In this chapter we present the background needed for the study of multi-fuzzy sets. We develop the theory of multi-fuzzy sets on the platform of fuzzy set theory. The knowledge about the developments of fuzzy sets is sufficient as a prerequisite and so we recall some basic definitions and results consisting of fuzzy sets, intuitionistic fuzzy sets, $L$-fuzzy sets, order homomorphisms, lattice valued lattices, fuzzy topology, fuzzy groups and fuzzy logic.

1.1 Fuzzy Sets

In Cantor’s set theory, a set is defined uniquely by its elements; an element of the universe is either in or outside the set. That is, the membership function of a set (crisp set) assigns a value of either 1 or 0 to each element in the universe. Zadeh \cite{107} extended the range of membership functions into the closed interval $[0, 1]$.

**Definition 1.1.1.** \cite{107} Let $X$ be a nonempty set. A fuzzy set $A$ of $X$ is a mapping

$$A : X \rightarrow [0, 1]$$
A : X → [0, 1], that is,

\[ A = \{(x, \mu_A(x)) : \mu_A(x) \text{ is the grade of membership of } x \text{ in } A, x \in X\}. \]

The set of all the fuzzy sets on \( X \) is denoted by \( \mathcal{F}(X) \).

Let \( A \) and \( B \) be fuzzy sets on a universal set \( X \), with the grade of membership of \( x \) in \( A \) and \( B \) denoted by \( \mu_A \) and \( \mu_B \) respectively. Zadeh \[107\] defined the following relations and operations:

- \( A = B \iff \mu_A(x) = \mu_B(x), \forall x \in X; \)
- \( A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \forall x \in X; \)
- \( \mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \forall x \in X; \)
- \( \mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X; \)
- \( \mu_A'(x) = 1 - \mu_A(x), \forall x \in X, \) where \( A' \) is the standard fuzzy complement of \( A \).

**Definition 1.1.2.** \[49\] A function \( t : [0, 1] \times [0, 1] \to [0, 1] \) is a \( t \)-norm if \( \forall a, b, c \in [0, 1]: \)

1. \( t(a, 1) = a; \)
2. \( t(a, b) = t(b, a); \)
3. \( t(a, t(b, c)) = t(t(a, b), c); \)
4. \( b \leq c \) implies \( t(a, b) \leq t(a, c). \)

Similarly, a \( t \)-conorm (\( s \)-norm) is a commutative, associative and non-decreasing mapping \( s : [0, 1] \times [0, 1] \to [0, 1] \) that satisfies the boundary condition:
\[ s(a, 0) = a, \text{ for all } a \in [0, 1]. \]

**Definition 1.1.3.**\(^3\)\(^8\) A function \( c : [0, 1] \to [0, 1] \) is called a complement (fuzzy) operation, if it satisfies the following conditions:

1. \( c(0) = 1 \) and \( c(1) = 0 \),

2. for all \( a, b \in [0, 1] \), if \( a \leq b \), then \( c(a) \geq c(b) \).

**Definition 1.1.4.**\(^3\(^8\) A \( t \)-norm \( t \) and a \( t \)-conorm \( s \) are dual with respect to a fuzzy complement operation \( c \) if and only if

\[
\begin{align*}
   c(t(a, b)) &= s(c(a), c(b)) \\
   \text{and} \\
   c(s(a, b)) &= t(c(a), c(b)),
\end{align*}
\]

for all \( a, b \in [0, 1] \).

**Definition 1.1.5.**\(^3\(^8\) Let \( n \) be an integer greater than or equal to 2. A function \( h : [0, 1]^n \to [0, 1] \) is said to be an aggregation operation for fuzzy sets, if it satisfies the following conditions:

1. \( h \) is continuous;

2. \( h \) is monotonic increasing in all its arguments;

3. \( h(0, 0, ..., 0) = 0 \);

4. \( h(1, 1, ..., 1) = 1 \).

**1.1.1 Zadeh’s Extension of Functions**

For the sake of simplicity we will use \( A(x) \) and \( f(A)(y) \) instead of \( \mu_A(x) \) and \( \mu_{f(A)}(y) \) respectively.
Definition 1.1.6. Let \( f : X \to Y \) be a crisp function. The fuzzy extension of \( f \) and the inverse of the extension are \( f : \mathcal{F}(X) \to \mathcal{F}(Y) \) and \( f^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X) \) defined by

\[
f(A)(y) = \bigvee_{y=f(x)} A(x), \ A \in \mathcal{F}(X), \ y \in Y
\]

and

\[
f^{-1}(B)(x) = B(f(x)), \ B \in \mathcal{F}(Y), \ x \in X.
\]

Theorem 1.1.7. \[14\] Let \( f \) be a function from \( X \) to \( Y \), then:

1. \( (f^{-1}(B))' = f^{-1}(B') \), for any fuzzy set \( B \) in \( Y \);
2. \( (f(A))' \subseteq f(A') \), for any fuzzy set \( A \) in \( X \);
3. \( B_1 \subseteq B_2 \) implies \( f^{-1}(B_1) \subseteq f^{-1}(B_2) \), where \( B_1, B_2 \) are fuzzy sets in \( Y \);
4. \( A_1 \subseteq A_2 \) implies \( f(A_1) \subseteq f(A_2) \), where \( A_1, A_2 \) are fuzzy sets in \( X \);
5. \( A \subseteq f^{-1}(f(A)) \), for any fuzzy set \( A \) in \( X \);
6. \( f(f^{-1}(B)) \subseteq B \), for any fuzzy set \( B \) in \( Y \).

