CHAPTER 4

Reliability of a $k$-out-of-$n$ system with repair by a service station attending a queue with postponed work

In this chapter the reliability of a repairable $k$-out-of-$n$ system is studied. Repair times of components follow a phase type distribution. In addition, the service facility offers service to external customers which on arrive according to a MAP. An external customer, who sees an idle server on its arrival, is immediately selected for service. Otherwise, the external customer joins the queue in a pool of postponed work of infinite capacity with probability 1 if the number of failed components in the system is $< M$ ($M \leq n - k + 1$) and if the number of failed components $\geq M$ it joins the pool with probability $\gamma$ or leaves the system forever. Repair times of components of the system and that of the external customers have independent phase type distributions. At a service completion epoch if the buffer has less than $L$ customers, a pooled customer is taken for service with probability $p$, $0 < p < 1$. If at a service completion epoch no component of the system is waiting for repair, a pooled customer, if any waiting, is immediately taken for service.

Thus in this chapter also we study the effect of allowing service to external customers in a $k$-out-of-$n$ system with single server. But different from chapters 2 and 3, here the external customers are never directed to an orbit instead, they join the queue in a pool or leaves the system forever. Also different from chapter 3, the capacity of the pool is assumed to be infinite and we give freedom for an external customer not to join the pool if he wishes. We expect that such a move will help us to utilize the server idle time more effectively.


79
4.1. MATHEMATICAL MODELLING

We obtain the system state distribution under the condition of stability. A number of performance characteristics are derived. A cost function involving $L$, $M$, $\gamma$ and $p$ is constructed and its behaviour investigated numerically.

4.1. Mathematical modelling

We consider a $k$-out-of-$n$ cold system in which the components have exponentially distributed lifetimes with parameter $\frac{1}{i}$, when there are $i$ operational components. There is a single server repair facility which gives service to failed components (main customers) and also to external customers. The external customers arrive according to a MAP with representation $(D_0, D_1)$ of order $m$. Repair times of main and external customers follow PH-distribution with representations $(\beta_1, S_1)$ of order $m_1$ and $(\beta_2, S_2)$ of order $m_2$, respectively.

Let $Y_1(t)$ be the number of external customers in the system including the one getting service, if any, and $Y_2(t)$ be the number of main customers in the system including the one getting service, if any, at time $t$. If an external customer, on arrival, finds a busy server and that $Y_2(t) < M$ ($M \leq n - k + 1$), it joins a pool of infinite capacity with probability 1; on the other hand if $Y_2(t) \geq M$ then with probability $\gamma$ it joins the pool or leaves the system forever.

If $Y_2(t) = 0$ at a service completion epoch then, with probability 1 a pooled customer, if any, gets service. If $0 < Y_2(t) \leq L - 1$, ($L \leq M$), at a service completion epoch, then with probability $p$ a pooled customer, if there is any, is given service. If $Y_2(t) > L - 1$ at a service completion epoch, then with probability 1 a main customer gets service. If $Y_1(t) = Y_2(t) = 0$ then an external customer arriving at time $t$ is taken for service.

Define

$$Y_3(t) = \begin{cases} 0 & \text{if a main customer is getting service at time } t \\ 1 & \text{if an external customer is getting service at time } t \end{cases}$$

Let $Y_4(t)$ and $Y_5(t)$ denote the phases of the arrival and service process respectively.
4.1. MATHEMATICAL MODELLING

Now $\mathcal{H} = \{(Y_1(t), Y_2(t), Y_3(t), Y_4(t), Y_5(t)) | t \geq 0\}$ forms a continuous time Markov chain which turns out to be a level independent quasi birth and death process with state space $\bigcup_{i=0}^{\infty} l(i)$ where $l(i)$ denotes the collection of states in level $i$ and are defined as

\[
l(0) = \{0\} \cup \{(0, j_1, 0, j_2, j_3) : 1 \leq j_1 \leq n - k + 1, 1 \leq j_2 \leq m, 1 \leq j_3 \leq m_1\}
\]

and for $i \geq 1$,

\[
l(i) = \{(i, j_1, 0, j_2, j_3) : 1 \leq j_1 \leq n - k + 1, 1 \leq j_2 \leq m, 1 \leq j_3 \leq m_1\}
\]

\[
\bigcup \{(i, j_1, 1, j_2, j_3) : 0 \leq j_1 \leq n - k + 1, 1 \leq j_2 \leq m, 1 \leq j_3 \leq m_2\}
\]

