end{bmatrix}
$$

(1.4)

With this transformation, time periodic non autonomous systems can be treated on the same footing as autonomous systems.

Non-autonomous systems that are not time periodic can also be converted to autonomous systems using transformations (1.3), with any $T > 0$. But the resulting solution will necessarily be unbounded because $\theta \to \infty$ as $t \to \infty$; hence most of the results about asymptotic states in autonomous theory will not be applicable.

The $C^1$ nature of the flow $\phi_t$ implies that $\phi_t(x, t_0) = \phi_t(y, t_0)$ if and only if $x = y$. Hence for given initial time $t_0$, a trajectory of a non autonomous system is uniquely specified by the initial state. However for $t_0 \neq t_1$, it is possible that $\phi_t(x, t_0) = \phi_t(y, t_1)$ for $x \neq y$, implying that, unlike autonomous systems the trajectories of a non autonomous system can intersect.
1.2.3 **Discrete-time Dynamical Systems**

Maps of the form $f: \mathbb{R}^n \to \mathbb{R}^n$ define a discrete time dynamical system whose state equation is given by

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2 \ldots$$  \hspace{1cm} (1.5)

where $x_n \in \mathbb{R}^n$, is called the state; $f$ maps $x_n$ to $x_{n+1}$. A sequence of points $\{x_n\}_{n=0}^{\infty}$, called the orbit of the discrete system, is obtained by repeated application of the map $f$ to the initial state $x_0$.

In many situations, the study of continuous time dynamical systems (flows) can be reduced to the study of discrete time dynamical systems (maps). Such a reduction was first demonstrated by Poincaré in celestial mechanics. The study of maps being more of a geometrical nature, the mapping techniques can be used to advantage in analysing continuous systems, without getting bogged down in the intricacies of solving the underlying differential equations, due to the correspondence between maps and flows.

1.2.4 **Conservative and Dissipative Systems**

Depending on the nature of the system under consideration, the properties of the system that are investigated into and the time scale for which the analysis is carried over, a real physical system is identified as conservative or dissipative. Let the equations of motion of the system be given by

$$\dot{x} = f(x).$$  \hspace{1cm} (1.6)

If $V$ is the volume of the phase space, the flux of flow from this region is given by

$$\text{flux} = \int_V \text{div} \ f \ dv.$$  \hspace{1cm} (1.7)
For a conservative system the phase space volume will remain preserved. Since volume $c$ is arbitrary in Eq. (1.7), a system will be conservative or dissipative according as $\text{div } \mathbf{f}$ is zero or negative.

The Hamiltonian function $H$, depending on the generalized position coordinates $q_i$, the canonically conjugate generalised momentum coordinate $p_i$ and time, can be used to describe the behaviour of a conservative system. This function given by

$$H = H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t)$$

(1.8)

is a solution of the $2n$ differential equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \quad (1.9)$$

Whenever $H$ is not an explicit function of time, $\frac{dH}{dt} = 0$ and $H(p, q) = E$, is a constant of motion.

While the total energy $E$ is a constant for a conservative system, it decreases with time in a dissipative system on account of factors such as friction, damping, etc. The role of dissipation is to change the equations of motion under time reversal. Hence the evolution of a dissipative system is irreversible. Another important property of dissipative systems is volume contraction in phase space. The work in this thesis is mainly centred around dissipative systems.

1.3 The Poincaré Maps and invariant Manifolds

Poincaré map, also known as the first return map is probably the most basic tool for studying the stability and bifurcations of steady state behaviour of dynamical
systems. It replaces the flow of a continuous time system with a map.

The $n$-th order nonautonomous system (1.2) defined in $\mathbb{R}^n$ is converted to an autonomous system of $(n + 1)$-th order in the cylindrical state space $\mathbb{R}^n \times S$ by Eq. (1.3). Consider the $n$-dimensional hyperplane $\Sigma$ defined by

$$\Sigma^{t_0} = \{(x, \theta) \in \mathbb{R}^n \times S / \theta = \theta_0\}$$  \hspace{1cm} (1.10)

The trajectory (1.4) intersects $\Sigma$ every $T$ second. Thus, if a map $P : \Sigma \rightarrow \Sigma$ (i.e., $\mathbb{R}^n \rightarrow \mathbb{R}^n$) is defined by $P(x) = \phi_T(x, t_0)$, $P$ is called the Poincaré map. Here $\phi_T$ is a diffeomorphism and hence, $P$ is one to one and differentiable. The map has the following two features:

1. $P(x)$ indicates where the flow takes $x$ after $t$ seconds. This is known as $T$-advance mapping.

2. The orbit of $\{P^k(x)\}_{k=1}^{\infty}$ is a sampling of a single trajectory every $T$ second.

i.e.,

$$P^k(x_0) = \phi_{kT}(x_0, t_0), \quad k = 0, 1, 2, \ldots$$  \hspace{1cm} (1.11)

The nonautonomous system (1.2) can usually be written in a more tractable way, incorporating time and possible parameters, as

$$\dot{x} = f(x) + \epsilon g(x, t, \mu); \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^n, \quad g(x, t) = g(x, t + T)$$  \hspace{1cm} (1.12)

where $\mu$ is some parameter and $g$ is $T$-periodic in $t$. Here, $\dot{x} = f(x)$ represents the unperturbed system which is assumed to be integrable. In such cases the mapping is denoted by

$$P^{t_0}_\epsilon : \Sigma^{t_0} \rightarrow \Sigma^{t_0}$$

In the unperturbed case, i.e., when $\epsilon = 0$, the Poincaré map $P_0 = P^{t_0}$ is identical on every section $\Sigma^{t_0}$, on account of the invariance of $f(x)$ under time translations.
Chapter 1. Introduction

The perturbed maps $P_{\epsilon 0}$ differ from section to section, but any two are diffeomorphic.

