CHAPTER-2

RECURRENCE OF MARKOV CHAINS - CERTAIN CONDITIONS FOR ERGODICITY AND APPLICATION OF THE RESULTS TO QUEUEING PROBLEMS

2.0 INTRODUCTION

Let \( \{X_n, n \geq 0\} \) be an irreducible time-homogeneous MC with transition probability matrix \( P = (P_{ij}) \). The quantity \( E(X_{n+1} - X_n \mid X_n = i) \) is of interest because it gives intuitively appealing and easily computed conditions for positive recurrence. The condition \( \limsup_{i \to \infty} E(X_{n+1} - X_n \mid X_n = i) < 0 \) was given by Pakes and the main result of this paper is sharpening of this criterion for periodic MC with transition probabilities satisfying \( \lim_{i \to \infty} p_{ij} = 0 \) for each \( j \). This is facilitated by considering a MC for which the matrix \( P \) can be written as the product of a finite number of stochastic matrices and obtaining an inequality satisfied by \( \limsup_{i \to \infty} E(X_{n+1} - X_n \mid X_n = i) \).

This inequality can also be used directly to get conditions for positive recurrence of such a MC and this is illustrated by considering a single server queueing problem with rotating servers.

Let \( \{X_n, n \geq 0\} \) be a MC with stationary transition probabilities \( P_{ij} \)'s where \( P_{ij} = P\{X_{n+1} = j \mid X_n = i\} \) and the set of non-negative integers as state space. Foster's (1953) theorem and its extension (which was first stated by Kingman (1961) and Pakes (1969)) give necessary and sufficient conditions for positive recurrence of \( \{X_n, n \geq 0\} \) when its state space forms a single irreducible class.
The theorem given by Pakes (1969) states that it is necessary and sufficient that there exists a non-negative sequence \( \{y_j, j \geq 0\} \) such that

\[
\sum_{j=0}^{\infty} p_{ij} y_j \leq y_i - 1 \quad \text{for } i \geq I,
\]

and

\[
\sum_{j=0}^{\infty} p_{ij} y_j < \infty \quad \text{for } i < I, \text{ for some } I < \infty. \tag{2.0.1}
\]

Pakes also defined \( \gamma_i = E(X_{n+1} - X_n \mid X_n = i) \), the conditional one step change in the state of process and choosing \( y_j = j \) in (2.0.1) showed that \( \limsup_{i \to \infty} \gamma_i < 0 \) with \( \gamma_i < \infty \) is sufficient for positive recurrence.

While only a special case of (2.0.1), this result is useful because, in applications, it gives simply obtained criteria for ergodicity which also turn out to be necessary in some cases. But if the MC is periodic, this gives a rather weak result and it is the main purpose of this paper to give a stronger criterion of the same type for periodic MCs with transition probabilities satisfying \( \lim_{i \to \infty} p_{ij} = 0 \) for each \( j \). This is facilitated by the concept of a MC in which transitions occur in phases and this concept itself can be used to obtain steady state conditions for some queueing models with service heterogeneity.

Section 2.1. provides certain theorems on MCs. The first result of this chapter is sharpening of this criterion for periodic MCs with transition probabilities satisfying \( \lim_{i \to \infty} P_{ij} = 0 \) for each \( j \). This is facilitated by considering a MC for which the matrix \( P \) can be written as the product of a finite number of stochastic matrices and obtaining an inequality satisfied by \( \limsup_{i \to \infty} E(X_{n+1} - X_n \mid X_n = i) \). This inequality can also be used
directly to get conditions for positive recurrence of such a MC and this is interpreted in Section 2.2 by considering a single server queueing problem with rotation of servers.

This chapter further finds sufficient conditions for the ergodicity and recurrence of irreducible and aperiodic MCs. The results relating to extension of Foster’s theorem and positive recurrence of MCs are provided in Section 2.3. The results presented in this chapter are reported in Manoharan (2000).

2.1 CERTAIN THEOREMS ON MARKOV CHAINS

In the following \( \{X_n, n \geq 0\} \) is a MC with stationary transition probabilities \( P_{ij} = P\{X_{n+1} = j \mid X_n = i\} \), with state space the set of non-negative integers forming a single irreducible class.