where $\{0\} = \{(0, j) : 1 \leq j \leq m\}$ represents the collection of states corresponding to $Y_1(t) = Y_2(t) = 0$. Let $J_1 = m + (n - k + 1)m_1$ be the dimension of level $l(0)$ and $J_2 = mm_2 + (n - k + 1)m(m_1 + m_2)$ be the dimension of levels $l(i)$ for $i \geq 1$. Arranging the states lexicographically we get the infinitesimal generator $Q$ of the process $\mathcal{H}$ as

\[
Q = \begin{bmatrix}
B_0 & B_1 & 0 & 0 & 0 & \cdots \\
B_2 & A_1 & A_0 & 0 & 0 & \cdots \\
0 & A_2 & A_1 & A_0 & 0 & \cdots \\
0 & 0 & A_2 & A_1 & A_0 & \cdots \\
& & & & & \vdots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\]

with

\[
B_0 = \begin{bmatrix}
B_0^{(1)} & B_0^{(6)} \\
B_0^{(7)} & B_0^{(2)} & B_0^{(6)} \\
B_0^{(8)} & B_0^{(2)} & B_0^{(6)} \\
& & & \ddots & \ddots & \ddots \\
& & & & B_0^{(8)} & B_0^{(3)} & B_0^{(6)} \\
& & & & & B_0^{(8)} & B_0^{(4)}
\end{bmatrix}
\]
where

\[
\begin{align*}
B_0^{(1)} &= D_0 - \lambda I_m, \\
B_0^{(2)} &= D_0 \oplus S_1 - \lambda I_{mm_1}, \\
B_0^{(3)} &= (D_0 + (1 - \gamma)D_1) \oplus S_1 - \lambda I_{mm_1}, \\
B_0^{(4)} &= (D_0 + (1 - \gamma)D_1) \oplus S_1, \\
B_0^{(5)} &= I_m \otimes (\lambda \beta_1), \\
B_0^{(6)} &= \lambda I_{mm_1}, \\
B_0^{(7)} &= I_m \otimes S_1^0, \\
B_0^{(8)} &= I_m \otimes (S_1^0 \beta_1).
\end{align*}
\]

\[
B_1 = \begin{bmatrix}
B_1^{(1)} & 0 & 0 \\
0 & I_{M-1} \otimes B_1^{(2)} & 0 \\
0 & 0 & I_{n-k-M+2} \otimes B_1^{(3)}
\end{bmatrix}
\]

where

\[
B_1^{(1)} = D_1 \otimes \beta_2, \\
B_1^{(2)} = D_1 \otimes I_{m_1}, \\
B_1^{(3)} = \gamma B_1^{(2)},
\]

\[
B_2 = \begin{bmatrix}
B_2^{(1)} & 0 \\
0 & I_{n-k+1} \otimes B_2^{(2)}
\end{bmatrix}
\]

where

\[
B_2^{(1)} = I_m \otimes S_2^0, \\
B_2^{(2)} = I_m \otimes (S_2^0 \beta_1).
\]
4.1. MATHEMATICAL MODELLING

where

\[ A_1^{(8)} = \begin{bmatrix} q(I_m \otimes (S_0^3 \beta_1)) & p(I_m \otimes (S_0^3 \beta_2)) \\ 0 & 0 \end{bmatrix}, \quad A_0^{(3)} = \begin{bmatrix} B_0^{(3)} \\ 0 \\ 0 \end{bmatrix} \]
4.2. Stability condition

The generator matrix \( A = A_0 + A_1 + A_2 \) is given by

\[
\begin{bmatrix}
A_1^{(1)} & 0 & 0  \\
0 & I_{L-1} \otimes A_2^{(2)} & 0  \\
0 & 0 & I_{n-k-L+2} \otimes A_2^{(3)}
\end{bmatrix}
\]

where

\[
A_2^{(1)} = I_m \otimes (S_2^0 \beta_2) \quad A_2^{(2)} = \begin{bmatrix} 0 & 0 \\ q(I_m \otimes (S_2^0 \beta_1)) & p(I_m \otimes (S_2^0 \beta_2)) \end{bmatrix}
\]

\[
A_2^{(3)} = \begin{bmatrix} 0 & 0 \\ I_m \otimes (S_2^0 \beta_1) & 0 \end{bmatrix}
\]

\( A_2^{(2)} \) and \( A_2^{(3)} \) are square matrices of order \( m(m_1 + m_2) \).
4.2. STABILITY CONDITION