The hyperplane $\Sigma$ has to be so chosen that the trajectories are transverse to the plane. This implies that if $x_0$ is a point of intersection

$$\Sigma = \{x \in \mathbb{R}^n / (x - x_0) \cdot f(x_0) = 0\} \quad (1.13)$$

which means that the trajectory is nowhere parallel to the surface, at points of intersection. The map may be denoted by a function of the form

$$y^{k+1} = F(y^k) \quad (1.14)$$

where $y^{k+1}, y^k \in \mathbb{R}^n$. In general, finding the mapping function $F$ is equivalent to solving the underlying differential equations for the flow, such as Eqns. (1.2) or (1.12).

The Poincaré map captures the dynamics of Eq. (1.12); $T$-periodic orbits of $f(x)$ correspond to final points of $P_{\epsilon 0}$, and $kT$-periodic subharmonics correspond to periodic cycles of period $k$. Considering a three dimensional system, the fixed and periodic points of the relevant two dimensional Poincaré map $P_{\epsilon 0}$, are more amenable to analysis than the corresponding periodic motions in the three dimensional flow of (1.12).

Let the map $P_{\epsilon 0}$ (abbreviated as $P$) possess a hyperbolic (non degenerate) fixed point $p$. This means that the linearized map $DP(p)$ has eigenvalues $\lambda_i$, with $|\lambda_i| \neq 1$ [20]. The stable manifold theorem asserts that in a neighbourhood $U(p)$ of $p$, there are smooth, local stable and unstable manifolds $W^s_{loc}(p)$ and $W^u_{loc}(p)$.

---

5 Stable manifold theorem: Suppose that $p$ is a hyperbolic fixed point of a $C^r$ diffeomorphism $P: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and that $E^s(p), E^u(p)$ are the stable and unstable eigenspaces of the associated linearized mapping $DP(p)$. Then, in a neighbourhood $U(p)$ there are $C^r$ invariant submanifolds $W^s_{loc}(p), W^u_{loc}(p)$ tangent to $E^s(p), E^u(p)$ at $p$. The global manifolds $W^s(p)$ and $W^u(p)$
tangent to the eigen spaces $E^s$ and $E^u$ of the linearized problem

$$\dot{\xi} = DP(p) \quad (1.15)$$

These subspaces have dimensions $s$ and $u$ (with $s + u = n$) where $s(u)$ is the number of eigen values with moduli less than (greater than) 1. These manifolds defined as the sets of points asymptotic to $p$ under forward (backward) iteration of $p$ are given by

$$W^s_{loc}(p) = \{ x \in U(p) / P^n(x) \in U(p), \forall n \geq 0 \text{ and } P^n(x) \to p as n \to \infty \},$$

$$W^u_{loc}(p) = \{ x \in U(p) / P^{-n}(x) \in U(p), \forall n \geq 0 \text{ and } P^n(x) \to p as n \to \infty \}. \quad (1.16)$$

The global manifolds are obtained by iteration of the local manifolds backwards (forwards):

$$W^s(p) = \bigcup_{n \geq 0} P^{-n} [W^s_{loc}(p)],$$

$$W^u(p) = \bigcup_{n \geq 0} P^n [W^u_{loc}(p)]. \quad (1.17)$$

For $f(x), x \in \mathbb{R}^3, s + u = 2$. Here, if both the eigen values lie outside (inside) the unit circle, then $W^s_p (W^u_p)$ is empty.

### 1.4 Lyapunov Exponents

Lyapunov exponents (LE) are a generalization of the eigen values. This measures the average rate of divergence of initially closeby trajectories. They can be used to analyse the stability of the different types of steady state behaviour, including chaotic solutions.

are injectively immersed copies of $\mathbb{R}^s, \mathbb{R}^u$, where $s$ and $u$ are dimensions of $E^s(p)$ and $E^u(p)$ respectively, with $s + u = n$. Moreover, if $P$ depends smoothly upon parameters $\mu \in \mathbb{R}^n$, so do $W^s_{loc}(p)$ and $W^u_{loc}(p)$ also.
Chapter I. Introduction

Let us first consider the case of a one-dimensional map \( f : J \to J \) where \( J \subset \mathbb{R} \) is some bounded interval on the real line \( \mathbb{R} \). Consider two close points \( x_0 \) and \( x_0 + \epsilon \), mapped by the function. The separation in the \( n \)-th iterate of these points is given by

\[
f^n (x_0 + \epsilon) - f^n (x_0) = \epsilon e^{n \lambda(x_0)} \tag{1.18}
\]

In the limit \( \epsilon \to 0 \) and \( n \to \infty \), the Lyapunov exponent is obtained as

\[
\lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \ln \left| \frac{df^n(x)}{dx} \right|_{x_0}.
\]

Using chain rule of differentiation, we can write

\[
\frac{df^n(x)}{dx} \bigg|_{x_0} = \prod_{i=0}^{n-1} \frac{df(x_i)}{dx} = \prod_{i=0}^{n-1} |f'(x_i)|. \tag{1.19}
\]

Then the Lyapunov exponent becomes

\[
\lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \tag{1.20}
\]

For continuous systems, the Lyapunov exponents are defined in terms of the linearized variational equation of the flow (1.1). The variational equation is

\[
\dot{\Phi} = Df(\phi_t(x_0)) \Phi, \quad \Phi_0 = I, \tag{1.21}
\]

where \( \phi_t(x_0) \) are the solutions of (1.1) and \( I \) is the unit matrix. Let \( \{m_i(t)\}_{i=1}^n \) be the eigen values of \( \Phi_t(x_0) \). The Lyapunov exponents are then given by

\[
\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \{|m_i(t)|\} \quad i = 1, 2, \ldots, n \tag{1.22}
\]

if the limit exists. In these settings, replacing \( \lim \) by \( \limsup \), existence of \( \lambda_i \) can be guaranteed for all \( t \).