Let

\[
\gamma_i = E [X_{n+1} - X_n \mid X_n = i], \quad \gamma^* = \limsup_{i \to \infty} \gamma_i \quad \text{and} \quad \gamma_* = \liminf_{i \to \infty} \gamma_i
\]

and as a general notation, when two or more stochastic matrices are considered simultaneously, let \( \gamma_i(P) \) denote the quantity \( \gamma_i \) with reference to the stochastic matrix \( P \).

We shall now assume that the matrix \( P = (P_{ij}) \) can be written as \( P = Q_1 Q_2 \ldots Q_k \), \( k \geq 1 \) where each of the \( Q_i \)'s is a stochastic matrix of countable dimension. Let \( q_{im}(j) \) be the \( (i, m) \)th element of \( Q_j \). Intuitively, this means that the transitions in the MC are the product of a chain of
k transitions, each obeying the conditional probability law (given the previous transition) given by the appropriate stochastic matrix $Q_j$. Then we have

**Theorem 2.1.1**

If $\lim_{i \to \infty} q_{im}(j) = 0$ for each $j$ and $m$ and $\gamma_j(Q_j) < \infty$, then, $\gamma^* \leq \sum_{j=1}^{k} \gamma_j^*(Q_j)$

and $\gamma^* \geq \sum_{j=1}^{k} \gamma_j^*(Q_j)$ whenever the sums on the right exist.

For the proof, we require the following lemma.

**Lemma 2.1.1**

Let $\{a_{ij}, i, j \geq 0\}$ be a non-negative real valued sequence with $\sum_{j=0}^{\infty} a_{ij} = c > 0$ and $\lim a_{ij} = 0$ for each $j$. Let $\{x_j, j \geq 0\}$ be a real valued sequence with $\limsup_{j \to \infty} x_j = x$ and $\liminf_{j \to \infty} x_j = x'$. Then $\limsup_{j \to \infty} y_j \leq cx$ and $\liminf_{j \to \infty} y_j \geq cx'$ where $y_i = \sum_{j=0}^{\infty} a_{ij} x_j$.

**Proof**

To prove the first assertion, if $x = \infty$ it is trivially true. If $|x| < \infty$, then $x_j \leq x + \delta/c$ for all $j \geq J$, say, then,

$$y_i \leq \sum_{j=0}^{J-1} a_{ij} x_j + (x + \delta/c) \sum_{j=J}^{\infty} a_{ij} \quad \text{since } a_{ij} \geq 0$$

$$= \sum_{j=0}^{J-1} a_{ij} (x_j - x - \delta/c) + c(x + \delta/c)$$

and since $\lim_{i \to \infty} a_{ij} = 0$, $\limsup_{i \to \infty} y_i \leq c(x + \delta/c) = cx + \delta$ and $\delta$ is arbitrary. Hence the result.
If \( x = -\infty \) then \( x_j \leq M \) for all \( j \geq J \), say and we get

\[
y_i \leq \sum_{j=0}^{J-1} a_{ij} x_j - M \sum_{j=J}^{\infty} a_{ij}
\]

\[
= \sum_{j=0}^{J-1} a_{ij} (x_j + M) - Mc
\]

\[
\limsup_{i \to \infty} y_i \leq -Mc, \ M \text{ arbitrarily large and hence}
\]

\[
\limsup_{i \to \infty} y_i = -\infty.
\]

The second assertion can be proved similarly.

**Proof of Theorem 2.1.1**

To prove the theorem, let \( Q_2, Q_3, \ldots, Q_k = R_2 \) so that \( P = Q_1 R_2 \). Then,

\[
\gamma_i = \sum_{j=0}^{\infty} (j-i)p_{ij} = \sum_{j=0}^{\infty} (j-i) \sum_{k=0}^{\infty} q_{ik}(1)r_{kj}(2)
\]

\[
= \sum_{k=0}^{\infty} q_{ik}(1) \sum_{j=0}^{\infty} (j-i) r_{kj}(2)
\]

\[
= \sum_{k=0}^{\infty} [\gamma_k(R_2) + k-i]q_{ik}(1)
\]

\[
= \sum_{k=0}^{\infty} \gamma_k(R_2)q_{ik}(1) + \gamma_i(Q_1)
\]

since \( \gamma_i(Q_1) < \infty \) and \( \gamma_i \) exists. Applying the lemma to the sum, \( \sum_{k=0}^{\infty} \gamma_k(R_2)a_{ik}(1) \),

