Regular graphs have elicited a lot of interest in both theory and applications. As we discussed in Section 2.1, all graphs pertinent to the degree-diameter problem are regular. Large random regular graphs have been observed to have several opti-
mal features such as high connectivity, low diameter and hamiltonicity [Robinson and Wormald, 1994; Wormald, 1999].

The *symmetry* requirement is very common in many network design problems. In Section 1.1, we discussed several applications where symmetry—in one form or the other—is an important design consideration.

In case of distributed indices, symmetry translates to a topology where each node is expected to take the same amount of bookkeeping cost in managing the distributed index. Bookkeeping cost arises from keeping track of other nodes and their addresses in DHT *finger tables*. Symmetry is addressed by modelling the index in the form of a graph in which every node has the same degree (number of edges incident on a node), *viz*; a *regular graph*.

In case of NCW, symmetry translates to building networks that are ideally *vertex transitive* (and/or *edge transitive*). Again, all vertex and edge transitive graphs are necessarily regular graphs. In case of CDNs, symmetry translates to designing networks such that the nodes and/or edges have uniform (or nearly uniform) communication load, which is to say nodes and/or edges have similar values of betweenness centrality.

In sections 2.2.2 and 2.2.3, we discussed topologies useful in DHTs, complex networks and communication networks, such as, rings, hypercubes, homogeneous isotropic networks and entangled networks, all of which are symmetric. Recently, there are several efforts in search of universal optimal structures for topology design. Structural and spectral analyses of regular graph families such as Cayley graphs and Ramanujan graphs show them to have high efficiency, high resilience
to failures and low susceptibility to congestion. Regular graphs emerge as the fundamental structures for designing optimal networks.

Thus, the underlying universal structure of a symmetric topology is the regular graph. The family of regular graph topologies forms an interesting subset of optimal topologies having its own unique properties that deserves to be studied separately. Further, almost all regular graphs are hamiltonian as shown by [Robinson and Wormald, 1994]. This implies almost all regular graphs are circular skip lists. In the previous chapters we observed that CSL are an important class of topologies useful across many application domains.

In this regard, we study regular graphs and bring out several properties and their relationships to the optimality in networks. The objective of this chapter is to explore theoretical underpinnings of regular graphs that are pertinent to network design. The important results from this work can be found in [Patil and Srinivasa, 2010].

### 7.1 Connected Regular Graphs: Basics

A graph $G (\mathcal{V}, \mathcal{E})$ is called $r$–regular if $\forall v \in \mathcal{V}, \text{degree}(v) = r$. The term $r$ is called the regularity of the graph $G$. We first examine properties of undirected regular graphs relevant to optimal network design. We will also briefly address regular directed graphs in Section 7.4.

In the following sections, whenever we refer to regular graphs, we shall be referring to undirected regular connected graphs, unless specified otherwise.
Further, we shall not be considering multiedges, as well as self loops.

We begin with the following observations about regular connected graphs.

**Postulate 1.** If a connected graph $G (V, E)$, where the number of nodes, $|V| = n$, $n > 2$, is $r$–regular, then $r \geq 2$.

**Postulate 2.** It is always possible to build a 2–regular connected graph for any given set of nodes $n > 2$. Such a 2–regular graph is in the form of a single hamiltonian circuit encompassing all $n$ nodes.

**Postulate 3.** The smallest number of nodes required to build an $r$–regular connected graph is $(r + 1)$, which is the clique with $(r + 1)$ nodes.

**Postulate 4.** It is not possible to build an $r$–regular graph with $n$ nodes, where both $r$ and $n$ are odd numbers, since the total degree of a graph is always even.

During the course of our work, we encountered some conjectures like the following: (1) every $r$–regular graph is also $(r - 1)$–regular. That is, it is possible to obtain an $(r - 1)$–regular graph from an $r$–regular graph by removing one or more edges; (2) since a connected 2–regular graph is a hamiltonian circuit, every connected $r$–regular graph with $r > 2$ has a hamiltonian circuit.