For the case of an equilibrium point \( x^* \), the Lyapunov exponents are obtained easily. Let \( \{\lambda_i\}_{i=1}^n \) and \( \{\eta_i\}_{i=1}^n \) be the eigen values and eigen vectors of \( Df(x^*) \).

\[^6\text{To avoid confusion, here eigen values are denoted by \( \lambda \) and Lyapunov exponents by \( \lambda \). Generally both are denoted by \( \lambda \).}\]
Chapter 1. Introduction

The solution of the variational equation (1.21) is

$$\Phi_t(x^*) = \exp\{Df(x^*)\}$$   \hspace{1cm} (1.23)

The eigenvalues and characteristic multipliers are related by $m_i(t) = \exp(\lambda_i t)$. Hence 1.22 gives

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \{\exp(\lambda_i t)\} = \lim_{t \to \infty} \frac{1}{t} \text{Re} (\lambda_i) = \text{Re} (\lambda_i)$$   \hspace{1cm} (1.24)

Thus for equilibrium points, the Lyapunov exponents are equal to the real parts of the eigenvalues, and indicate the rate of (phase) space contraction ($\lambda_i < 0$) or expansion ($\lambda_i > 0$) in the direction of the respective eigen vectors $\eta_i$.

Suppose $x_0$ is not an equilibrium but some point in the basin of attraction of $x^*$; i.e., $\phi_t(x_0) \to x^*$ as $t \to \infty$. As the Lyapunov exponent is defined in the limit as $t \to \infty$, finite transients can be neglected. As such, the Lyapunov exponents for an attractor and any point in the basin of attractor are the same for nonchaotic attractors.

In the case of a limit cycle, the Lyapunov exponents $\lambda_i$ and the characteristic multipliers are related by

$$\lambda_i = \frac{1}{T} \ln |m_i|, \hspace{1cm} i = 1, 2, \ldots$$   \hspace{1cm} (1.25)

where $T$ is the minimal period of the limit cycle.

The details of techniques for calculating LE are, given in chapter II.

1.5 Attractors

Dissipative dynamical systems are bestowed with the property that the phase space volume occupied by an ensemble of states continuously decreases with time. Over a long time this results in simplification of the topological structure of the
trajectories in phase space. Quite often, this would mean that the final behaviour of an initially complex dynamical system with a large number of degrees of freedom, is confined to a subspace of lower dimensions. The asymptotic behaviour (i.e., as \( t \to \infty \)), of a dynamical system is referred to as the steady state. The system behaviour before attaining steady state is termed transients.

The attractor is an attracting limit set.

A point \( y \) is a limit point of \( x \) if for every neighbourhood \( U \) of \( y \), \( \phi_t(x) \) repeatedly enters \( U \) as \( t \to \infty \).

The limit set \( L(x) \) of \( x \) is the set of all limit points of \( x \). Limit sets are compact and invariant under \( \phi_t \); i.e., for all \( x \in L \) and all \( t, \phi_t(x) \in L(x) \).

A limit set is an attracting set, if there exists an open neighbourhood \( U \) of \( L \) such that \( L(x) = L \), for all \( x \in U \). The terms attractor and attracting limit set are usually used interchangeably. Summarizing the above, the following properties can be attributed to an attractor:

(a) it is a compact set.
(b) it is an invariant set, under \( \phi_t \).
(c) the attractor has zero volume in the \( n \)-dimensional phase space. i.e., it has a shrinking neighbourhood.
(d) the flow is recurrent- that is, trajectories emanating from any (open) subset of the attractor set \( L \), hits that subset repeatedly for arbitrarily large values of \( t \); the flow is nowhere transient.
(e) the flow cannot be decomposed; i.e., \( L \) cannot be split into two non-trivial invariant parts.

The set of all points in the state space, which asymptotically are attracted
Figure 1.1: Basins of attraction of a simple damped pendulum along with the equilibrium points.

to the attractor, forms the basin of attraction. Formally, the basin of attraction $B(L)$ of an attracting set $L$ is defined as the union of all the open neighbourhoods $U$ of $L$ such that $L(x) = L$ for all $x \in U$. Every trajectory starting in the basin of attraction $B(L)$ settles on the attractor $L$ asymptotically. The basins of attraction of a simple damped pendulum are shown in Fig. 1.1.

The different types of steady state behaviour that are usually encountered are


1.5.1 Equilibrium points

Equilibrium points, known also as fixed points, are possessed only by autonomous systems. For nonautonomous systems, since the vector field is time dependent,
fixed points do not exist. It is given by a constant solution $\phi_t(x^*) = x^*$ for all $t$ in Eq. (1.1). At the equilibrium point, the vector field vanishes. Except for a very few pathological cases [24] the equilibrium points are solutions of the equation $f(x^*) = 0$.

If $x^*$ is a fixed point of Eq. (1.1), the local behaviour of the flow near $x^*$ is determined by linearizing $f$ at $x^*$. In particular, the vector field

$$\delta \dot{x} = Df(x^*) \delta x$$

(1.26)

governs the time evolution of perturbation near the equilibrium point.

The trajectory with initial condition $x^* + \delta x(t)$ is

$$\phi_t(x^* + \delta x_0) = x^* + \delta x(t)$$

(1.27)

$$= x^* + \delta x_0 \exp[Df(x^*) t].$$

Here the second term has the expansion

$$\delta x_0 \exp[Df(x^*) t] = c_1 \eta_1 e^{\lambda_1 t} + c_2 \eta_2 e^{\lambda_2 t} + \cdots + c_n \eta_n e^{\lambda_n t},$$

(1.28)

where $\{\lambda_i\}_{i=1}^n$ and $\{\eta_i\}_{i=1}^n$ are the eigen values and eigen vectors respectively of $Df(x^*)$; $\{c_i\}_{i=1}^n$ are scalar constants satisfying the given initial condition. The real part of $\lambda_i$ (i.e., $\text{Re}(\lambda_i)$) gives the local rate of expansion of the phase space. $\text{Re}(\lambda_i) < 0$ would therefore imply a space contraction in the neighbourhood of $x^*$ in the direction of $\eta_i$.