\( a_{ik} = q_{ik}(1), \ c = 1, \ \gamma_k(R_2) = x_k \), we get

\[
\gamma^* \leq \gamma^*(Q_1) + \gamma^*(R_2) \text{ and } \gamma^* \geq \gamma^*(Q_1) + \gamma^*(R_2)
\]

when the sums exist. Since \( R_2 = Q_2 \ldots Q_k \) the process can be repeated and hence we get the result.
Let $\gamma_i^{(r)} = E(X_{n+r} - X_n \mid X_n = i)$. This was defined by Marlin (1973) who obtained an expression for it. If the MC $\{X_n\}$ is aperiodic, it is obvious that if $p_{ij}$ in (2.0.1) is replaced by $p_{ij}^{(r)}$ and the conditions are required to hold for some $r \geq 1$, the result is still true, for then we have: $\{X_{nr}, n \geq 0\}$ (which, for each $r$, has an irreducible state space since $X_n$ is aperiodic) is positive recurrent and hence so is $\{X_n, n \geq 0\}$. It also follows from this that $\lim \sup_{i} \gamma_i^{(r)} < 0$ (with $\gamma_i^{(r)} < \infty$) is sufficient for positive recurrence. In itself this is a rather trivial generalization, but we can now show that, at least when $\lim p_{ij} = 0$ for each $j$, $\lim \sup_{i} \gamma_i^{(r)} < 0$ is a stronger condition for positive recurrence than $\lim \sup_{i} \gamma_i < 0$ even when it holds for only a specified $r > 1$.

**Corollary 2.1.1**

If $\lim_{i} p_{ij} = 0$ for each $j$, and $\gamma_i < \infty$, then

$$\lim \sup_{i} \gamma_i^{(r)} \leq r\gamma^* \text{ and } \lim \inf_{i} \gamma_i^{(r)} \geq r\gamma^*.$$  

**Proof**

Choose $k = r$, $Q_i = P$ or $j = 1, 2, \ldots, r$ and apply the theorem.

We now assume that $\{X_n\}$ is a periodic MC with an irreducible state space and $s$ cyclic subclasses. Denote the subclasses by $c_j$, $j = 1, 2, \ldots, s$. Since the MC is positive recurrent, if any of the subclasses is finite, we will assume that all the $s$ subclasses are non-finite.
As an illustration, consider the classical random walk on non-negative integers with $p_{i,i+1} = p_i$, $p_{i,i-1} = 1 - p_i$ for $i > 0$ and $p_{01} = 1$. This is a periodic MC of period 2 with the even numbered states and the odd numbered states forming the two classes. Here $\gamma_i = 2p_i - 1$ for $i > 0$ and the random walk is positive recurrent if $\limsup_{i \to \infty} p_i < \frac{1}{2}$. But, when $\lim p_i$ does not exist, this gives a rather weak condition. For example, let $p_i = p$ if $i$ is odd and $p_i = q$ if $i$ is even, $p \neq q$. Then $\limsup_{i \to \infty} p_i = \max(p, q)$ so that the condition states: $\max(p, q) < \frac{1}{2}$, whereas it is sufficient if $p+q < 1$, as will be shown below.

Theorem 2.1.2

Let $\{X_n, n \geq 0\}$ be a periodic MC with $s$ periodic subclasses (within an irreducible state space). Let $\lim p_{ij} = 0$ for each $j$. Then it is positive recurrent if

$$\sum_{j=1}^{s} \gamma^*(c_j) < 0,$$

where $\gamma^*(c_j) = \limsup_{i \to \infty} \gamma_i$.

Proof

The matrix $P^s$ gives the transition probabilities for $\{X_n, n \geq 0\}$ and consists of $s$ closed classes corresponding to the $s$ subclasses of the MC $\{X_n\}$. Since the stationary distribution (if it exists) corresponding to the $j^{th}$ class in $P^s$ is given by

$$\pi_k = s \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} p_{ik}^{(m)}$$

for $k \in c_j$, $\pi_k > 0$ if only if $\{X_n\}$ is positive recurrent.