1.2 Lattices and Lattice Valued Mappings

One of the important concepts in all of mathematics is that of a relation. Among various kinds of relations, equivalence relations, functions and order relations have major role in our study. Here we concentrate on the latter concept.

Definition 1.2.1. (See [8]) A partially ordered set (or poset) is a set in which a binary relation \( x \leq y \) is defined, which satisfies for all \( x, y, z \) the following conditions:

P1. For all \( x \), \( x \leq x \). (Reflexivity)
1.2. LATTICES AND LATTICE VALUED MAPPINGS

P2. If \( x \leq y \) and \( y \leq x \), then \( x = y \).  \(\text{(Antisymmetry)}\)

P3. If \( x \leq y \) and \( y \leq z \), then \( x \leq z \).  \(\text{(Transitivity)}\)

Definition 1.2.2. (See [8]) A mapping \( f \) from a poset \( P \) into a poset \( Q \) is called order preserving, if \( x \leq y \) implies \( f(x) \leq f(y) \). A mapping \( g \) from \( P \) into \( Q \) is called an order reversing function (antitone) if and only if \( x \leq y \) implies \( g(y) \leq g(x) \).

Definition 1.2.3. (See [8]) A lattice is a partially ordered set in which \( x \land y = \inf(x, y) \) and \( x \lor y = \sup(x, y) \) exist for any pair of elements \( x \) and \( y \). A sublattice of a lattice \( L \) is a subset \( X \) of \( L \) such that \( a, b \in X \) implies \( a \land b \in X \) and \( a \lor b \in X \). A lattice \( L \) is complete when each of its subsets \( X \) has a l.u.b (sup) and a g.l.b (inf) in \( L \). A lattice \( L \) is said to be distributive, if  
\[
\begin{align*}
    x \land (y \lor z) &= (x \land y) \lor (x \land z) \\
    x \lor (y \land z) &= (x \lor y) \land (x \lor z),
\end{align*}
\]
for every \( x, y, z \in L \).

A lattice \( L \) is said to have a lower bound \( 0_L \), if \( 0_L \leq x, \forall x \in L \). Analogously, \( L \) is said to have an upper bound \( 1_L \), if \( x \leq 1_L, \forall x \in L \). We say \( L \) is bounded, if \( L \) has both a lower bound \( 0_L \) and an upper bound \( 1_L \). In such a lattice we have the identities \( 0_L \land x = 0_L, 0_L \lor x = x, 1_L \land x = x \) and \( 1_L \lor x = 1_L \). Any finite lattice is bounded as well as complete. An element \( a \in L \) is called a complement of an element \( b \in L \), if \( a \land b = 0_L \) and \( a \lor b = 1_L \). A lattice \( L \) is said to be complemented if \( L \) is bounded and every element in \( L \) has a complement. In a bounded distributive lattice, complements are unique, if they exist.

Definition 1.2.4. (See [104]) A complete lattice \( L \) is called infinitely distributive, if it satisfies the conditions:
\[
a \land \bigvee_{b \in B} B = \bigvee_{b \in B} (a \land b) \quad \text{and} \quad a \lor \bigwedge_{b \in B} B = \bigwedge_{b \in B} (a \lor b), \forall a \in L, \forall B \subseteq L.
\]

Proposition 1.2.5. (See [104]) A complete lattice \( L \) is infinitely distributive if and
only if
\[ \bigvee A \land \bigvee B = \bigvee_{a \in A, b \in B} (a \land b) \quad \text{and} \quad \bigwedge A \lor \bigwedge B = \bigwedge_{a \in A, b \in B} (a \lor b), \forall A, B \subseteq L. \]

**Proposition 1.2.6.** (See [8]) In any poset \( P \), the operations of meet and join satisfy the following laws, whenever the expressions exist:

**L1.** \( x \land x = x, \ x \lor x = x. \) (Idempotent)

**L2.** \( x \land y = y \land x, \ x \lor y = y \lor x. \) (Commutative)

**L3.** \( x \land (y \land z) = (x \land y) \land z, \ x \lor (y \lor z) = (x \lor y) \lor z. \) (Associative)

**L4.** \( x \land (x \lor y) = x \lor (x \land y) = x. \) (Absorption)

Moreover \( x \leq y \) is equivalent to each of the conditions:

\( x \land y = x \) and \( x \lor y = y. \) (Consistency)

**Note 1.2.7.** (See [8]) A semilattice is a set \( L \) with a binary operation \( ' \ast ' \) which is idempotent, commutative and associative. Let \( P \) be any poset in which any two elements have a meet. Then \( P \) is a semilattice with respect to the binary operation \( \land \). Such semilattices are called meet-semilattices. Join-semilattices are defined in a similar manner. Any system \( L \) with two binary operations which satisfy the conditions L1, L2, L3 and L4 is a lattice, and conversely.