If $\text{Re}(\lambda_i) < 0$ for all $\lambda_i$, then all sufficiently small perturbations tend to zero as $t \to \infty$: $x^*$ is asymptotically stable or is an attractor. When some $\text{Re}(\lambda_i) > 0$, $x^*$ is not stable; it is either unstable (all $\text{Re}(\lambda_i) > 0$) or nonstable (some $\text{Re}(\lambda_i) < 0$ and remaining $\text{Re}(\lambda_i) > 0$). Whereas an unstable equilibrium point becomes stable in reverse time, a nonstable equilibrium point remains nonstable itself under time reversal. A nonstable equilibrium point is called a saddle point.
Chapter 1. Introduction

Equilibrium points where all the eigen values have nonzero real part are called *hyperbolic*. They are structurally stable; i.e., under small perturbation of the vector field, they still exist and the perturbed equilibrium point has the same stability type. Generally, nonhyperbolic equilibrium points are structurally unstable and hence are not observed in experiments or simulations. It may be said that the nonhyperbolic fixed points have zero Lebesgue measure. The number of eigen values with \( \text{Re}(\lambda_i) > 0 \) is called the index (of instability). This measures the instability of the equilibrium point in the sense that fixed points with zero index have no instability, and those with larger index are more unstable.

The four basic types of fixed points for a three dimensional state space are:

1. **Node:** \( \text{Im}(\lambda_i) = 0 \) and \( \lambda_i < 0 \); \( i = 1, 2, 3 \). All trajectories in the neighbourhood of a node are attracted to the fixed point without spiralling around it.

   **b. Spiral node:** \( \text{Im}(\lambda_1) = 0 \); \( \text{Re}(\lambda_1) < 0 \); \( \lambda_2 \) and \( \lambda_3 \) are complex with \( \lambda_3 = \lambda_2^* \) (complex conjugate). For this type of fixed points, trajectories spiral toward the node. These are also called attracting spirals.

   The index of a node is zero.

2. **Repeller:** All eigen values are real positive. All trajectories in the neighbourhood of such a fixed point are straightaway repelled from it.

   **b. Spiral repellor:** Here \( \lambda_1 \) is real positive; \( \lambda_2, \lambda_3 \) complex with \( \lambda_3 = \lambda_2^* \) with \( \text{Re}(\lambda_2) > 0 \). Trajectories spiral around the fixed point, as they are pushed away from its vicinity. These are alternatively called *repelling spirals* or *sources*.

   Repellors are index -3 equilibrium points. The spiral repellors (attractors)
are also referred to as unstable (stable) focus.

Equilibrium points with index 1 or 2 are in general termed saddle points.

3. a. Saddle point—index-1: All the eigen values are real, with one positive and the other two negative. Trajectories approach the saddle point on a surface called the inset or stable manifold, and move away from the saddle point along a curve called the outset or unstable manifold.

b. Spiral saddle point—index-1: $\lambda_1 > 0; \lambda_2$ and $\lambda_3$ form a complex conjugate pair with $\text{Re}(\lambda_2) < 0$, $\text{Re}(\lambda_3) < 0$. Trajectories spiral around the saddle point as they approach it along the inset surface.

4. a. Saddle point—index-2: For such points $\lambda_1 > 0$, $\lambda_2 > 0; \lambda_3 < 0$. Trajectories approach the saddle point along the inset which is a curve. The trajectories diverging from the saddle point do so on the outset surface.

b. Spiral saddle point—index-2: Here with $\lambda_3 < 0$, $\lambda_1$ and $\lambda_2$ form a complex conjugate pair with $\text{Re}(\lambda_2) = \text{Re}(\lambda_3) > 0$. Trajectories spiral around the fixed point as they diverge from it, along the outset surface.

1.5.2 Periodic Solutions and Limit Cycles

A periodic solution $\phi_t(x^*, t_0)$ of a dynamical system is characterised by

$$\phi_t(x^*, t_0) = \phi_{t+T'}(x^*, t_0),$$  \hspace{1cm} (1.29)

for all $t$ and some minimal period $T' > 0$.

Such a periodic solution has a fundamental component of frequency $f = \frac{1}{T'}$ and evenly spaced harmonics of frequency $nf$, $n = 1, 2, 3, \ldots$. Some of these harmonics may be absent.
In a non-autonomous system $T'$ will be typically a multiple $n$ of the forcing period $T(n = 1, 2, 3, \ldots)$. Such a solution is termed period-$n$ solution. When $n > 1$, it is known as the $n^{th}$ subharmonic. For a linear system, the only steady state may be a period-1 sinusoidal solution; no subharmonics can occur.

If the periodic solution $\phi_t(x^*)$ of an autonomous system is isolated, it is called a limit cycle. Limit cycle is a self sustained oscillation. It cannot occur in linear systems. The attractor corresponding to the limit cycle is the closed curve traced out by $\phi_t(x^*)$ over one period. This limit set is a diffeomorphic copy of the circle $S$. The terms limit cycle attractor (limit set of limit cycle) and limit cycle are usually used interchangeably.

The stability of a periodic solution is determined by the Floquet multipliers, also known as the characteristic multipliers. These are generalization of the eigen values at an equilibrium point.

A periodic solution corresponds to a fixed point on the Poincaré map $P$. Hence, the stability of the periodic solution is decided by the stability of the fixed point in the Poincaré map. The stability of a fixed point $x^*$ of the Poincaré map is determined by linearizing $P$ at $x^*$.

The local behaviour of $P$ near $x^*$ is governed by the linear discrete-time system

$$\delta x_{k+1} = D P(x^*) \delta x_k \quad k = 1, 2, \ldots$$

(1.30)

where $\delta x_k$ is the vector distance between $x^*$ and the $k^{th}$ iterate of a point sufficiently close to $x^*$; $D P(x^*)$ is the monodromy matrix. $q$, the dimension of $D P(x^*)$, equals $n$ (phase space dimensionality) for non-autonomous and $(n - 1)$ for autonomous systems. The orbit of $P$ approximated to first order for an initial
condition $x_0 = x^* + \delta x_0$ is

\[
x_k = x^* + \delta x_k \\
= x^* + D \ P(x^*)^k \delta x_0 \\
= x^* + \sum_{i=1}^{q} c_i \eta_i m_i^k
\]

where $\{m_i\}_{i=1}^{q}$ and $\{\eta_i\}_{i=1}^{q}$ represent the eigen values and eigen vectors respectively of $DP(x^*)$; $\{c_i\}_{i=1}^{q}$ are scalar constants satisfying the initial conditions. These eigen values $m_i$ are the Floquet multipliers of the fixed point, and decide the phase contraction ($|m_i| < 1$) and expansion ($|m_i| > 1$) near $x^*$, in the direction of $\eta_i$ for one iteration of the map.