The stationary probability vector \( \pi \) of \( A \), partitioned as

\[
\pi = (\pi(0), \pi(1), \pi(2), \ldots, \pi(n - k + 1))
\]

where the subvector \( \pi(0) \) contains \( mm_2 \) entries and the subvectors \( \pi(i) \) for \( 1 \leq i \leq n - k + 1 \) contains \( m(m_1 + m_2) \) entries, satisfies the equations

\[
\pi(0)A_1^{(7)} + \pi(1)A_1^{(7)} = 0
\]
\[
\pi(0)A_1^{(5)} + \pi(1)A_2 + \pi(2)A_1^{(8)} = 0
\]
\[
\pi(i)A_1^{(6)} + \pi(i + 1)A_2 + \pi(i + 2)A_1^{(8)} = 0, \quad 1 \leq i \leq L - 2
\]
\[
\pi(i)A_1^{(6)} + \pi(i + 1)A_3 + \pi(i + 2)A_1^{(9)} = 0, \quad L - 1 \leq i \leq n - k - 1
\]
\[
\pi(n - k)A_1^{(6)} + \pi(n - k + 1)A_4 = 0
\]

together with the normalizing condition

\[
\pi e = 1.
\]

The equations from (4.1) to (4.5) implies

\[
\pi(0) = \left[ \pi(1)A_1^{(7)} \right] \left[ (-A_1)^{-1} \right]
\]
\[
\pi(1) = \left[ \pi(0)A_1^{(5)} + \pi(2)A_1^{(8)} \right] \left[ (-A_2)^{-1} \right]
\]
The invertibility of the matrices \((D_0 + D_1 - \lambda I_m) \oplus (S_2 + S_2^0 \beta_2), (D_0 + D_1 - \lambda I_m) \oplus S_1, (D_0 + D_1 - \lambda I_m) \oplus (S_2 + p(S_2^0 \beta_2)), (D_0 + D_1 - \lambda I_m) \oplus S_2\) follows from the fact that they are strictly diagonally dominant. The invertibility of the matrix \((D_0 + D_1) \oplus S_1\) can be proved as follows.

Suppose that \((D_0 + D_1) \oplus S_1\) is not invertible, then there exists a non-negative vector \(u \neq 0\) such that

\[ u((D_0 + D_1) \oplus S_1) = 0 \quad \text{(ie)} \quad u((D_0 + D_1) \otimes I_{m_1} + I_m \otimes S_1) = 0 \]

Multiplying both sides of the above equation with \(e_m \otimes I_{m_1}\), we get

\[ [u(I_m \otimes S_1)](e_m \otimes I_{m_1}) = 0, \quad \text{since} \quad [(D_0 + D_1) \otimes I_{m_1}](e_m \otimes I_{m_1}) = 0. \]

(ie) \(u[e_m \otimes S_1] = 0\)

If we partition \(u\) as \(u = (u_1, u_2, \ldots, u_m)\), where each \(u_i\) is a row vector containing \(m_1\) elements, the above equation implies that

\[(u_1 + u_2 + \ldots + u_m)S_1 = 0\]

Now, since \(S_1\) is invertible, this implies that

\[ u_1 + u_2 + \ldots + u_m = 0 \]
4.3. STATIONARY DISTRIBUTION

since each $u_i \geq 0$, above equation implies $u_i = 0 \quad \forall i$

$\Rightarrow u = 0$

which contradicts the assumption that $u \neq 0$.

Hence $(D_0 + D_1) \oplus S_1$ is invertible.

Similarly $(D_0 + D_1) \oplus S_2$ is invertible.

The matrices $\hat{A}_2$, $\hat{A}_3$ and $\hat{A}_4$ have the general form

\[
\begin{bmatrix}
H_1 & 0 \\
H_2 & H_3
\end{bmatrix}
\]

where $H_1$ and $H_3$ are invertible. The inverse of such a matrix is given by

\[
\begin{bmatrix}
H_1^{-1} & 0 \\
-(H_3^{-1} H_2 H_1^{-1}) & H_3^{-1}
\end{bmatrix}
\]

which makes it easier to find the inverses $(\hat{A}_2)^{-1}$, $(\hat{A}_3)^{-1}$ and $(\hat{A}_4)^{-1}$.

The equations from (4.6) to (4.11) are well suited for Block Gauss-Seidel iteration procedure which can now be used to find the vector $\pi$.