### 1.2.1 Operations on Lattices

**Definition 1.2.8.** (See [104]) If \( \{ L_j : j \in J \} \) is a family of lattices, then the product \( \prod_{j \in J} L_j \) is a lattice if for arbitrary \( x, y \in \prod_{j \in J} L_j \), the join \( x \lor y \) and the meet \( x \land y \) of \( x, y \) are defined as for every \( j \in J \) and for every \( x_j, y_j \in L_j \):

\[ (x \lor y)_j = x_j \lor y_j \]
and

$$(x \land y)_j = x_j \land y_j$$

or, equivalently, $x \leq y$ is defined by

$$x_j \leq_j y_j, \forall j \in J,$$

where $\leq$ and $\leq_j$ are the order relations in $\prod_{j \in J} L_j$ and $L_j$ respectively.

**Proposition 1.2.9.** (See [104]) Let $\{L_j : j \in J\}$ be a family of posets. Then:

1. $\prod_{j \in J} L_j$ is a poset if and only if $\forall j \in J$, $L_j$ is a poset;
2. $\prod_{j \in J} L_j$ is a lattice if and only if $\forall j \in J$, $L_j$ is a lattice;
3. $\prod_{j \in J} L_j$ is a complete lattice if and only if $\forall j \in J$, $L_j$ is a complete lattice;
4. $\prod_{j \in J} L_j$ is a distributive lattice if and only if $\forall j \in J$, $L_j$ is a distributive lattice;
5. $\prod_{j \in J} L_j$ is an infinitely distributive lattice if and only if $\forall j \in J$, $L_j$ is an infinitely distributive lattice.

**Definition 1.2.10.** (See [8]) Let $\theta: L \to M$ be a function from a lattice $L$ to a lattice $M$. Then $\theta$ is order preserving (isotone) when $x \leq y$ implies $\theta(x) \leq \theta(y)$; a join-morphism (join homomorphism) when

$$\theta(x \lor y) = \theta(x) \lor \theta(y) \text{ for all } x, y \in L.$$ (1.i)

and a meet-morphism (meet homomorphism) when

$$\theta(x \land y) = \theta(x) \land \theta(y) \text{ for all } x, y \in L.$$ (1.ii)

$\theta$ is a lattice morphism (lattice homomorphism) when (1.i) and (1.ii) hold.
A lattice homomorphism is called: (i) an isomorphism if it is a bijection, (ii) an epimorphism if it is onto, (iii) a monomorphism if it is one-one, (iv) an endomorphism if \( L = M \), (v) an automorphism if it is an isomorphism and \( L = M \).

**Definition 1.2.11.** (See [104]) Let \( L \) and \( M \) be complete lattices and \( h : L \to M \) be a mapping. The map \( h \) is called a complete join preserving or arbitrary join preserving map, if for any \( A \subseteq L \)

\[
h(\bigvee A) = \bigvee_{x \in A} h(x); \quad (1.iii)
\]
a complete meet preserving or arbitrary meet preserving map if for any \( A \subseteq L \),

\[
h(\bigwedge A) = \bigwedge_{x \in A} h(x). \quad (1.iv)
\]

The map \( h \) is a complete lattice homomorphism when (1.iii) and (1.iv) hold.

**Definition 1.2.12.** (See [104]) Let \( L \) be a lattice. A mapping \( \cdot' : L \to L \) is called an order reversing involution, if for all \( a, b \in L \):

1. \( a \leq b \Rightarrow b' \leq a' \);
2. \( (a')' = a \).

The symbols \( \cdot' \) and \( \cdot' \) are used in this thesis for order reversing involutions.

**Definition 1.2.13.** [89] Let \( \cdot' : M \to M \) and \( \cdot' : L \to L \) be order reversing involutions. A mapping \( h : M \to L \) is called an order homomorphism, if it satisfies the conditions:

1. \( h(0_M) = 0_L \);
2. \( h(\bigvee a_i) = \bigvee h(a_i) \);
3. \( h^{-1}(b') = (h^{-1}(b))' \),

where \( h^{-1} : L \to M \) is defined by, for every \( b \in L \),

\[
h^{-1}(b) = \bigvee \{ a \in M : h(a) \leq b \}.
\]
Proposition 1.2.14. \(89\) If \(': \, M \to M\) and \(': \, L \to L\) are order reversing involutions and \(h : \, M \to L\) is an order homomorphism, then for every \(a, a_i \in M\) and \(b, b_i \in L\)

(1) \(h^{-1}(0_L) = 0_M\);

(2) \(h^{-1}(1_L) = 1_M\);

(3) \(a_1 \leq a_2\) implies \(h(a_1) \leq h(a_2)\), that is, \(h\) is an order preserving map;