When $\Sigma$ is transverse to the periodic solution, the characteristic multipliers are independent of the orientation and position of the cross-section $\Sigma$. As with equilibria, these eigen values determine the stability type. When all $m_i$ lie within a unit circle in the complex plane the limit cycle is asymptotically stable. If all the multipliers lie outside the unit circle, the solution is a repelling (unstable) limit cycle. When some $m_i$ lie outside and others inside the unit circle, the solution is a saddle cycle (non stable). Periodic solutions with a characteristic multiplier lying on the unit circle are non-hyperbolic; this corresponds to structurally unstable limit cycles, in the sense that a slight perturbation of the dynamic system will result in limit cycles with stability types different from the unperturbed system. If the multipliers are negative, near the fixed point $x^*$, successive return points fall alternatively on opposite sides of the fixed point. For the autonomous case one of the eigen values is always unity; the remaining $(n - 1)$ eigen values of the monodromy matrix constitute the characteristic multipliers. But for the non-autonomous case, all eigen values are characteristic multipliers also.
1.5.3 Quasiperiodic attractor

A quasiperiodic solution is the sum of periodic solutions, each of whose frequency is one of the various sums and differences of a finite set of base frequencies. While these base frequencies are not uniquely defined, the number of such base frequencies is fixed. A quasiperiodic solution can be represented as

\[ x(t) = \sum_i h_i(t) \]  

where \( h_i \) has a minimal period \( T_i \) and frequency \( f_i \left( = \frac{1}{T_i} \right) \). Further, there exists a finite set of base frequencies \( \{ \hat{f}_1, \hat{f}_2, \cdots, \hat{f}_n \} \) with the following properties:

(a) the set is linearly independent,

(b) the set forms a finite integral base for the \( f_i \), i.e., for every \( i \), \( f_i = \sum_{j=1}^{p} k_j f_j \).

A quasiperiodic solution with \( N \) base frequencies is called \( N \)-periodic. If \( N = 1 \), we have a periodic solution. The simplest case of a quasiperiodic solution is the 2-periodic solution given by

\[ x(t) = h_1(t) + h_2(t), \]

where \( T_1 \) and \( T_2 \) (\( f_1 \) and \( f_2 \)) are incommensurate. The spectrum of \( x(t) \) consists of two sets of harmonics. The first set, \( h_1(t) \) consists of the fundamental frequency \( f_1 \) and higher harmonics; the other set consisting of the fundamental frequency \( f_2 \) and higher harmonics, corresponds to \( h_2(t) \).

If there are two positive integers \( p \) and \( q \) with no common divisor such that

\[ \frac{p}{q} = \frac{f_2}{f_1} \]

where \( f_1 \) and \( f_2 \) are the two independent frequencies of the system, then the frequencies are said to be commensurate or equivalently, the frequency ratio is
rational. Quasiperiodic behaviour results when the ratio is irrational. Quasiperiodic behaviour is also referred to as *conditionally periodic* or *almost periodic*. The system's behaviour is periodic if the ratio is rational. Such motions are described as *phase-locked* or *mode-locked* or *frequency-locked*.

Behaviour with two frequencies can be described by trajectories confined to the surface of a torus (two-torus); for $N$-frequency quasiperiodicity an $N$ torus would be required. Considering two-frequency quasiperiodic behaviour, one frequency (say) $f_1$, is associated with the motion of the trajectories around the large circumference of the torus; the other frequency $f_2$ is then associated with the motion around the small cross section. If one of the frequencies is that of a driving force or a modulating term or some disturbance, then that frequency is under our control and is taken as $f_1$, because sampling the behavior of the system at a fixed phase of the controlled frequency term, makes construction of Poincaré section easier.

1.5.4 Chaos

Chaotic attractor is the next type of attractor. Chaos can be described as the bounded steady state behavior that is not an equilibrium point, not a periodic solution (limit cycle) nor a quasiperiodic state. This steady state behaviour is unpredictable due to the sensitive dependence on initial conditions, even though the dynamics is deterministic. The inherent inaccuracies in prescribing the initial states of a system, amplifies with time and attain the size of the (bounded) space occupied by the system. This unpredictability cannot be avoided by increasing the resolution of the measurement; it is an inherent property of the system.

Formally chaos can be defined in the following way:
The dynamics is chaotic on an invariant compact set $\Lambda$, if the dynamics has sensitive dependence on $\Lambda$ and $\Lambda$ is topologically transitive.

Let $x, x_1, x_2$ be some points in the $\mathbb{R}^n$ space, and let $d(x_1, x_2, x_3)$ be the distance between $x_1$ and $x_2$. The dynamics of the flow $\phi_t$ has sensitive dependence on the invariant compact set $\Lambda$, if there exists an $\epsilon > 0$ such that for any $x \in \Lambda$ and any neighbourhood $N$ of $x$, there exists a $x' \in N$ and $\epsilon > 0$ so that $d(\phi_t(x), \phi_t(x')) > \epsilon$. Invariant means that $\Lambda$ is not a transient property.

For Chaos to happen on $\Lambda$ there is another requirement that the set be mixed up properly. Chaotic trajectories are bounded. dynamical systems, expanding in the state space has sensitive dependence on initial conditions. For boundedness, the chaotic system should contract also in some directions. This alternating expansions in some directions, followed by contractions in some others, guarantees the proper mixing of the set $\Lambda$. For a dissipative system the contraction outweighs the expansion. Lyapunov exponents measure the average expansion and contraction occurring in the state space. For a system to be chaotic at least one LE has to be positive, indicating phase space expansion.