If the first conjecture were to be true, it has applications in handling failures in distributed systems. If one or more connections fail, and it is not possible to construct an $r$–regular graph with the set of nodes, we should always be able to build an $r - 1$ regular graph and retain the symmetric degree centrality property.

If the second conjecture were to be true, we can guarantee that an $r$–regular graph with $n$ nodes with $r > 2$ has a diameter less than or equal to $\frac{n}{2}$. Also, a
hamiltonian undirected graph is at least 2–connected. That is, there are at least two edge-independent (as well as, node-independent) paths between any pair of nodes in the graph.

However, we can refute both these conjectures using the above postulates. Further, Figure 7.1 clearly refutes the second conjecture.

### 7.2 Extending Regular Graphs

In this section, we describe a simple algorithm to construct regular graphs of arbitrary number of nodes and arbitrary regularity by a procedure called extension. We also address the question of how to extend a given regular graph to accommodate arrival of new nodes without affecting the regularity.

**Theorem 1.** If \( r \) is even (i.e. \( r = 2q \), where \( q \in \mathbb{N} \)), it is always possible to build an \( r \)–regular graph over \( n \) nodes, where \( n \geq r + 1 \).
Proof. Let $G_k = (V, E)$ be a $2q$–regular graph over $k$ nodes where $k \geq 2q + 1$. In order to add a new node $v'$ into $V$, first find a matching $^1 m$ of $q$ edges. For every edge, $e_{ij} \in m$, do the following: (1) disconnect it from one of its end nodes, say $j$, and connect the free end to $v'$, thus effecting the edge $e_{iv'}$. (2) add an edge between $v'$ and $j$, $e_{v'j}$. Note that at the end of step (2), the degrees of every node pair $(i, j)$ remain $2q$, whereas the degree of $v'$ increases by 2. Thus, after $q$ such operations, the degree of $v'$ is $2q$, which is the same as all other nodes in the graph. Thus, we get $G_{k+1}$, a $2q$–regular graph with $k + 1$ nodes.

The crucial step in the above construction is to find a matching of size $q$. The existence of a matching is a sufficient condition for extension. The existence of a matching of size $q$ is easily proved based on the following known results: (1) a graph, $G$, with minimum degree, $\delta \geq 2$, has a cycle, $C$, of length, $|C| \geq \delta + 1$, and (2) a graph $G$ with $|C| \geq l$, $l \in \mathbb{N}$, has a matching of length at least $\left\lfloor \frac{l}{2} \right\rfloor$.

From the above results, Since a $2q$–regular graph has a matching of length at least, $q$, we can extend it indefinitely by 1 node. Hence the theorem.

Figure 7.2 illustrates Theorem 1 being applied on a 4–regular graph with 5 nodes. It is extended by one node by rearranging two edges and adding two more edges. Another way of looking at the above algorithm is this: for each edge in matching $m$, “break” the edge in the middle, and connect the new node between the dangling ends of the edge. At the end of this process, we get a regular graph with a new node.

The extension algorithm essentially needs to find a cycle $C$, whose length is

$^1$A matching on a graph $G$ is a subset, $m$, of $E$ such that no two edges in $m$ share a vertex.
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|C| \geq 2q (such a cycle has two matchings each of size at least q). This can be done by a modified version of the standard depth first search (DFS) algorithm. Thus the complexity of the above algorithm is, \(O(|V| + |E|)\).

Extension of regular graphs has implications in network design. For example, in DHT design it is quite common to find questions of being able to extend a DHT in the face of node arrivals. We can see that it is possible to extend DHT topologies while keeping it regular.

For odd regularities, the technique used in Theorem 1 does not work. We need to extend an odd regular graph by two nodes at a time. For this, we can extend Theorem 1 for odd regularities as shown below in Theorem 2. Figure 7.3 illustrates the extension procedure for odd regular graphs.