Now the stability condition can be stated as follows:
The process $\mathcal{H}$ will be positive recurrent if and only if $\pi A_0 e < \pi A_2 e$, where

\[
\pi A_0 e = \pi(0) \left[ (D_1 e_m) \otimes e_{m_2} \right] + \left[ \sum_{i=1}^{M-1} \pi(i) + \sum_{i=M}^{n-k+1} \gamma \pi(i) \right] \left[ (D_1 e_m) \otimes e_{m_1} \right] \\
\pi A_2 e = \pi(0) \left[ e_m \otimes S_2^0 \right] + \left[ \sum_{i=1}^{n-k+1} \pi(i) \right] \left[ 0 \quad e_m \otimes S_2^0 \right]
\]

4.3. Stationary distribution

Since the model is studied as a level independent QBD Markov Process, its stationary distribution (when it exists) has a matrix geometric solution. Under the assumption of the existence of the stationary distribution, let the stationary vector $x$ of $Q$ be partitioned by
4.3. STATIONARY DISTRIBUTION

the levels as \( x = (x(0), x(1), x(2), \ldots) \). Then \( x(i) \)'s are given by

\[
x(i) = x(1)R^{i-1} \quad \text{for } i \geq 2
\]

where \( R \) is the minimal non-negative solution to the matrix quadratic equation

\[
R^2 A_2 + RA_1 + A_0 = 0.
\]

The vectors \( x(0) \) and \( x(1) \) are obtained by solving the equations

\[
x(0)B_0 + x(1)B_2 = 0
\]

\[
x(0)B_1 + x(1)[A_1 + RA_2] = 0
\]

subject to the normalizing condition

\[
x(0)e + x(1)(I - R)^{-1}e = 1.
\]

To compute the \( R \) matrix numerically we used the logarithmic reduction algorithm (see Latouche and Ramaswami [41]).

Departure process of external customers:

We define the departure process of external customers as the sequence of times \( \{\tau_m : m \geq 0\} \) at which the external units leave the system due to a service completion with \( \tau_0 \equiv 0 \). To study this sequence, it is enough to study the interdeparture times of external customers \( \{\bar{\tau}_m = \tau_m - \tau_{m-1} : m \geq 1\} \). Since the random variables \( \bar{\tau}_1, \bar{\tau}_2, \ldots \) are identically distributed when the process \( \mathcal{H} \) is positive recurrent, we focus on \( \bar{\tau}_1 \) and determine its distribution under the assumption of positive recurrence of \( \mathcal{H} \).

Let \( F(t) = P(\bar{\tau}_1 \leq t) \) be the distribution function of \( \bar{\tau}_1 \) and \( \Phi(\theta) = E[e^{-\theta\bar{\tau}_1}], Re(\theta) \geq 0 \), be its Laplace-Stieltjes transform.
4.3. STATIONARY DISTRIBUTION

Conditioning on the state of the process \( \mathcal{H} \) at time \( \tau_0 \), we can write

\[
F(t) = \sum_{i=0}^{\infty} x(i) F_i(t),
\]

(4.12)

\[
\Phi(\theta) = \sum_{i=0}^{\infty} x(i) \Phi_i(\theta),
\]

(4.13)

where \( F_0(t) \) and \( \Phi_0(\theta) \) are column vectors with \( J_1 \) entries, \( F_i(t) \) and \( \Phi_i(\theta) \) are column vectors with \( J_2 \) entries for \( i \geq 1 \). The entries of \( F_i(t) \) and \( \Phi_i(\theta) \) are defined as the conditional distribution functions and conditional Laplace-Stieltjes transforms respectively of \( f_i \), given that the state of the process \( \mathcal{H} \) at time \( \tau_0 \) is in the level \( l(i) \) for \( i \geq 0 \). Since the process \( \mathcal{H} \) is level independent, we see that

\[
F_2(t) = F_3(t) = F_4(t) = \ldots = F_1(t)
\]

and

\[
\Phi_2(\theta) = \Phi_3(\theta) = \Phi_4(\theta) = \ldots = \Phi_1(\theta).
\]

After arranging the state in the level \( l(i), i \geq 1 \), lexicographically, we rename them as \((i, 1), (i, 2), \ldots (i, J_2)\) and the states in the level \( l(0) \) as \((0, 1), (0, 2), \ldots (0, J_1)\). Now to find \( F_i(t) \) and \( \Phi_i(\theta) \) we suppose that at time \( \tau_0 \) the process \( \mathcal{H} \) is in the state \((1, j), 1 \leq j \leq J_2 \). Then since the transitions in the level independent QBD process \( \mathcal{H} \) due to the arrival process of external customers will not affect the departure process, the time \( \tau_1 \) can be thought of as the time until absorption in a finite continuous time Markov chain \( \mathcal{H}_1 \) with state space \( \Delta \cup \{1, 2, \ldots, J_2\} \), where \( \Delta \) is an absorbing state, and with infinitesimal generator

\[
Q_1 = \begin{bmatrix}
0 & 0 \\
A_{e} & \bar{A}_1
\end{bmatrix}
\]
4.3. STATIONARY DISTRIBUTION