### 1.6 Dimension

The attractor of a dynamical system could be defined as $n$-dimensional, if it is diffeomorphic to an open subset of $\mathbb{R}^n$, i.e., if in a neighbourhood of every point, it looks like an open subset of $\mathbb{R}^n$. This is the topological dimension of a manifold. Thus, the dimension of a limit cycle is 1, since it looks like an interval in $\mathbb{R}^1$. Similarly a torus is two dimensional since, it resembles an open subset of $\mathbb{R}^2$. In this way, the equilibrium point has zero dimension.

Contrasted to the above regular attractors, the geometry of a chaotic attractor
is quite strange. So chaotic attractors are also called strange attractors, though the converse is not always true. (There are strange attractors which are not chaotic.) The neighbourhood of any point in a strange attractor has a fine structure and does not resemble any Euclidean (regular) space. Thus strange attractors are not manifolds and possess no integer dimension. Such attractors are said to have fractal (meaning, fractional) dimensions; and the attractor itself is referred to as a fractal. There are several ways to characterise the fractal dimension. We discuss below, four relatively simple characterisations of the dimension of an attractor.

1.6.1 Capacity Dimension

This is a quantity defined to describe the geometry of the attractor. Suppose an attractor \( L \) is covered by volume elements, each of diameter \( \epsilon \). Let \( N(\epsilon) \) be the number of such volume elements required to cover \( L \) completely. As \( \epsilon \to 0 \), the sum of volume elements approaches the volume of \( L \). If \( D \) is the dimension of \( L \), \( N(\epsilon) \) increases as \( \epsilon^{-D} \); i.e., \( N(\epsilon) = k\epsilon^{-D} \), for some constant \( k \). From this equation the capacity (dimension) is given by

\[
D_c = \lim_{\epsilon \to 0} \frac{\ln(\epsilon)}{\ln(1/\epsilon)},
\]

(1.35)

For manifolds, \( D_c \) is an integer. If the attractor is a fractal, the dimension \( D_c \) is a non-integer.

1.6.2 Information Dimension

This is a probability based dimension, defined in terms of the relative frequency with which a trajectory visits any region of the attractor. If \( N(\epsilon) \) is the number of volume elements, each of the same diameter \( \epsilon \), the information dimension \( D_I \)
is defined by

\[ D_I = \lim_{\epsilon \to 0} \frac{S(\epsilon)}{\ln(1/\epsilon)} \]  

(1.36)

where \( S(\epsilon) \) is the entropy, defined as the amount of information needed to specify

the state of the system to an accuracy of \( \epsilon \), when the state is known to be on the

attractor. It is defined in terms of the relative probability (frequency) with which

a typical trajectory enters the \( i \)-th volume element and is given by

\[ S(\epsilon) = - \sum_{i=1}^{N(\epsilon)} P_i \ln P_i. \]  

(1.37)

The entropy \( S(\epsilon) \) and the information dimension \( D_I \) are related by

\[ S(\epsilon) = k \left( \frac{1}{\epsilon} \right)^{D_I}, \]  

for sufficiently small \( \epsilon \). The equation suggests that the amount of information

needed to specify the state of a system with an accuracy \( \epsilon \) is inversely proportional

to the \( D_I \)-th power of \( \epsilon \).

1.6.3 Correlation Dimension

This is defined as

\[ D_G = \lim_{\epsilon \to 0} \frac{\ln C(\epsilon)}{\ln \epsilon} \]  

(1.39)

where \( C(\epsilon) \) is the correlation function defined by

\[ C(\epsilon) = \lim_{\epsilon \to 0} \frac{1}{N(N-1)} \sum_{i,j=1}^{N} H(|x_i - x_j|) \]  

(1.40)

In this definition \( H \) is the Heaviside step function \( H(x) = 0 \) if \( x < 0 \) and \( H(x) = 1 \)

if \( x \geq 0 \), and \( N \) is the total number of points in the attractor.
1.7 Stability and bifurcations

When the parameters or perturbations in 1.12 varies, steady state behaviour may drastically change. A system is said to be stable, if on slightly changing the parameters or the form of the equation defining the flow, the asymptotic phase portrait characteristics are not changing. In this context, generally two types of stabilities are defined viz., *Lyapunov stability* and *structural stability*.

Let \( \mathbf{x} = \mathbf{x}^* \in \mathbb{R}^n \) be a point where the dynamical system

\[
\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \quad f \in C'(E), \quad E \subset \mathbb{R}^n, \quad \mu \in \mathbb{R}^n
\]

is in the equilibrium configuration. We will say that \( \mathbf{x}^* \) is Lyapunov stable if for every neighbourhood \( U \) of \( \mathbf{x}^* \) there exists a smaller neighbourhood \( U_1 \) of \( \mathbf{x}^* \) where \( U_1 \subset U \), such that every solution starting in \( U_1 \) will remain in \( U \) for all \( t > 0 \).

Structural stability is associated with perturbations of the system. If the qualitative behaviour remains the same for all nearby vector fields of \( f \), then the dynamical system 1.41 or the vector field \( f \) is said to be structurally stable. i.e., A vector field \( f \) is said to be structurally stable, if for any sufficiently close vector field \( \mathbf{g} \in C'(E) \), \( f \) and \( \mathbf{g} \) are topologically equivalent on \( E \). If the field \( f \) is not structurally stable, it belongs to a *bifurcation set* defined in an open set \( E \subset \mathbb{R}^n \). A vector field \( f \in C'(E) \) is topologically equivalent to another vector field \( \mathbf{g} \in C'(E) \), if there is an \( \epsilon > 0 \) such that for all \( f \) and \( \mathbf{g} \), the \( C' \) norm \( ||f - g||_1 < \epsilon \). Then there is an orientation preserving homeomorphism \( H : E \rightarrow E \), which maps trajectories of \( \dot{\mathbf{x}} = f(\mathbf{x}) \) onto trajectories of \( \dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}) \). When the vector field \( f \) is structurally stable, the dynamical system \( \dot{\mathbf{x}} = f(\mathbf{x}) \) also is structurally stable. The qualitative structure of the solution set or of the global phase portrait of the dynamical system changes as the vector field passes through a point in the bifurcation set.