Figure 7.2: Extending a 5-node 4-regular graph by one node.

\begin{itemize}
  \item \textbf{Step-0}: A 4-regular graph with 5 nodes.
  \item \textbf{Step-1}: A 4-regular graph with 6 nodes.
  \item \textbf{Step-2}: A 4-regular graph with 7 nodes.
  \item \textbf{Step-3}: A 4-regular graph with 8 nodes.
  \item \textbf{Step-4}: A 4-regular graph with 9 nodes.
\end{itemize}
Theorem 2. Given an $r$–regular graph with $n \geq r + 1$ nodes where $r$ is odd (i.e. $r = 2q + 1$, where $q \in \mathbb{N}$) and $n$ is even, there exists an $r$–regular graph with $n + 2$ nodes.

Proof. Given a $(2q + 1)$–regular graph $G = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = k$, where $k$ is even, we show how to extend it by adding two nodes $v'$ and $v''$ respectively to $\mathcal{V}$.

Find two size $q$ matchings $m'$ and $m''$ in the graph, such that $m' \cap m'' = \emptyset$. Use $m'$ to connect to $v'$ and $m''$ to $v''$, as described previously. Now, both $v'$ and $v''$ have a degree of $2q$. Finally, add an edge between $v'$ and $v''$. Thus, we have a $(2q + 1)$–regular graph with $k + 2$ nodes.

Let us note that the union of the two matchings, $m' \cup m''$, need not be a matching. Further, we can prove that it is always possible to find matching $m'$ and $m''$, using arguments similar to the ones in Theorem 1.

Note that Theorem 2 represents the minimal extension we can make to a graph with odd regularity. If an $r$–regular graph with $n$ nodes has to exist where $r$ is odd, then $n$ should be even and can only be extended in steps of 2. Again, the above
algorithm needs to find a cycle $C$ of length, $|C| \geq 2q$. Thus, the complexity is, $O(|V| + |E|)$.

**Extending regular graphs by “merging”:** While the minimal extension theorems show theoretical underpinnings for extending regular graphs, in practice it is often desirable to build regular graphs much faster than extending them by the minimal extension possible. For example, if there are a large number of nodes over which a DHT has to be constructed, it is often desirable to start the DHT construction process parallely involving several subsets of nodes. These subsets can then merge into one another to finally result in a single DHT over the entire network. An example of this is GHT [Ratnasamy et al., 2003], a geographic hash table implemented over wireless sensor networks, where locality plays a major role in the DHT performance. It is easier for local nodes to form networks amongst themselves and then merge into a global network. Similarly, merging is also necessary when a network has to repair itself from a partitioning. Here, we address merging of regular graphs without losing the regularity property.

**Theorem 3.** Given any $r$–regular graph with $n$ nodes and $r \geq 2$, there exists an $r$–regular graph of $n + r - 1$ nodes.

**Proof.** We prove this by merging an $(r-2)$–regular clique with an $r$–regular graph. Note that an $(r-2)$–regular clique has $r-1$ nodes and is fully connected. So the combined graph would have $n + r - 1$ nodes.

Let $G^r$ be the given graph and $C^{r-2}$ be the $(r-2)$–regular clique to be merged with $G^r$. In order to perform the merging, take each node $v_i$ of $C^{r-2}$ in sequence.
For each such $v_i$ pick an arbitrary edge from $G^r$ and insert $v_i$ in the “middle” of the edge. We now note that $v_i$, which already had degree $r - 2$, now has degree $r$ after inserting it in the middle of an existing edge. Once all nodes from $C^{r-2}$ are inserted, we get one combined graph with regularity $r$.

However, we need to address a caveat here. Suppose there is an edge $(u, v) \in G^r$. This edge can take insertions from nodes $v_i, v_j \in C^{r-2}$ iff $(v_i, v_j)$ is not an edge in $C^{r-2}$. Failing which, we would have multiple edges between $v_i$ and $v_j$ in the combined graph.

In order to ensure that the formation of multi-graphs can always be prevented, $G^r$ needs to have at least as many edges as the number of nodes in $C^{r-2}$. Since $C^{r-2}$ is an $(r-2)$–regular clique, it has $r - 1$ nodes. If the number of nodes in $G^r$ is $n$, then it has $\frac{nr}{2}$ edges. Since $n$ has a lower bound of $(r + 1)$ (Postulate 3), the
inequality $\frac{mr}{2} > r - 1$ is satisfied trivially.