And if $\limsup_{i \to \infty} \mathbb{E}(X_{(n+1)s} - X_{ns} \mid X_{ns} = i) < 0$ for some $j$, that class has a stationary distribution and hence does $\{X_n\}$.
Suppose the matrix $P^s$ is replaced by the product $Q = P_1 P_2 P_3 \ldots P_s$ where, in $P_j$, the rows corresponding to the states in $c_j$ are retained as they are in $P$ and if $i \notin c_j$, then all the elements of the $i^{\text{th}}$ row are taken as zero except the element in the $[i/2]^{\text{th}}$ column. This is taken as $1([x]$ is the largest integer smaller than or equal to $x$). Such a large clearly leaves the rows in $P^s$ corresponding to the states in $c_1$ unchanged (i.e. the transition probabilities, in $Q$, from the states in $c_1$ are the same as in $P^s$) so that

$$\limsup_{i \to \infty} E(X_{(n+1)s} - X_{ns} \mid X_{ns} = i) = \limsup_{i \to \infty} \gamma_i(Q)$$

(2.1.1)

In the $k^{\text{th}}$ column of $P_j$ the elements corresponding to the rows of $c_j$ are retained as they are and all other elements are zero excepting the $2k^{\text{th}}$ and $(2k+1)^{\text{th}}$ elements.

Hence, if $p_{ik}(j)$ is the $(i, k)^{\text{th}}$ element of $P_j$, $\lim p_{ik}(j) = 0$. Applying Theorem 2.1.1,

$$\limsup_{i \to \infty} \gamma_i(Q) \leq \sum_{j=1}^{s} \gamma^*(P_j)$$

(2.1.2)

But

$$\gamma_i(P_j) = E(X_{n+1} - X_n \mid X_n = i) \quad \text{if} \quad i \in c_j$$

$$= [i/2] - i \leq i/2 \quad \text{if} \quad i \in c_j$$

so that $\gamma^*(p_j) = \limsup_{i \to \infty} E(X_{n+1} - X_n \mid X_n = i) = \gamma^*(c_j)$ even if it is equal to $-\infty$.

Hence, from (2.1.2),

$$\limsup_{i \to \infty} \gamma_i(Q) \leq \sum_{j=1}^{s} \gamma^*(c_j)$$

and since $\limsup_{i \to \infty} \gamma_i(Q) \leq \limsup_{i \to \infty} \gamma_i(Q)$ it is $< 0$ if $\sum_{j=1}^{s} \gamma^*(c_j) < 0$ and hence from (2.1.1) and the remarks preceding it, $\{X_n\}$ is positive recurrent. This completes the proof.
The inequality \( \limsup_{i \to \infty} \gamma_i(Q) \leq \limsup_{i \in \mathbb{C^j}} \gamma_i(Q) \) actually holds as an equality since, if \( i \in \mathbb{C^1} \), \( \gamma_i(Q) \leq -i/2 + \sum_{i=0}^{\infty} (j-[i/2])(P_{2j} \ldots P_s)^{i/2} \) and as \( i \to \infty \) the first quantity \( \to -\infty \) whereas, the superior limit of the second is \( \leq \sum_{j=2}^{\infty} \gamma^*(c_j) < \infty \) so that \( \gamma_i(Q) \to \infty \) if \( i \notin \mathbb{C^1} \).

Reverting to the example of the random walk, we see that the condition reduces to \( \limsup_{i \text{ even}} \gamma_i + \limsup_{i \text{ odd}} \gamma_i < 0 \) or \( \limsup_{i \to \infty} p_i + \limsup_{i \to \infty} p_i < 1 \) or \( p + q < 1 \).

(Since almost two elements are non zero in any column, \( \lim p_{ij} = 0 \) for each \( j \)).

Apart from its usefulness in providing Theorem 2.1.2, Theorem 2.1.1 can also be used directly to obtain simple conditions for ergodicity in specially structured MCs. This will be illustrated later on in this chapter (it may be noted that the following result can be obtained directly without using Theorem 2.1.1).

2.2 APPLICATION TO A SINGLE SERVER QUEUE WITH SERVICE HETEROGENEITY

The practice of rotation of servers is very common and the usual assumption of a single service distribution for all time is not valid when the servers have different service rates, Scott (1972, 1973) has considered models in which two (or more) servers serve in rotation, each handling service for a fixed number of consecutive busy cycles each time he takes charge of the service counter. A different assumption will be made here and steady state condition will be obtained. There are \( s \) servers, who were in rotation and give service to \( N_1, N_2, \ldots, N_s \) customers,
i.e., during each tenure, the \( j \)th server gives service to \( N_j \) customers. The arrivals are in a Poisson stream with intensity \( \lambda > 0 \) and the \( j \)th server has service time distribution \( B_j(\cdot) \) with mean \( 1/\mu_j < \infty \); \( \rho_j = \lambda/\mu_j \).