An undamped simple pendulum is a typical example of a structurally unstable
Chapter 1. Dynamics

Introduction

The system is given by

\[ \dot{x} = y, \]
\[ \dot{y} = -\omega^2 \sin x \]

with \((\pm n\pi, 0)\) as the equilibrium points, called centres. These are all non-hyperbolic. A perturbation of the dynamical system by introducing a small damping, results in drastic changes in the global nature of the phase portrait; the centre transforms to a saddle or focus. But a damped pendulum is structurally stable in that, small variations in the damping results in no changes in the global phase space characteristics.

In general, dynamical systems with nonhyperbolic equilibrium points and/or nonhyperbolic periodic orbits are structurally unstable. In \(n\) dimensional space, with \(n \geq 3\), a class of structurally stable dynamical systems is called the Morse-Smale systems, with the following properties:

1. the number of equilibrium points and periodic orbits is finite and each is hyperbolic,
2. all stable and unstable manifolds which intersect, do so transversally; and
3. the non-wandering set consists of equilibrium points and periodic orbits only.

While all Morse-Smale systems are structurally stable, all structurally stable systems in \(\mathbb{R}^n\) with \(n \geq 3\) are not necessarily Morse-Smale systems.

1.7.1 Bifurcations

A Bifurcation is the sudden change undergone by the qualitative behaviour of the solution set of a dynamical system. This happens when the vectorfield passes
through a point in the bifurcation set or when the parameter $\mu$ passes through a bifurcation value $\mu_0$. The number of parameters that must change to cause a bifurcation is called the *co-dimension* of the bifurcation. Co - dimension of a manifold is the difference between the dimension of the space in which the manifold lives and the dimension of the manifold itself.

The bifurcation event can be considered to be a crossing of curves or surfaces (of dimension $n$) in the parameter space. The crossing or intersection is said to be transverse, if the sum of the co-dimensions of the intersecting manifolds equals the co - dimension of the manifold that constitutes the intersection. Bifurcation in which the transversality condition is satisfied are called *generic*. Generic bifurcations have the property that the bifurcation is structurally stable. And, in real physical problems, only generic bifurcations can be observed, usually. But non-generic ones can be detected, if there are suitable constraints on the system, such as the imposition of certain symmetries.

When the vector field passes through a point in the bifurcation set or when the parameter passes through a bifurcation value $\mu_0$, the dynamical system

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^n, \quad f \in C'(E \times J), \quad E \subset \mathbb{R}^n, \quad \mu \in \mathbb{R}$$

undergoes a bifurcation. We discuss briefly, the different types of possible bifurcations in a dynamical system in the following sections.

### 1.7.2 Bifurcations at Non Hyperbolic Equilibrium Points

The generic bifurcation undergone by a non hyperbolic equilibrium point under one control parameter is the *Saddle Node* bifurcation. The nongeneric variants, transcritical and pitch-fork bifurcations also can happen, for the genericity of
which, more than one control parameter is required. These are codimension 1 bifurcations. The Saddle Node (SNB) bifurcation is also called the fold bifurcation or the tangent bifurcation (TB).

For the dynamical system \( \dot{x} = f(x, \mu) \), if \( f(x_0, \mu_0) = Df(x_0, \mu_0) = 0 \), then \( (x_0, \mu_0) \) is a non hyperbolic critical point and \( \mu_0 \) is a bifurcation value. The critical point is said to be of multiplicity \( m \) when \( D^k f(x_0, \mu_0) = 0 \) for \( k = 0, 1, \ldots, (m-1) \) and \( D^m f(x_0, \mu_0) \neq 0 \). In such a case at most \( m \) critical points can be made to bifurcate.

Suppose that \( f(x_0, \mu_0) = 0 \) and that the \( n \times n \) matrix \( A = Df(x_0, \mu_0) \) has a simple eigen value \( \lambda = 0 \) with eigen vector \( v \) and that \( A^T \) has an eigen vector \( w \) corresponding to the eigen value \( \lambda = 0 \). Further, let \( A \) have \( k \) eigen values with negative real part and \((n - k - 1)\) eigen values with positive real part. And let

\[
  w^T f_\mu(x_0, \mu_0) \neq 0, \quad w^T \left[ D^2 f(x_0, \mu_0)(v, v) \right] \neq 0, \quad (1.43)
\]

where \( Df \) denotes the matrix of partial derivatives of the components of \( f \) with respect to the components of \( x \), and \( f_\mu \) denotes the vector of partial derivatives of the components of \( f \) with respect to the scalar \( \mu \). Then there is a smooth curve of equilibrium points of Eq. (1.41) in \( \mathbb{R}^n \times \mathbb{R} \) passing through \( (x_0, \mu_0) \), and tangent to the hyperplane \( \mathbb{R}^n \times \{\mu_0\} \). The system (1.41) experiences a SN bifurcation at the equilibrium point \( x_0 \) as the parameter \( \mu \) passes through the bifurcation value \( \mu_0 \).

If (1.43) becomes

\[
  w^T f_\mu(x_0, \mu_0) = 0, \quad w^T \left[ Df(x_0, \mu_0)v \right] \neq 0 \quad (1.44)
\]

and

\[
  w^T \left[ D^2 f(x_0, \mu_0)(v, v) \right] \neq 0,
\]

then the system (1.41) undergoes a transcritical bifurcation at the equilibrium point.
If the condition (1.43) is changed as

$$w^T f_{x_0, \mu} = 0, \quad w^T \left[ D^2 f (x_0, \mu) (v, v) \right] \neq 0$$

the system 1.41 experiences a pitchfork bifurcation [24].