We can extend Theorem 3 by allowing the merging of any $(r - 2)$–regular graph (not necessarily a clique) with an $r$–regular graph as long as the number of nodes in the former is no more than the number of edges in the latter. It is possible to merge graphs with a larger number of nodes than that specified above, but the maximum node constraint is a safe upper bound below which, merging is always possible.

### 7.3 Regularity, Connectivity and Hamiltonicity

The *edge-connectivity* of a graph is the smallest number of edges (or the edge cutset) whose removal would partition the graph. Edge connectivity has direct correspondence with the resilience of a network in the face of connection failures. In the following sections, when we refer to the connectivity of a graph, we are referring to edge-connectivity only. We shall not be addressing the counterpart of node-connectivity.

Menger’s theorem states that the connectivity of a graph is equal to the maximum number of pairwise independent paths between any two nodes in the graph. Also, as we had noted earlier, the presence of a hamiltonian circuit guarantees connectivity of at least 2. In this section, we discuss some observations relating regularity, connectivity and hamiltonicity of graphs.
7.3.1 Regularity and Connectivity

We observed earlier that a regularity of $r$ need not mean $r$–connectivity (Figure 7.1). We can find graphs that are $r$–regular but $k$–connected where $k < r$.

**Postulate 5.** The maximal connectivity of an $r$–regular graph is $r$. This follows from the following two facts: (1) the maximal connectivity of a graph is equal to the minimal degree in the graph [Harary, 1962], and (2) in a regular graph all nodes have the same degree equal to the regularity; hence the minimal degree is the regularity.

**Postulate 6.** A connected 2–regular graph is exactly 2–connected. This follows from the fact that all connected 2–regular graphs are circles, which are 2–connected.

**Theorem 4.** Given an $r$–regular, $k$–connected graph ($k \leq r$) over $n$ nodes with the following constraints: $r$ is even (i.e. $r = 2q$, where $q \in \mathbb{N}$) and $n \geq 2r$ (i.e. $n \geq 4q$); it is always possible to extend this graph by any number of nodes without altering the regularity $r$ or edge connectivity $k$.

**Proof.** A $k$–connected, $2q$–regular graph $G^{k,2q}$ is one that is $2q$–regular and there exists at least one set of $k$ edges, whose removal partitions the graph into two disconnected components.

Let the graph be represented as two components $C_1$ and $C_2$ connected with $k$ bridges across them. From the extension theorem (Theorem 1), we can extend either $C_1$ or $C_2$ arbitrarily without adding any edges across $C_1$ and $C_2$. This keeps the graph $2q$–regular and $k$–connected after extension.
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However, for the extension theorem to work for any component $C_i$, we see that $C_i$ should have at least $2q$ nodes. This brings us to the constraint that the number of nodes $n \geq 4q$. Given this, at least one of $C_1$ or $C_2$ is guaranteed to have at least $2q$ nodes and can be arbitrarily extended.

![Figure 7.5: Extension without changing the connectivity](image)

Since the smallest number of nodes, $n$, to create a $2q$–regular graph is only $2q + 1$, it is interesting to explore whether we can extend $k$–connected, $2q$–regular graphs having $n < 4q$ nodes. As of now, we do not have any proof or refutation for this conjecture.

The significance of the above theorem for network design is as follows. Often, it is cheaper to build $r$–regular graphs with a connectivity less than $r$. For example, in NCW, while it is ideal to have the maximum possible connectivity $r$, it might not be cost effective; hence, building/extending a network with connectivity $k < r$ might be an useful alternative. This is also useful when a distributed
system is modelled as a metric space and there is a distance function or a cost associated with each edge that is proportional to the (physical or logical) distance between the incident nodes. In such situations, it is easier to maintain \( r \)-regularity using short-range connections, rather than connecting nodes in far away corners of the network. The connectivity, \( k \), which even though is less than \( r \), may be the minimal connectivity guarantee that the design offers for such a network.

The extension theorem for \( k \)-connected regular graphs is significant in this respect, where connectivity can be maintained (or at least is not compromised) when extending the graph. The extension theorem for \( k \)-connected graphs can be extended to odd regularity in an analogous fashion as follows.