It is convenient to assume that there are \( N_1 + N_2 + \ldots + N_s = k \) servers, each giving a single service. The duration of these \( k \) services will be called a service cycle. It is clear that if service cycle ending points are taken as regeneration points (the times of departure of the \( 0^{th} \), \( k^{th} \), \( 2k^{th} \), customers) the queue length process \( \{Q_n, n \geq 0\} \) at such points forms a time-homogeneous MC. It has transition probability matrix \( P = P_1 P_2 P_3 \ldots P_k \), where \( P_j \) denotes the transition probability matrix for the queue length from the end of the \( (j-1)^{th} \) service to the end of the \( j^{th} \) service in a service cycle. Then we have

**Theorem 2.2.1**

The MC \( \{Q_n, n \geq 0\} \) is positive recurrent if \( (l/k) \sum_{j=1}^{k} \rho_j < 1 \).

**Proof**

Let \( \gamma_i(P) = E(Q_{n+1} - Q_n \mid Q_n = i) \). Then the condition \( \gamma^*(P) (= \limsup_{i \to \infty} \gamma_i(P)) < 0 \)

And \( \gamma(P) < \infty \) is sufficient for positive recurrence of \( \{Q_n\} \). Let \( p_{im}(j) \) be the \((i, m)^{th} \) element of \( P_j \). Then \( p_{im}(j) = 0 \) if \( i > m + 1 \) so that \( \lim_{i \to \infty} p_{im}(j) = 0 \) and,

\( \gamma_i(P_j) = \rho_j \) if \( i = 0 \) and \( \rho_j - 1 \) if \( i > 0 \) so that \( \gamma^*(P_j) = \rho_j - 1 \). Applying Theorem 2.1.1, we have \( \gamma^*(P) \leq \sum_{j=1}^{k} \gamma^*(P_j) = \sum_{j=1}^{k} (\rho_j - 1) < 0 \) by hypothesis.

Hence \( Q_n \) is positive recurrent.
Remarks 2.2.1

In the corollary to Theorem 2.1.1 and in Theorem 2.1.2 we assumed that
\[ \lim_{i \to \infty} p_{ij} = 0 \] for each \( j \). The question that naturally arises is: what happens when this condition is not satisfied? In this case there are two possibilities:

Case 1

\[ \liminf_{i \to \infty} p_{ij} > 0 \] for some \( j \). Then \( p_{ij} > \delta \) for all \( i \geq 1 \), say and then,

\[
P_{ji}^{(m)} \geq \sum_{i=1}^{\infty} p_{ji}^{(m-1)} p_{ij} > \delta \left[ 1 - \sum_{i=0}^{l-1} p_{ij}^{(m-1)} \right]
\]

\[
(1/n) \sum_{m=1}^{n} p_{ji}^{(m)} > \delta \left[ 1 - \sum_{i=0}^{l-1} (1/n) \sum_{m=1}^{n} p_{ji}^{(m-1)} \right]
\]

and since \( \{X_n\} \) is irreducible, as \( n \to \infty \) the R.H.S. tends to a quantity > 0 and hence \( \{X_n\} \) is positive recurrent.

Case 2

\[ \liminf_{i \to \infty} p_{ij} = 0 \] and \( \limsup_{i \to \infty} p_{ij} > 0 \) for some \( j \). In this case, the present approach does not permit any conclusions regarding the ergodicity of the chain.

2.3 EXTENSION OF FOSTER'S THEOREM AND POSITIVE RECURRENCE OF MARKOV CHAINS

Crabill (1968) has given a set of sufficient conditions for the positive recurrence of a special type of irreducible and aperiodic denumerable MC that arises in the study of
queuing systems. The proof consisted of fulfilling the conditions of a well known theorem of Foster (1953). Crabill made a proposition on a condition that would be sufficient for the positive recurrence of an arbitrary irreducible aperiodic MC. In this section, an extension of Foster’s theorem is proved to enable Crabill’s (1968) proposition. It may be noted that a multidimensional extension of Foster’s theorem has been independently stated, but not proved, by Kingman (1961).

Let \( \{X_n\}^\infty_0 \) be an irreducible and aperiodic MC having as its 0 state space the nonnegative integers. Let \( ((P_{ij})) \) (i, j = 0, 1, ...) be the matrix of transition probabilities and let \( ((P_{ij}^{(n)})) \) be its \( n \) th power.