One dimensional systems of the form

$$\dot{x} = \mu - x^2$$

undergoes S.N bifurcations. The corresponding bifurcation diagram is shown in Fig. 1.2.a. Dynamical systems having the structure $\dot{x} = \mu x - x^2$ are typical candidates for transcritical bifurcation. Fig. 1.2.b, schematically shows a transcritical bifurcation. Equations of the form $\dot{x} = \mu x \pm x^3$ give rise to pitch-fork bifurcations; the $+$ ($-$) sign resulting in unstable-symmetric (stable-symmetric) pitch-fork bifurcations (PBF). The bifurcation diagrams for the two variants of PBF are shown.

Figure 1.2: Bifurcations from equilibrium point: (a) SN bifurcation; (b) transcritical bifurcation; (c) and (d) are pitch fork bifurcations.
1.7.3 Hopf Bifurcations and Bifurcations of Limit Cycles from a multiple Focus

Hopf bifurcation is the second type of generic bifurcation that an equilibrium point can undergo. When the matrix $Df(x_0, \mu_0)$ has a pair of pure imaginary eigen values and no other eigen values with zero real part, there results a Hopf bifurcation at the equilibrium point. For every $\mu$ near $\mu_0$, there will be a unique equilibrium point $x_\mu$ near $x_0$; however as the eigen values of $Df(x_\mu, \mu)$ cross the imaginary axis at $\mu = \mu_0$, the dimensions of the stable (and unstable) manifold change, resulting in a change in the local phase portrait of (1.41). Generically, a Hopf bifurcation occurs, where a periodic orbit is created as the equilibrium point becomes unstable.

Below $\mu = \mu_0$, the equilibrium point (focus) becomes unstable, the state space showing expansion. Above $\mu_0$, the state space contracts; at $\mu_0$, there is a phase area (volume) preservation and the stable limit cycle is formed. If the focus is attracting type, the limit cycle generated by Hopf bifurcation will be unstable.

The stability of the limit cycle formed by the bifurcation can be rigorously stated in terms of the Lyapunov number $\sigma$ of the focus. If $\mu = \mu_0$ is the bifurcation value, then for $\sigma < 0$ the system has a unique stable cycle for $\mu > \mu_0$ and no limit cycle for $\mu \leq \mu_0$. If $\sigma > 0$, the system possesses a unique unstable limit cycle for $\mu < \mu_0$ and no limit cycle for $\mu \geq \mu_0$. The case $\sigma < 0$ gives rise to supercritical Hopf bifurcation. Here the critical point repels the limit cycle. When $\sigma > 0$, we have a subcritical bifurcation where the critical point absorbs the unstable limit cycle. A typical bifurcation diagram is shown in Fig. 1.3.
Figure 1.3: Hopf bifurcation from an equilibrium point: (a) shows supercritical and (b) the subcritical cases. In (b), a stable equilibrium coexists with stable and unstable periodic orbits.
When the critical point is a multiple focus of Eq. (1.41) with multiplicity $m$, a bifurcation to at most $m$ limit cycles can take place, all of which may not show themselves unless under special conditions.

### 1.7.4 Bifurcations at Non Hyperbolic Periodic orbits

Many fascinating types of bifurcations can occur at a non-hyperbolic periodic orbit, possessing two or more characteristic multipliers, with one always on the unit circle. When the second multiplier also lies on the unit circle, the periodic orbit $\Gamma$ has a two-dimensional central manifold $W^c(\Gamma)$ and the simplest types of bifurcations that can occur on this manifold are the SN, transcritical and pitchfork bifurcations. (The nomenclature is identical for bifurcations of equilibrium points and limit cycles). The pitch-fork bifurcation is also called the *cusp-bifurcation*. Of these, the SN is the generic type. This happens when the derivative of the Poincaré map $DP(x_0)$ at a point $x_0 \in \Gamma$, has one eigen value $+1$. If $DP(x_0)$ has one eigen value equal to $-1$, the *flip bifurcation* (also called *period doubling*) occurs. Finally if $DP(x_0)$ has a pair of complex conjugate eigen values on the unit circle, then $\Gamma$ generically bifurcates into an invariant two-torus; this, known as the Hopf bifurcation for the Poincaré map is the analogue of Hopf bifurcation of equilibrium points. A schematic drawing of this bifurcation is shown in Fig. 1.4.

The S.N bifurcation also called the cyclic fold, is often associated with the *hysteretic jump*, where for a limited range of the parameter, two stable closed orbits coexist, separated by a repelling cycle. In the presence of constraints such as symmetries, transcritical and pitch-fork bifurcations also can take place, for the genericity of which, more than one control parameter would be required.
The flip (period doubling) bifurcation is a familiar mechanism of stability loss. This has no analogue in bifurcations of equilibrium points. In this, a stable limit cycle loses stability as $\mu$ changes, a new stable limit cycle with period twice of the original is formed. A period doubled trajectory intersects itself in two dimensions. Since no such crossing is possible, period doubling requires at least a three dimensional state space. This can occur in supercritical (subtle) and subcritical (catastrophic) forms. The bifurcation diagrams for both these cases are shown in Fig .1.5.

When $DP (x_0)$ has a pair of complex conjugate eigen values on the unit circle, the formerly stable limit cycle bifurcates generically into a two-torus. This is called also as the Neimark or secondary Hopf bifurcation. These bifurcations also require a three dimensional state space. The phenomenon can be generalised to an $n$-dimensional torus bifurcating into an $n + 1$ dimensional torus. In some cases the limit cycle bifurcates into another periodic orbit of period $kT$, with $k > 2$. 

Figure 1.4: Neimark bifurcation or secondary Hopf bifurcation.
Figure 1.5: Period doubling bifurcations. (a) is the supercritical and (b) the subcritical form.

The resulting motion lies on a two-torus, but does not fill the whole surface. This phenomenon is called phase-locking.

We consider the different analytical techniques with which transition from periodic to chaotic behaviour can be studied and the various routes for this, in the next chapter.