**Theorem 5.** Given an \( r \)-regular, \( k \)-connected graph, where \( r \) is odd, \( n \geq 2r \) and \( n \) is even, there exists an \( r \)-regular, \( k \)-connected graph with \( n + 2 \) nodes.

**Proof.** The proof is similar to the one above, except that in this case we consider the extension theorem for odd regularity (Theorem 2).

Both the above extension theorems require the number of nodes, \( n \), to be at least twice the regularity, \( r \). This value is much higher than the minimum required value of \( r + 1 \) for forming an \( r \)-regular graph. However, this is not a limiting constraint since in most real world networks \( n \gg r \).

We see that the smallest connected \( r \)-regular graph, which is a clique of \( r + 1 \) nodes is exactly \( r \)-connected. If the connectivity has to decrease, it means more nodes need to be added to this graph increasing the value of \( n \).

This brings us to the conjecture that given a value of \( r \) there is a maximum
value of \( n \) below which a regular graph of \( r \)-regularity has to be at least \( k \)-connected. It would be interesting to find this threshold. As of now we do not have any proof or refutation for this conjecture. [Imase et al., 1985] have developed some interesting results in this connection by studying regular directed graphs. They derive relations linking the number of nodes, \( n \), regularity, \( r \), diameter, \( d \), and edge connectivity, \( \lambda \). They show, for example, that the edge connectivity, \( \lambda \) is equal to \( r \) in an \( r \)-regular graph if \( n \geq r^{d-1} + d^2 - 2 \). They also develop similar results for connectivity less than \( r \) given a diameter \( d \).

![Figure 7.6: Extension while maintaining connectivity](image)

A side effect of Theorems 4 and 5 is also a guarantee that we can always build and extend \( r \)-regular graphs that are \( r \)-connected, whenever \( r \) is even. This is explained by the following theorem.

**Theorem 6.** Starting from an \( r \)-regular, \( r \)-connected graph, it is always possible to build an \( r \)-regular, \( r \)-connected graphs over any \( n \), when \( r \) is even (i.e., \( r = 2q \), where \( q \in \mathbb{N} \)).

**Proof.** Note that this theorem does not trivially entail from Theorem 4 for \( k \)-
connected graphs due to the constraint there on the number of nodes, \( n \geq 4q \). This is because, the smallest \( 2q \)-regular, \( 2q \)-connected graph is a clique with \( 2q + 1 \) nodes. Therefore, here, \( n \geq 2q + 1 \).

We prove the theorem using the extension theorem for even regular graphs as follows. Consider the way extension is done. When a new node needs to be added to the graph, first, \( q \) edges in the existing graph are “broken”. To guarantee extensibility, this is done after ensuring that a matching of \( q \) edges is available.

Therefore, each of the \( q \) pairs of nodes now have one path less between them. As a result, the connectivity of the graph reduces to \( 2q - 1 \) (or \( r - 1 \)). Next, the \( 2q \) dangling ends of the broken edges are connected to the new node to obtain the extended graph of regularity \( 2q \). Now, each of the \( q \) pairs of nodes have a new path between them through the newly added node, thus restoring their pairwise \( 2q \) (or \( r \)) edge independence. Also, the newly added node is connected to \( 2q \) distinct nodes in the graph, each of which has \( 2q \) edge independent paths to all nodes. Thus, the connectivity of the graph remains \( 2q \) (or \( r \)). Hence the result follows.

Note that in this case of \( r \)-connectivity, there is no constraint on the number of nodes \( n \) to be at least \( 2r \). We can extend this result to odd regular graphs analogously.

**Theorem 7.** Starting from an \( r \)-regular, \( r \)-connected graph, it is always possible to build \( r \)-regular, \( r \)-connected graphs over any \( n+2 \) using the extension theorem.

**Proof.** The proof is similar to the above proof, except that we consider the exten-
Hence, when connectivity is \( r \), extension without compromising on the connectivity is always possible. However, for regular graphs where the connectivity is less than \( r \), there is a constraint on the minimum number of nodes in order to guarantee extensibility.