**THEOREM 2.3.1**

The MC is ergodic if there exists a non-negative solution of the inequalities

\[
\sum_{j=0}^{\infty} P_{ij} y_j \leq y_i - 1, \quad (i = N, N+1, ...) \tag{1}
\]

such that

\[
\sum_{j=0}^{\infty} P_{ij} y_j < \infty, \quad (i = 0, 1, ..., N-1) \tag{2}
\]

where N is any fixed positive integer.

**Proof**

Under the hypotheses, \( \lim_{i \to \infty} P_{ij}^{(n)} = \pi_j \) exists and is independent of i. Furthermore \( \pi_j \) is either positive of all j or zero for all j. The chain is ergodic iff \( \pi_j > 0 \) for all j. (See Karlin (1966) Chapters 1 and 2). Thus it suffices to prove the existence of a nonnegative integer j such that \( \pi_j > 0 \).
Let \( \lambda_i = \sum_{j=0}^{\infty} p_{ij} y_j \) for \( i = 0, 1, \ldots, N-1 \). Then \( \lambda_i \) are finite. Define the sequence \( \{ y_i^{(n)} \} \) (\( i = 0, 1, 2, \ldots, j : n = 1, 2, \ldots \)) by

\[
y_i^{(n+1)} = \sum_{j=0}^{\infty} p_{ij}^{(n)} y_j, \quad y_i^{(1)} = y_i
\]

Then

\[
y_i^{(n+2)} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} p_{ij}^{(n)} p_{jk} y_k,
\]

\[
\leq \sum_{j=0}^{N-1} p_{ij}^{(n)} \lambda_j + \sum_{j=N}^{\infty} p_{ij}^{(n)} (y_j - 1)
\]

\[
\leq \sum_{j=0}^{N-1} (1 + \lambda_j) p_{ij}^{(n)} + y_i^{(n+1)} - 1.
\]

We see that if \( y_i^{(n)} \) is finite, then so is \( y_i^{(n+1)} \). But \( y_i^{(1)} \) is finite for each \( i \), so \( y_i^{(n)} \) is finite for each \( i \) and \( n \). Iterating the last inequality gives,

\[
y_i^{(n+2)} \leq \sum_{j=0}^{N-1} (1 + \lambda_j) \sum_{m=1}^{n} p_{ij}^{(m)} + y_i^{(2)} - n.
\]

Dividing by \( n \) and letting \( n \to \infty \) gives the result,

\[
0 \leq \sum_{j=0}^{N-1} (1 + \lambda_j) \pi_j - 1 \quad \text{or} \quad \sum_{j=0}^{N-1} (1 + \lambda_j) \pi_j \geq 1
\]

Thus at least one of the \( \pi_j \)'s is positive, and so the chain is ergodic.

Define the sequences \( \{ \gamma_i \} \) and \( \{ \beta_i \} \) by

\[
\gamma_i = \mathbb{E}\{ X_{n+1} \mid X_n = i \}, \quad (i = 0, 1, \ldots)
\]

\[
\beta_i = \mathbb{E}\{ X_{n+1} \mid X_n = i \} = \sum_{j=1}^{\infty} j p_{ij} \quad (i = 0, 1, \ldots)
\]
We have $\beta_i - i = \gamma_i$. Crabill (1968) has proposed that if, in an irreducible and aperiodic MC, $\limsup_{i \to \infty} \gamma_i < 0$ then the chain is ergodic.

We now proceed to prove that this is true.

**THEOREM 2.3.2**

If $|\gamma_i| < \infty$ for all $i$ and if $\limsup_{i \to \infty} \gamma_i < 0$ in an irreducible and aperiodic MC, then it is ergodic.

**Proof**

By definition of $\limsup$, there exists $\alpha > 0$ such that $\gamma_i \leq -\alpha$ for all $i \geq N$. We then have $\beta_i \leq i - \alpha$ for all $i \geq N$.

Define the sequence $\{y_i\}_{i=0}^{\infty}$ by $y_i = i / \alpha$. Then for $i \geq N$, we have

$$\sum_{j=0}^{\infty} P_{ij} y_j = \beta_i / \alpha \leq (i - \alpha) / \alpha = y_i - 1,$$

The hypotheses also imply that $\beta_i < \infty$ for all $i$ and we have

$$\sum_{j=0}^{\infty} P_{ij} y_j < \infty, \quad (i = 0, 1, \ldots, N-1)$$

Hence the conditions of Theorem 2.3.1 are fulfilled, and the chain is ergodic.