(4) \(b_1 \leq b_2\) implies \(h^{-1}(b_1) \leq h^{-1}(b_2)\), that is, \(h^{-1}\) is an order preserving map;

(5) \(a \leq h^{-1}(b)\) if and only if \(h(a) \leq b\) if and only if \(h^{-1}(b') \leq a'\);

(6) \(h^{-1}(\lor b_i) = \lor h^{-1}(b_i)\), that is, \(h^{-1}\) is an arbitrary join preserving map;

(7) \(h^{-1}(\land b_i) = \land h^{-1}(b_i)\), that is, \(h^{-1}\) is an arbitrary meet preserving map;

(8) \(a \leq h^{-1}(h(a))\);

(9) \(h(h^{-1}(b)) \leq b\).

Proposition 1.2.15. \(103\) Let \(f : \, L_1 \to L_2\) be a union (join) preserving map. If \(f\) is injective, then

\[
f^{-1}(f(a)) = a, \forall a \in L_1
\]

and if \(f\) is surjective, then

\[
f(f^{-1}(b)) = b, \forall b \in L_2.
\]

1.2.2 L-fuzzy Sets

Definition 1.2.16. \(27\) Let \(X\) be a nonempty ordinary set and \(L\) be a partially ordered set. An \(L\)-fuzzy set on \(X\) is a mapping \(A : \, X \to L\), that is, the family of all the \(L\)-fuzzy sets on \(X\) is just \(L^X\) consisting of all the mappings from \(X\) to \(L\).
Equality of $L$-fuzzy sets and inclusion of $L$-fuzzy sets are defined in similar to the respective relations on fuzzy sets. For any $A, B \in L^X$, the membership functions of $A \cup B$ and $A \cap B$ are defined as follows:

\[
\mu_{A \cup B}(x) = \mu_A(x) \lor \mu_B(x) \quad \text{and} \\
\mu_{A \cap B}(x) = \mu_A(x) \land \mu_B(x), \quad \text{for all } x \in X.
\]

**Definition 1.2.17.** [38][12] Let $X$ be a nonempty ordinary set, $L$ be a complete lattice, $\alpha \in L$ and $A \in L^X$. An $\alpha$-level set (or $\alpha$-cut) of a fuzzy set $A$ is a crisp set

\[ A_{[\alpha]} = \{ x \in X : \alpha \leq A(x) \}. \]

$A_{\alpha}$ is also denoted by $\alpha$-level set of the fuzzy set $A$.

### 1.3 Intuitionistic Fuzzy Sets

Throughout this thesis intuitionistic fuzzy set means Atanassov intuitionistic fuzzy sets. It is a generalization of the notion of Zadeh’s fuzzy sets with the condition that the sum of degrees of membership and nonmembership is less than or equal to one.

**Definition 1.3.1.** [3] An Intuitionistic Fuzzy Set on $X$ is a set

\[ A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \}, \]

where $\mu_A(x) \in [0, 1]$ denotes the membership degree and $\nu_A(x) \in [0, 1]$ denotes the non-membership degree of $x$ in $A$ and

\[ \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X. \]
Definition 1.3.2. Let $L$ be a complete lattice with an order reversing involution $\mathcal{N} : L \to L$. An intuitionistic $L$-fuzzy set (lattice valued intuitionistic fuzzy set) is an object of the form

$$A = \{(x, \mu_1(x), \mu_2(x)) : x \in X\},$$

where $\mu_1$ and $\mu_2$ are functions $\mu_1 : X \to L$, $\mu_2 : X \to L$, such that for all $x \in X$,

$$\mu_1(x) \leq \mathcal{N}(\mu_2(x)).$$

### 1.4 Fuzzy Topology

Among various branches of Mathematics, Topology is one of the first subjects where the notion of fuzzy sets was applied. Chang [14] introduced the concept of fuzzy topology, and subsequently Lowen [45] proposed a modified definition of fuzzy topology. Wong [92], Conrad [16] and Mira [51] studied various aspects of fuzzy topology.

Definition 1.4.1. A fuzzy topology is a family $\tau$ of fuzzy sets in $X$ which satisfies the following conditions:

1. $\phi, X \in \tau$;
2. If $A, B \in \tau$, then $A \cap B \in \tau$;
3. If $A_i \in \tau$ for each $i \in I$, then $\bigcup_{i \in I} A_i \in \tau$.

Note 1.4.2. The ordered pair $(X, \tau)$ is called a fuzzy topological space (or fts for short). Fuzzy sets in $\tau$ are called $\tau$-open fuzzy sets in $X$, simply open fuzzy sets in $X$. A fuzzy set $A \in \mathcal{F}(X)$ is called $\tau$-closed if and only if its complement $A'$ is $\tau$-open. The collection of all constant fuzzy sets in $X$ is a fuzzy topology on $X$.