The process $\mathcal{H}_1$ is obtained from the process $\mathcal{H}$ as follows: since the process $\mathcal{H}$ is at $(1,j)$, $1 \leq j \leq J_2$ at time $\tau_0$, we suppose that the process $\mathcal{H}_1$ starts at state $j$. Now corresponding to each transition in $\mathcal{H}$ brought by the arrival process of external customers, that is the transitions governed by the matrices $D_0$ and $D_1$, we suppose that no transition occurs in $\mathcal{H}_1$. That is corresponding to these transitions in $\mathcal{H}$, there is a sojourn in $\mathcal{H}_1$. Corresponding to those transitions in $\mathcal{H}$ between states in the same level, which are not governed by the matrix $D_0$, there is a transition in $\mathcal{H}_1$ governed by the matrix $\bar{A}_1$. The moment there is a transition in the process $\mathcal{H}$ which results in a decrease of level of $\mathcal{H}$ by 1 unit, the departure of an internal customer occurs and we suppose that an absorption to the state $\Delta$ occurs in $\mathcal{H}_1$ with rates governed by the column matrix $A_2 e$.
4.3. STATIONARY DISTRIBUTION

Thus given that the process $\mathcal{H}$ is in state $(1, j), 1 \leq j \leq J_2$, the time $\tau_1$ is the time until absorption in the process $\mathcal{H}_1$ with generator matrix $Q_1$ and with initial probability vector $\alpha_1 = (0, \alpha_1)$ where $\alpha_1$ is a row vector containing $J_2$ entries whose $j^{th}$ entry is 1 and all other entries are zeros; that is $\tau_1$ has a PH distribution with representation $(\alpha_1, \bar{A}_1)$. Hence the $j^{th}$ entry of the column matrix $F_1(t)$, namely $F_{1j}(t)$ is given by

$$F_{1j}(t) = 1 - \alpha_1[\exp(\bar{A}_1 t)]e.$$  

Note that $\alpha_1[\exp(\bar{A}_1 t)]e$ is the $j^{th}$ entry of the column matrix $(\exp(\bar{A}_1 t))e$. Thus we have

$$F_1(t) = e - [\exp(\bar{A}_1 t)]e \quad (4.14)$$

Also the $j^{th}$ entry of $\Phi_1(\theta)$, namely $\Phi_{1j}(\theta)$ is given by

$$\Phi_{1j}(\theta) = \alpha_1(\theta I - \bar{A}_1)^{-1}A_2 e$$

and therefore

$$\Phi_1(\theta) = (\theta I - \bar{A}_1)^{-1}A_2 e. \quad (4.15)$$

Now to find $F_0(t)$ and $\Phi_0(\theta)$ we proceed in a similar way. Suppose that at time $\tau_0$ the process $\mathcal{H}$ is in state $(0, j), 1 \leq j \leq J_1$. Then the time $\tau_1$ can be thought of as the time until absorption in the process $\mathcal{H}_2$ with state space

$$\Delta \cup \{(0, 1), (0, 2), \ldots, (0, J_1), 1, 2, \ldots, J_2\},$$

where $\Delta$ is an absorbing state, and with infinitesimal generator

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & B_0 & B_1 \\ A_2 e & 0 & \bar{A}_1 \end{bmatrix}$$

Like the process $\mathcal{H}_1$, the process $\mathcal{H}_2$ is constructed from the process $\mathcal{H}$ as follows: since the process $\mathcal{H}$ is assumed to be in state $(0, j)$ at time $\tau_0$, we suppose the process $\mathcal{H}_2$ starts in the state $(0, j), 1 \leq j \leq J_1$. Now corresponding to each transition in the process $\mathcal{H}$
from \((0,j_1)\) to \((0,j_2)\), there is a transition in the process \(H_2\) from \((0,j_1)\) to \((0,j_2)\) at the rate \((B_0)_{j_1,j_2}\). Corresponding to each transition in \(H\) from \((0,j_1)\) to \((1,j_2)\), \(1 \leq j_1 \leq J_1; 1 \leq j_2 \leq J_2\), there is a transition in \(H_2\) from \((0,j_1)\) to \(j_2\) at the rate \((B_1)_{j_1,j_2}\). After the process \(H\) reaches the level \(l(1)\), corresponding to each transition in \(H\) brought by the arrival process of external customers, we suppose that there is no transition in the process \(H_2\). Corresponding to those transitions in \(H\) within the same level, which are not governed by the matrix \(D_0\), there is a transition in \(H_2\) governed by the matrix \(A_1\). When a transition which results in a decrease of level by 1 unit occurs in the process \(H\), the departure of an external customer occurs and we suppose that an absorption to the state \(\Delta\) occurs in the process \(H_2\); with absorption rates governed by the column matrix \(A_2e\). Thus the conditional distribution of \(\bar{n}_1\) given that at time \(\tau_0\) the process \(H\) is in state \((0,j)\), \(1 \leq j \leq J_1\), is PH-type with representation \((\alpha_2, \bar{A}_1)\) where \(\alpha_2\) is a row vector containing \(J_1 + J_2\) entries whose \(j^{th}\) entry is 1 and all other entries are zero; and

\[
\bar{A}_1 = \begin{bmatrix} B_0 & B_1 \\ 0 & A_1 \end{bmatrix}.
\]