Connectivity of regular graphs also has important applications in consensus problems in distributed systems, where nodes are allowed to fail in a Byzantine fashion (failure due to arbitrariness in behavior) [Lamport et al., 1982].

### 7.3.2 Regularity and Hamiltonicity

We have similar observations with respect to regularity and hamiltonicity; and the absence of it.

In an undirected graph of \( n \) nodes, the presence of a hamiltonian circuit bounds the diameter of the graph to \( \left\lceil \frac{n}{2} \right\rceil \). It also guarantees 2–connectivity. For directed graphs, a hamiltonian circuit is the smallest regular graph (with one directed edge per node) that keeps the graph strongly connected. Hamiltonicity hence plays an important role in network design.

[Bermond et al., 1998] study hamiltonian decompositions in wrapped butterfly networks (WBNs). WBNs comprise of several edge-disjoint hamiltonian decompositions. Hamiltonian networks are advantageous for algorithms that make use of the ring structure (many DHT algorithms do this). A hamiltonian decomposition allows the load to be equally distributed, also making the network robust.
Similarly, topologies based on hypercubes of dimension $2d$ can be decomposed into $d$ hamiltonians.

[Gould, 1991] provides a detailed survey of the hamiltonian cycle problem and reports several sufficiency conditions for hamiltonicity. Given a set of $n$ nodes and a regularity number $r$, it is quite natural to expect the question whether it is possible to construct a hamiltonian $r$–regular graph over these $n$ nodes. To answer this, we start with the following observation.

**Postulate 7.** The smallest $r$–regular graph, i.e. a clique of $n = r + 1$ nodes is hamiltonian.

**Theorem 8.** Starting from an $r$–regular hamiltonian graph, it is always possible to build an $r$–regular graph over $n$ nodes, while preserving hamiltonicity, when $r$ is even and $n \geq r + 1$.

*Proof.* Let $G$ be an $r$–regular hamiltonian, and let $h = v_1, v_2, \ldots v_n, v_1$ be a hamiltonian cycle in $G$.

Now extend $G$ using the extension theorem for even graphs (Theorem 1) as follows. While extending $G$, using a matching $m$, let at least one edge $(v_i, v_j)$, which is in $h$, be in $m$. After extension, since $v'$ will be incident on two edges in $h$, it will now be a part of the extended hamiltonian cycle, $h + v'$. We have thus successfully inserted an extra node into the hamiltonian cycle and retained the hamiltonicity property in the new graph. 

We can analogously prove the corresponding result for odd regular graphs using the extension theorem for odd regularity (Theorem 2).
Theorem 9. It is always possible to build a hamiltonian $r$-regular graph over $n$ nodes, where $r$ is odd, $n$ is even and $n \geq r + 1$.

While hamiltonicity is a desirable property in applications such as DHT and NCW, there are applications where not having a hamiltonian circuit is desirable. An example is of hybrid networks that have a mix of fixed and mobile components. For such networks it is desirable to not have a single overarching data structure over the entire network. It may be desirable to model a DHT for hybrid networks as a set of distinct components connected by bridges, with an appropriate key distribution mechanism. Changes to a DHT structure in the face of churn is usually expensive. In hybrid networks, it is desirable to isolate the mobile part of the network from the fixed part. All these may require to be performed in a way that does not affect the symmetry property of the DHT. In such cases the absence of hamiltonicity becomes an important property.

For addressing such issues, we begin with the following observation.

Postulate 8. It is not possible to have a connected 2–regular graph that is non-hamiltonian. In a 2–regular graph, all nodes have a degree 2. Unless there are an infinite number of nodes, it is not possible to have anything other than a logical cycle encompassing all the nodes, to build a connected 2–regular graph.

Corollary 1. Given an $r$–regular, non-hamiltonian graph with $n$ nodes, where $r$ is even and $n \geq 2r$, it is always possible to extend this graph over any number of nodes and retain the regularity and non-hamiltonian properties.