Definition 1.4.3. $\delta \subseteq \mathcal{F}(X)$ is a fuzzy topology on $X$ if and only if:

1. $\alpha \in \delta$, for every constant $\alpha \in \mathcal{F}(X)$;
(2) \( A \cap B \in \delta \), for every \( A, B \in \delta \);

(3) \( \bigvee_{i \in I} A_i \in \delta \), for every \( A_i \in \delta \).

**Definition 1.4.4.** [14] A fuzzy set \( U \) in a fts \((X, \tau)\) is a neighborhood of a fuzzy set \( C \) if and only if there exists an open fuzzy set \( O \) such that \( C \subseteq O \subseteq U \). Let \( A \) and \( B \) be fuzzy sets in a fts \((X, \tau)\), and let \( B \subseteq A \). Then \( B \) is called an interior fuzzy set of \( A \) if and only if \( A \) is a neighborhood of \( B \). The union of all interior fuzzy sets of \( A \) is called the interior of \( A \) and is denoted by \( A^0 \).

\( A^0 \) is open and is the largest open fuzzy set contained in \( A \). The fuzzy set \( A \) is open if and only if \( A = A^0 \) (see [14]). Closure of \( A \) is the meet of all the closed subsets containing \( A \) and is denoted by \( \overline{A} \) (see [45]).

**Definition 1.4.5.** [14] Let \( f: \mathcal{F}(X) \to \mathcal{F}(Y) \) be a fuzzy extension of \( f: X \to Y \) and \( f^{-1}: \mathcal{F}(Y) \to \mathcal{F}(X) \) be the inverse of the extension. \( f: (X, \tau) \to (Y, \rho) \) is said to be fuzzy continuous, if for each fuzzy set \( B \in \rho \), then the fuzzy set \( f^{-1}(B) \in \tau \).

**Theorem 1.4.6.** [14] Let \((X, \tau)\) and \((Y, \rho)\) be fuzzy topological spaces and let \( f \) be a function from \( X \) into \( Y \). Then, \( f \) is fuzzy continuous if and only if \( f^{-1}(C) \) is closed in \( X \), for each closed fuzzy set \( C \) in \( Y \).

**Proposition 1.4.7.** [14] If \( f: (X, \tau) \to (Y, \rho) \) and \( g: (Y, \rho) \to (Z, \delta) \) are fuzzy continuous, then \( g \circ f: (X, \tau) \to (Z, \delta) \) is fuzzy continuous.

**Definition 1.4.8.** [14] A family \( \mathcal{U} \) of fuzzy sets is a cover of a fuzzy set \( A \) if and only if \( A \subseteq \bigcup \{U: U \in \mathcal{U}\} \). It is an open cover if and only if each member of \( \mathcal{U} \) is an open fuzzy set. A subfamily of \( \mathcal{U} \) is called a subcover of \( A \), if it is an open cover of \( A \).

**Definition 1.4.9.** [14] A fuzzy topological space \((X, \tau)\) is compact if and only if each open cover has a finite subcover.

**Proposition 1.4.10.** [14][45] Let \((X, \tau)\) is compact and \( f \) a fuzzy continuous mapping from \((X, \tau)\) onto \((Y, \rho)\), then \((Y, \rho)\) is compact.
1.5 Fuzzy Algebra

Fuzzy approach to algebraic concepts started with Rosenfeld’s [71] paper on fuzzy groups. That paper led to extensive study of fuzzy subsystems of various algebraic structures. Das [17] studied the inter-relationship between the fuzzy subgroup and its $\alpha$-level subsets. Fuzzy normal subgroups were studied by Liu [44], Wu [93], and Mukherjee and Bhattacharya [57]. In this section we review some definitions and results in that theory of fuzzy algebra.

1.5.1 Fuzzy Subgroups

**Definition 1.5.1.** [71] A fuzzy set $A$ of a group $G$ is called a fuzzy subgroup of $G$ if

1. $\min\{A(x), A(y)\} \leq A(xy)$, and
2. $A(x^{-1}) \leq A(x), \forall x, y \in G$.

Combine the two conditions we can write $\min\{A(x), A(y)\} \leq A(xy^{-1}), \forall x, y \in G$. It follows immediately from this definition that $A(x) \leq A(e)$ and $A(x^{-1}) = A(x), \forall x, y \in G$, where $e$ is the identity element of $G$. A fuzzy subset $A$ of a group $G$ is called a fuzzy sub-groupoid of $G$, if $\min\{A(x), A(y)\} \leq A(xy), \forall x, y \in G$.

**Proposition 1.5.2.** [71] If $\{A_i : i \in I\}$ is a family of fuzzy subgroups of a group $G$, then $\bigcap A_i$ is a fuzzy subgroup of $G$. But the union of two fuzzy subgroups of $G$ need not be a fuzzy subgroup of $G$.

A fuzzy set $A$ in $X$ is said to have the sup property if, for any subset $S \subseteq X$, there exists $s_0 \in S$ such that $A(s_0) = \sup_{s \in S} A(s)$.