Hence the \(j^{th}\) entry of the column matrix \(F_0(t)\), namely \(F_{0j}(t)\) is given by

\[
F_{0j}(t) = 1 - \alpha_2[\exp(\bar{A}_1 t)][e_{j_1 + j_2}]
\]

and therefore

\[
F_0(t) = e_{j_1} - [I_{J_1} 0_{J_1 \times J_2}][\exp(\bar{A}_1 t)][e_{j_1 + j_2}]
\]

Also the \(j^{th}\) entry of \(\Phi_0(\theta)\), namely \(\Phi_{0j}(\theta)\) is given by

\[
\Phi_{0j}(\theta) = \alpha_2(\theta I - \bar{A}_1)^{-1} \begin{bmatrix} 0 \\ A_2e \end{bmatrix}
\]
and therefore
\[ \Phi_0(\theta) = \begin{bmatrix} I_{J_1} & 0_{J_1 \times J_2} \end{bmatrix} (\theta I - \bar{A}_1)^{-1} \begin{bmatrix} 0 \\ A_2 e \end{bmatrix} \]

Now
\[ \theta I - \bar{A}_1 = \begin{bmatrix} \theta I - B_0 & -B_1 \\ 0 & \theta I - \bar{A}_1 \end{bmatrix} \]

therefore
\[ [\theta I - \bar{A}_1]^{-1} = \begin{bmatrix} (\theta I - B_0)^{-1} & (\theta I - B_0)^{-1}B_1(\theta I - \bar{A}_1)^{-1} \\ 0 & (\theta I - \bar{A}_1)^{-1} \end{bmatrix} \]

which gives
\[ \Phi_0(\theta) = \begin{bmatrix} I_{J_1} & 0_{J_1 \times J_2} \end{bmatrix} \begin{bmatrix} (\theta I - B_0)^{-1}B_1(\theta I - \bar{A}_1)^{-1}A_2 e \\ (\theta I - \bar{A}_1)^{-1}A_2 e \end{bmatrix} \]

that is,
\[ \Phi_0(\theta) = (\theta I - B_0)^{-1}B_1(\theta I - \bar{A}_1)^{-1}A_2 e \]

(4.17)

Now,
\[ F(t) = \sum_{i=0}^{\infty} x(i) F_1(t) \]

\[ = x(0)F_0(t) + \sum_{i=1}^{\infty} x(i)[F_1(t) \]

\[ = x(0)F_0(t) + x(1)(I - R)^{-1}F_1(t) \]

\[ = x(0) \left( e_{J_1} - \left[ I_{J_1} 0_{J_1 \times J_2} | \exp(\bar{A}_1 t) | e_{J_1 + J_2} \right] \right) \]

\[ + x(1)(I - R)^{-1}[e - \exp(\bar{A}_1 t)] e \]

\[ = [x(0)e_{J_1} + x(1)(I - R)^{-1}e] - [x(0) 0] \exp(\bar{A}_1 t) [e_{J_1 + J_2}] \]

\[ - x(1)(I - R)^{-1} \exp(\bar{A}_1 t) e \]
4.3. STATIONARY DISTRIBUTION

\[ F(t) = 1 - \left\{ x(o) 0 \exp(\bar{A}_1 t) e + x(1)(I - R)^{-1} \exp(\bar{A}_1 t)e \right\} \] (4.18)

The above relation shows that \( F(t) \) is the distribution function of a PH distribution with representation \((\alpha_3, \bar{A}_1)\) where \( \alpha_3 = \langle x(0) 0 x(1)(I - R)^{-1} \rangle \) is a row vector containing \((J_1 + 2J_2)\) elements and \( \bar{A}_1 = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_1 \end{bmatrix} \).