Proof. The proof for this can be seen as a corollary to Theorem 4. In a non-
hamiltonian graph, there exists at least one edge, \( e' \), that needs to be traversed twice before completing a circuit that encompasses all the nodes in the graph.

Even though the removal of this edge does not partition the graph, we can consider this edge to divide the graph into two parts \( C_1 \) and \( C_2 \), where \( C_1 \) is the part that was traversed before the first traversal of \( e' \) combined with the part traversed after the second traversal; and \( C_2 \) is the part traversed in between the first traversal and the second.

We can readily see that \( C_2 \) is a connected component and we can also show that \( C_1 \) is a connected component since it has at least one node of the bridge in common between the first and third traversals. Once we show that \( e' \) logically divides the graph into two, with at least one part containing at least \( r \) nodes, we can use Theorem 4 to extend the graph.

We can analogously extend this to graphs with odd regularity.

**Corollary 2.** Given an \( r \)-regular, non-hamiltonian graph over \( n \) nodes, where \( r \) is odd, \( n \) is even, and \( n \geq 2r \), there exists an \( r \)-regular, non-hamiltonian graph with \( n + 2 \) nodes.

### 7.4 Regular Directed Graphs

Until now, we have only addressed regular graphs that are undirected. Directed graphs involve several issues like differences between indegree and outdegree and strong connectivity that make them harder to analyze than undirected graphs.
Since many networks are modelled naturally as directed graphs, we briefly explore the symmetry property for directed graphs in this section.

A directed graph is \( r \)-inregular if all nodes have an indegree of \( r \); similarly, a directed graph is \( r \)-outregular if all nodes have an outdegree of \( r \). Finally, if a directed graph is \( r \)-regular if it is both \( r \)-inregular and \( r \)-outregular.

An undirected graph can be rewritten as a directed graph by representing each undirected edge as a pair of directed edges in opposite directions. Given this, we face the question of whether designing a diameter optimal \( r \)-regular directed graph necessarily entails representing directed edges in pairs, making the graph into an undirected graph.

Figure 7.7 refutes such an assertion. Given 8 nodes and 8 undirected edges (or 16 pairs of directed edges), the best we can do for regularity-preserving diameter reduction, is to form a 2-regular hamiltonian cycle having a diameter 4. This is shown in Figure 7.7(a). However, when the pairwise coupling of the directed edges are broken, we can obtain a directed 2-regular graph with diameter 3, as shown in Figure 7.7(b).
Further, we can make the following observations over directed \( r \)-regular graphs.

**Postulate 9.** The smallest number of nodes required for a strongly connected, directed \( r \)-regular graph is \( r+1 \). Such a graph is in the form of a clique, where every node has \( r \) outgoing edges connecting every other node in the graph, directly.

**Theorem 10.** An \( r \)-regular directed graph over \( n \) nodes, where \( n \geq r + 1 \), can be extended by a single node, regardless of whether the regularity is odd or even.

**Proof.** The extension theorem for directed graphs is similar to Theorem 2 concerning odd regular undirected graphs. Consider an \( r \)-regular directed graph \( G \) and an incoming node \( v \). We can think of \( v \) as having \( r \) holes for incoming connections and \( r \) dangling outgoing edges looking for connections. We now break \( r \) edges from \( G \) and connect them to \( v \), such that they create holes on \( r \) distinct nodes in \( G \). The \( r \) outgoing edges from \( v \) are then connected to these holes.

Note that any node in \( G \) can host both an outgoing connection, as well as an incoming connection from \( v \). This requires a maximum of only \( r \) nodes in \( G \), which is satisfied trivially.

In directed regular graphs, we do not face incompatibility between regularity and number of nodes. If the graph is \( r \)-regular in both indegree and outdegree, the total degree of every node is always even. This makes it possible to have \( r \)-regular directed graphs over any number of nodes, as long as the number of nodes are at least \( r + 1 \).

The connectivity of a directed graph is the size of the smallest edge cutset, whose removal will no longer make the graph strongly connected. Given this, we
can prove results analogous to Theorem 4 for directed graphs as well. Similarly, hamiltonian circuits in directed graphs can be extended in the same spirit as of Theorem 8.