**Proposition 1.5.3.** (See [71]) Let $G_1$ and $G_2$ be groups, $f$ be a group homomorphism from $G_1$ into $G_2$. 
• Let $A$ be a fuzzy subgroup in $G_1$ that has the sup property. Then $f(A)$ is a fuzzy subgroup of $G_2$;

• Let $B$ be a fuzzy subgroup in $G_2$. Then $f^{-1}(B)$ is a fuzzy subgroup of $G_1$.

**Proposition 1.5.4.** [17] If $A$ is a fuzzy subgroup of a group $G$, then each $\alpha$-level subset $A_{[\alpha]}$ is subgroup of $G$, for $\alpha \in [0, 1]$.

**Definition 1.5.5.** [44, 57, 93] A fuzzy subgroup $A$ of a group $G$ is called a normal fuzzy subgroup if and only if $A(xy) = A(yx), \forall x, y \in G$.

**Proposition 1.5.6.** [57] If $A$ is a normal fuzzy subgroup of a group $G$, then each $\alpha$-level subgroups of $A$ is normal in $G$, for $\alpha \in [0, 1]$.

**Proposition 1.5.7.** [2] The intersection $\cap A_i$ of an arbitrary family of normal fuzzy subgroups of a group $G$ is a normal fuzzy subgroup of $G$.

### 1.5.2 Lattice Valued Lattices

**Definition 1.5.8.** [50] Let $(M, \vee_M)$ be a join-semilattice and $(L, \wedge_L, \vee_L)$ be a complete lattice with the least element $0_L$ and the greatest element $1_L$. A mapping $A : M \rightarrow L$ is called an $L$-fuzzy sub-semilattice ($L$-fuzzy semilattice) of $M$ if all the $p$-level sets ($p \in L$) of $A$ are sub-semilattices of $M$. The set of all $L$-fuzzy subsets of $M$ is denoted by $\mathcal{F}_L(M)$.

**Proposition 1.5.9.** [50] Let $(M, \vee_M)$ be a (join) semilattice and $(L, \wedge_L, \vee_L)$ be a complete lattice with the least element $0_L$ and the greatest element $1_L$. $A \in \mathcal{F}_L(M)$ is an $L$-fuzzy sub-semilattice of $M$ if and only if

$$A(x) \wedge_L A(y) \leq A(x \vee_M y), \forall x, y \in M.$$

**Definition 1.5.10.** [81] Let $(M, \wedge_M, \vee_M)$ be a lattice and $L$ be a complete lattice with the least element $0_L$ and the greatest element $1_L$. The mapping $A : M \rightarrow L$ is
called a lattice-valued fuzzy lattice ($L$-fuzzy lattice) if all the $p$-level sets ($p \in L$) of $A$ are sublattices of $M$.

**Proposition 1.5.11.** [81] Let $A : M \to L$ be an $L$-fuzzy lattice, and let $p, q \in L$. If $p \leq q$, then the $q$-level set

$$A_q = \{ x \in M : q \leq A(x) \}$$

is a sublattice of the $p$-level set

$$A_p = \{ x \in M : p \leq A(x) \}.$$  

**Proposition 1.5.12.** [81] Let $(M, \land_M, \lor_M)$ be a lattice and $(L, \land_L, \lor_L)$ a complete lattice with $0_L$ and $1_L$. Then the mapping $A : M \to L$ is an $L$-fuzzy lattice if and only if both of the following relations hold for all $x, y \in M$:

1. $A(x) \land_L A(y) \leq A(x \land_M y)$;
2. $A(x) \land_L A(y) \leq A(x \lor_M y)$.

### 1.6 Fuzzy Logic

In a classical logic system, every proposition is either true or false. That is, truth value are either 0 or 1. The classical two-valued logic can be extended into three-valued logic in various ways. A logic system having three or more truth values is called many valued logic. Formal many-valued logics, which form the basis for formal logic, were first studied by the Polish mathematician Lukasiewicz [46] in 1920. He developed a series of many-valued logical systems, from three valued to infinite-valued. Later Goguen [28] connected fuzzy sets with many-valued logic and proposed a formal fuzzy logic system. In 1998 Hájek [29] introduced an axiomatic system (Basic logic) for fuzzy logic and found out the common features of various fuzzy logics. Lukasiewicz
logic, Gödel logic (see [29]) and product logic (see [29]) are basic logics and all these
logics generalize syntax of classical logic only in adding a new connective called strong
conjunction (denoted by &).