Now

\[
\Phi(\theta) = \sum_{i=0}^{\infty} x(i) \Phi_i(\theta) \\
= x(0) \Phi_0(\theta) + x(1)(I - R)^{-1} \Phi_1(\theta) \\
= x(0)(\theta I - B_0)^{-1} B_1 (\theta I - \bar{A}_1)^{-1} A_2 e \\
+ x(1)(I - R)^{-1} (\theta I - \bar{A}_1)^{-1} A_2 e
\]

\[
\Phi(\theta) = [x(0)(\theta I - B_0)^{-1} B_1 + x(1)(I - R)^{-1}] (\theta I - \bar{A}_1)^{-1} A_2 e \] (4.19)

Thus we can conclude that the interdeparture time \( \bar{t}_1 \) has a PH-distribution.

4.3.1. System performance measures.

1. System reliability which is defined as the probability that there is at least \( k \) operational components is given by

\[ \theta_1 = x(0)e^{(0)} + x(1)(I - R)^{-1}e^{(1)} \]

where \( e^{(0)} \) is a column vector whose last \( mm_1 \) entries are 0s and all other entries are 1s and \( e^{(1)} \) is a column vector whose last \( m(m_1 + m_2) \) entries are 0s and all other entries are 1s.

2. Probability that system is down \( P_{down} = 1 - \theta_1 \).
4.3. STATIONARY DISTRIBUTION

(3) Expected number of pooled customers

\[ \theta_3 = \sum_{i=1}^{\infty} \sum_{j_1=1}^{n-k+1} \sum_{j_2=1}^{m} \sum_{j_3=1}^{m_1} ix(i, j_1, 0, j_2, j_3) \]

\[ + \sum_{i=1}^{\infty} \sum_{j_1=0}^{n-k+1} \sum_{j_2=1}^{m} \sum_{j_3=1}^{m_2} ix(i + 1, j_1, 1, j_2, j_3) \]

(4) Expected loss rate of external customers

\[ \theta_4 = (1 - \gamma)[x(0)e^{(2)} + x(1)(I - R)^{-1}e^{(3)}] \]

where \( e^{(2)} \) and \( e^{(3)} \) are column vectors given by

\[ e^{(2)} = \begin{bmatrix} 0 \\ e_{n-k-M+2} \otimes ((D_1e_{m_1}) \otimes e_{m_1}) \end{bmatrix} \]

\[ e^{(3)} = \begin{bmatrix} 0 \\ e_{n-k-M+2} \otimes [(D_1e_{m_1}) \otimes e_{m_1}] \\ (D_1e_{m_1}) \otimes e_{m_2} \end{bmatrix} \]

(5) Expected number of transfers from the pool when there is at least 1 main customer present, per unit time

\[ \theta_5 = \sum_{i=1}^{\infty} \sum_{j_1=2}^{L} \sum_{j_2=1}^{m} \sum_{j_3=1}^{m_1} x(i, j_1, 0, j_2, j_3) p_{S_1^0}(j_3) \]

\[ + \sum_{i=2}^{\infty} \sum_{j_1=1}^{L-1} \sum_{j_2=1}^{m} \sum_{j_3=1}^{m_2} x(i, j_1, 1, j_2, j_3) p_{S_2^0}(j_3) \]

(6) Expected number of failed components

\[ \theta_6 = x(0)e^{(4)} + x(1)(1 - R)^{-1}e^{(6)} \]
4.4. A COST FUNCTION AND NUMERICAL ILLUSTRATIONS

where \( e^{(4)} \) and \( e^{(5)} \) are column matrices given by

\[
e^{(4)} = \begin{bmatrix}
0 \\
e^{(6) \otimes e_{m1}}
\end{bmatrix}, \quad e^{(5)} = \begin{bmatrix}
0 \\
e^{(6) \otimes e_{m(m1+m2)}}
\end{bmatrix}
\]

\( e^{(6)} = [1, 2, \ldots, n - k + 1]^T \)

(7) Probability that the server is found busy with an external customer

\[
\theta_7 = \sum_{i=1}^{n-k+1} \sum_{j_1=0}^{m_1} \sum_{j_2=1}^{m_2} x(i, j_1, 1, j_2, j_3)
\]

(8) Probability that the server is found idle,

\[
\theta_8 = \sum_{j=1}^{m} x(0, j)
\]

(9) Probability that the server is found busy, \( P_{\text{busy}} = 1 - \theta_8 \)

(10) Traffic intensity \( \rho = \frac{\pi_{A0} \gamma}{\pi_{A2} c} \)

4.4. A cost function and numerical illustrations

Let \( C_1 \) be the cost per unit time incurred if the system is down, \( C_2 \), be the holding cost per unit time per customer in the pool, \( C_3 \) be the cost due to loss of 1 customer and \( C_4 \) profit obtained by serving an external unit when there is at least one main customer present, and \( C_5 \) be the holding cost per unit time of one failed component. We construct a cost function as

\[
C = P_{\text{down}} C_1 + \theta_3 C_2 + \theta_4 C_3 - \theta_5 C_4 + \theta_6 \cdot C_5
\]