It may be remarked here that, Crabill's Theorem is a direct consequence of Theorem 2.3.2 above.

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Crabill (1968) used another theorem of Foster (1953) to obtain sufficient conditions for the recurrence of the imbedded MC that he analyzed. A detailed proof of this theorem is given in page 75 of Karlin (1966).

We now extend this proof in the following manner.

**THEOREM 2.3.3**

A sufficient condition for the recurrence of an irreducible and aperiodic MC is that there exists a sequence \( \{y_i\} \) such that

\[
\sum_{j=0}^{\infty} y_j P_{ij} \leq y_i \quad (i \geq N)
\]

\( y_i \to \infty \) as \( i \to \infty \), and \( N \) is a fixed positive integer.

**Proof**

An irreducible and aperiodic MC will be recurrent iff the probability of eventually entering the set of states \( C = \{0, 1, \ldots, N-1\} \) from any other state is unity, for if this condition holds then the probability of eventual return to \( C \) from any state in \( C \) is unity. Then, if we reason as in page 54 of Karlin (1966) it is clear that there is a state in \( C \) that is visited infinitely often with probability unity.

Accordingly, consider the modified MC having as its transition matrix \( (\tilde{P}_{ij}) \) where \( \tilde{P}_{ij} = \delta_{ij} \) for \( i = 0, 1, \ldots, N-1 \) and \( \tilde{P}_{ij} = P_{ij} \) for \( i \geq N \). This is a chain having \( C \) as a recurrent class and all remaining states are transient. Let \( \pi'_i = \lim_{n \to \infty} \tilde{P}_{ij}^{(n)} \) and \( \pi'_i(C) = \sum_{j=0}^{N-1} \pi'_j \). We show that \( \pi'_i(C) = 1 \).
We have
\[ \sum_{j=0}^{\infty} \tilde{P}_{i, j} y_j \leq y_i, \quad i \geq 0, \]
so that
\[ \sum_{j=0}^{\infty} \tilde{P}^{(n)}_{i, j} y_j \leq y_i. \]
We may assume that \( y_i > 0 \) for all \( i \). Choose \( M(\varepsilon) \) such that \( 1/y_i \leq \varepsilon \) for \( i \geq M(\varepsilon) \).

Fix \( i^* \geq N \). Then
\[ \sum_{j=0}^{M-1} \tilde{P}^{(n)}_{i, j} y_j + \sum_{j=M}^{\infty} \tilde{P}^{(n)}_{i, j} y_j \leq y_{i^*}, \]
so that
\[ \sum_{j=0}^{M-1} \tilde{P}^{(n)}_{i, j} y_j + \min_{j \geq M} \{ y_j \} \sum_{j=M}^{\infty} \tilde{P}^{(n)}_{i, j} y_j \leq y_{i^*}, \]
and hence
\[ \sum_{j=0}^{M-1} \tilde{P}^{(n)}_{i, j} y_j + \min_{j \geq M} \{ y_j \} \left[ 1 - \sum_{j=0}^{M-1} \tilde{P}^{(n)}_{i, j} y_j \right] \leq y_{i^*}. \]
Now for \( j \notin C \), \( \tilde{P}^{(n)}_{i, j} \to 0 \) as \( n \to \infty \), so that we have
\[ \sum_{j=0}^{M-1} \pi_{i, j} y_j + \min_{j \geq M} \{ y_j \} \left[ 1 - \pi_{i, \ast}(C) \right] \leq y_{i^*}. \]
We finally have
\[ 1 - \pi_{i, \ast}(C) \leq \varepsilon K, \]
where \( K = y_{i^*} - \sum_{j=0}^{N-1} \pi_{i, j} y_j \). But the left-hand side of the last inequality is independent of \( \varepsilon \), so \( \pi_{i, \ast}(C) = 1 \) and the original process is recurrent.
THEOREM 2.3.4

A sufficient condition that an irreducible and aperiodic MC be recurrent is that there exist a positive integer $N$ such that

$$
\gamma_i = E\{X_{n+1} - X_n | X_n = i\}, \quad (i \geq N)
$$

Proof

The hypotheses imply that $\beta_i \leq i$ and $i \geq N$ and the result follows on letting $\gamma_i = i$ in Theorem 2.3.3.

We now proceed to the next chapter to provide certain results relating to ergodicity and recurrence of MCs on an arbitrary state space and busy period distribution of a queueing system.