1.6.1 Basic Logic

The basic logic BL [29] was introduced by Hájek as a family of logics including
Łukasiewicz logic, Gödel logic, Product logic etc. It has two basic binary connectives
→, & and the truth constant 0 (nullary connective). Other connectives are defined
as follows:

\[ \neg \phi \text{ is } \phi \rightarrow 0; \]
\[ \phi \land \psi \text{ is } \phi \& (\phi \rightarrow \psi); \]
\[ \phi \lor \psi \text{ is } ((\phi \rightarrow \psi) \rightarrow \psi) \land ((\psi \rightarrow \phi) \rightarrow \phi); \]
\[ \phi \equiv \psi \text{ is } (\phi \rightarrow \psi) \& (\psi \rightarrow \phi); \]
\[ \neg 0 \text{ is } \neg 0. \]

An evaluation of a propositional variable or a formula is a mapping \( e \) assigning a truth
value \( e(\phi) \in [0, 1] \) for each propositional variable (or formula) \( \phi \), with the conditions:

\[ e(0) = 0; \]
\[ e(\phi \rightarrow \psi) = e(\phi) \Rightarrow e(\psi); \]
\[ e(\phi \& \psi) = e(\phi) \otimes e(\psi). \]

**Definition 1.6.1.** [29] The basic logic BL is axiomatized by the inference rule
modus ponens (MP) and the following axioms:

(A1) \((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi));\)

(A2) \((\phi \& \psi) \rightarrow \phi;\)

(A3) \((\phi \& \psi) \rightarrow (\psi \& \phi);\)
(A4) \((\phi \& (\phi \to \psi)) \to (\psi \& (\psi \to \phi))\);

(A5a) \((\phi \to (\psi \to \chi)) \to ((\phi \& \psi) \to \chi)\);

(A5b) \(((\phi \& \psi) \to \chi) \to (\phi \to (\psi \to \chi))\);

(A6) \(((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)\);

(A7) \(\overline{0} \to \phi\).

1.6.2 Łukasiewicz Logic

Definition 1.6.2. \([46,47]\) Łukasiewicz logic \(L\) is defined by the following axioms:

(L1) \(\phi \to (\psi \to \phi)\);

(L2) \((\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))\);

(L3) \((\neg \phi \to \neg \psi) \to (\psi \to \phi)\);

(L4) \(((\phi \to \psi) \to \psi) \to (((\psi \to \phi) \to \phi)\),

and the deduction rule modus ponens, (that is, \(\phi\) and \(\phi \to \psi\) deduce \(\psi\)).

Remark 1.6.3. \([29]\) Łukasiewicz logic is an involutive (ie. \(\neg \neg \phi = \phi\)) Basic Logic. In Łukasiewicz logic, it has one and only one basic connective \(\to\) and has a truth constant 0. The negation and strong conjunction are defined as \(\neg \phi\) is \(\phi \to \overline{0}\) and \(\phi \& \psi\) is \(\neg (\phi \to \neg \psi)\).

1.6.3 Gödel Logic

Definition 1.6.4. \([23,29]\) Godel Logic \(G\) is a Basic Logic BL satisfying the condition:

(G) \(\phi \to (\phi \& \phi)\).
Remark 1.6.5. \[29\] In the Godel Logic \((\phi \land \phi) \equiv (\phi \& \phi)\) (ie. Godel logic interpreting \& as infimum \&). Axiom A2 (ie. \((\phi \& \psi) \rightarrow \phi\)) and axiom G (ie. \(\phi \rightarrow (\phi \& \phi)\)) together implies \& is idempotent. In the Godel logic we can replace axiom A4 as

\[
(\phi \land (\phi \rightarrow \psi)) \rightarrow (\psi \land (\psi \rightarrow \phi)).
\]

Also we have the identity

\[
(\phi \land \psi) \equiv (\phi \land (\phi \rightarrow \psi)).
\]

1.6.4 Product Logic

Definition 1.6.6. \[29\] Product Logic \(\Pi\) is a Basic Logic BL satisfying the conditions:

\((\Pi_1) \neg \neg \chi \rightarrow ((\phi \odot \chi \rightarrow \psi \odot \chi) \rightarrow (\phi \rightarrow \psi));\)

\((\Pi_2) \phi \land \neg \phi \rightarrow 0,\) where \(\odot\) is the product conjunction.

1.6.5 Algebras of Logic

Definition 1.6.7. \[20, 29\] A residuated lattice is a structure \((L, \land, \lor, \otimes, \Rightarrow, 0, 1)\) satisfying the following axioms:

(a) \((L, \land, \lor, 0, 1)\) is a bounded lattice;

(b) \((L, \otimes, 1)\) is a commutative semigroup with unit element 1;

(c) \((\otimes, \Rightarrow)\) is an adjoint pair, that is,

\[
z \leq (x \Rightarrow y) \text{ if and only if } x \otimes z \leq y, \text{ for all } x, y, z \in L \text{ (residuation ).}
\]

An MTL-algebra \[24, 25, 29, 108\] is a residuated lattice \(L\) satisfying the equation:

(d) \((x \Rightarrow y) \lor (y \Rightarrow x) = 1, \) for all \(x, y \in L\) (pre-linearity).

A BL-algebra \[29\] is an MTL-algebra \(L\) satisfying the equation:

(e) \((x \land y) = x \otimes (x \Rightarrow y), \) for all \(x, y \in L\) (divisibility).
An MV-algebra \[12, 13\] is a BL-algebra in which the negation is an involution, that is,

\((f1)\) \((x \Rightarrow 0) \Rightarrow 0 = x\), for all \(x \in L\), or equivalently,

\((f2)\) \((x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x\), for all \(x, y \in L\).

An IMTL-algebra is an MTL-algebra \(L\) satisfying the equation:

\((g)\) \((x \Rightarrow 0) \Rightarrow 0 = x\), for all \(x \in L\) (regularity).

An NM-algebra is an IMTL-algebra \(L\) satisfying the equation:

\((h)\) \(((x \otimes y) \Rightarrow 0) \lor ((x \land y) \Rightarrow (x \otimes y)) = 1\), for all \(x, y \in L\).

A Boolean algebra \[29, 98\] is a bounded distributive lattice \((L, \land, \lor, 0, 1)\) with a unary operation ‘ satisfying the equations:

\((i)\) \(x \lor x' = 1\);

\((j)\) \(x \land x' = 0\), for all \(x \in L\).

A Heyting algebra is a residuated lattice \(L\) satisfying the equation:

\((k)\) \(x \otimes x = x\), or equivalently \(x \otimes y = x \land y\), for all \(x, y \in L\).

Pavelka \[61\] and Turunen \[84, 85\] studied various algebraic structures of residuated lattice.

**Lemma 1.6.8.** \[29\] If \((L, \land, \lor, \otimes, \Rightarrow, 0, 1)\) is a BL-algebra, then for every \(x, y, z \in L\):

1. \(x \otimes (x \Rightarrow y) \leq y\) and \(x \leq (y \Rightarrow (x \otimes y))\);

2. if \(x \leq y\), then :

   (a) \((x \otimes z) \leq (y \otimes z)\),

   (b) \((z \Rightarrow x) \leq (z \Rightarrow y)\),
\( (c) \ (y \Rightarrow z) \leq (x \Rightarrow z); \)

3. \( x \leq y \) if and only if \( x \Rightarrow y = 1; \)

4. \( (x \lor y) \otimes z = (x \otimes z) \lor (y \otimes z); \)

5. \( x \lor y = ((x \Rightarrow y) \Rightarrow y) \land ((y \Rightarrow x) \Rightarrow x). \)

### 1.7 Rough Sets

A binary relation \( R \) on a set \( X \) is called an equivalence relation if it is reflexive, symmetric and transitive. For an \( x \in X \), an equivalence class \([x]_R \) consists of all elements \( y \in Y \) such that \( xRy \). An equivalence relation induces a partitioning of the universe. Using such partitions one can approximate subsets of the universe whenever the information is vague. In the early 1980’s, Pawlak [62, 63] developed a set theory in this way and he called such sets as rough sets.

**Definition 1.7.1.** (See [38] ) Let \( X \) be the universal set, \( R \) be an equivalence relation on \( X \), \( X/R \) be the family of all equivalence classes induced on \( X \) by \( R \) and \([x]_R \) be the equivalence class in \( X/R \) containing \( x \) in \( X \). The lower approximation of a crisp subset \( A \) of \( X \) is defined by

\[
R(A) = \bigcup \{ [x]_R : [x]_R \subseteq A, x \in X \}
\]

and the upper approximation of \( A \) is defined by

\[
\overline{R}(A) = \bigcup \{ [x]_R : [x]_R \cap A \neq \emptyset, x \in X \}.
\]

The pair \((X, R)\) is called an approximation space and \( R(A) = (R(A), \overline{R}(A)) \) is called the rough set representation of \( A \).
1.7. ROUGH SETS

Note 1.7.2. The upper and lower approximations can also be written as

\[ R(A) = \{ x \in X : [x]_R \subseteq A \} \quad \text{and} \quad \overline{R}(A) = \{ x \in X : [x]_R \cap A \neq \emptyset \}. \]

The set difference \( \overline{R}(A) - R(A) \) denotes the rough description of the boundary of \( A \). Vaguely speaking, a set is said to be rough if its boundary region is non-empty, otherwise the set is crisp. Accuracy of approximation \( \alpha_R(X) = \frac{|R(A)|}{|\overline{R}(A)|} \), where \( |A| \) denotes the cardinality of \( A \). Note that, \( 0 \leq \alpha_R(A) \leq 1 \). If \( \alpha_R(A) = 1 \), then \( R(A) \) is crisp and if \( \alpha_R(A) < 1 \), then \( R(A) \) is rough with respect to \( R \).

Theorem 1.7.3. Let \( X \) be the universe and \( R \) be an equivalence relation on \( X \). For every multi-fuzzy sets \( A, B \in X \), the rough approximation operators \( R \) and \( \overline{R} \) possess the following properties:

1. \( R(X) = X = \overline{R}(X) \);
2. \( R(\emptyset) = \emptyset = \overline{R}(\emptyset) \);
3. \( R(A) \subseteq A \subseteq \overline{R}(A) \);
4. \( R(A \cap B) = R(A) \cap R(B) \);
5. \( \overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B) \);
6. \( R(A) \cup R(B) \subseteq R(A \cup B) \);
7. \( \overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B) \);
8. \( A \subseteq B \) implies \( R(A) \subseteq R(B) \) and \( \overline{R}(A) \subseteq \overline{R}(B) \).