The common parameters for the following tables are: \( n = 35, k = 10, \gamma = 0.5, p = 0.5 \)

\[
\beta_1 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, \quad S_1^0 = \begin{bmatrix} 3.0 \\ 6.0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} -4.0 & 1.0 \\ 1.0 & -7.0 \end{bmatrix}.
\]

\[
\beta_2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \quad S_2^0 = \begin{bmatrix} 4.0 \\ 9.5 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -5.0 & 1.0 \\ 1.0 & -10.5 \end{bmatrix}.
\]
4.4. A COST FUNCTION AND NUMERICAL ILLUSTRATIONS

\[
D_0 = \begin{bmatrix}
-5.5 & 3.5 \\
1.0 & -3.5
\end{bmatrix} \quad D_1 = \begin{bmatrix}
1.0 & 1.0 \\
1.0 & 1.5
\end{bmatrix}
\]

Arrival rate = 2.34615, Correlation = -0.00029.

\( C_1 = 1000.0, C_2 = 10.0, C_3 = 25.0, C_4 = 75.0, C_5 = 15.0, \)

Table 1 shows that when the component failure rate \( \lambda \) is small, increase in \( 'L' \) has not much effect on the probability that the server is found idle. But when \( \lambda \) is 2.5, the probability \( \theta_7 \) decreases as \( L \) increases. The reason for this can be obtained from Table 3 which shows that when \( \lambda \) is 2.5, expected number of pooled customer decreases as \( L \) increases. An intuitive reasoning for such a behaviour is that as \( L \) increases a pooled customer has a better chance of being selected for service. Note that as we have taken \( p = 0.5 \), when the number of failed components is \( < L \), there is equal probability of selecting a pooled customer for service. Also note that the average service rate is greater than average arrival rate. Table 1 also shows that when \( \lambda = 0.1 \) and 1.5, increase in \( 'M' \) has not much effect on \( \theta_7 \) but when \( \lambda = 2.5 \), \( \theta_7 \) increases with increase in \( M \). As in the previous case, the reasoning for this can be obtained from Table 3 which shows that when \( \lambda \) is 2.5, expected number of pooled customers increases as \( M \) increases.

Table 2 shows when \( \lambda = 0.1 \), increase in \( L \) and \( M \) has not much effect on \( \theta_6 \). But when \( \lambda = 2.5 \), \( \theta_7 \) increases with increase in \( L \) as well as in \( M \).

Table 4 shows that only when \( \lambda = 2.5 \), variations in \( L \) and in \( M \) has a considerable effect on \( \rho \). When \( \lambda = 2.5 \), \( \rho \) decreases as \( L \) increases and \( \rho \) increases as \( M \) increases. This can be explained in the same way as the variation in \( \theta_7 \).

Table 5 shows that cost increases as \( M \) increases towards \( n - k + 1 \), decreases as \( L \) increases towards \( M \).

In tables 6 and 7 we compare the model in this chapter with the model where no external customers are allowed.

Let case 1 denote \( k \)-out-of-\( n \) system where no external customers are allowed and case 2 denote the model discussed in this chapter. Table 6 shows that compared to the increase
in the server busy probability, the increase in the system breakdown probability is small. To make these statements more clear, as in chapters 2 and 3, we consider a cost function:

$$\text{ID}_{\text{cost}} = C_{11}P_{\text{down}} + C_{12}P_{\text{busy}}$$

where $C_{11}$ is the cost per unit time due to the system breakdown and $C_{12}$ is the profit per unit time due to the server becoming busy.

Table 7 shows that by allowing external customers as described in this chapter, there is a decrease in the value of $\text{ID}_{\text{cost}}$ even when $C_{11}$ is 1000 times larger than $C_{12}$, except when $\lambda = 2.5$. Which shows atleast numerically that our goal of idle time utilization without affecting the system reliability is achieved through the model in this chapter.

**Table 1. Variation in probability that the server is found busy with an external customer $\theta_7$**

<table>
<thead>
<tr>
<th>$L$</th>
<th>$\lambda = 0.1$</th>
<th>$\lambda = 1.5$</th>
<th>$\lambda = 2.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M = 10$</td>
<td>$M = 15$</td>
<td>$M = 20$</td>
</tr>
<tr>
<td></td>
<td>$M = 10$</td>
<td>$M = 15$</td>
<td>$M = 20$</td>
</tr>
<tr>
<td>3</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>5</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>7</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>9</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>10</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>12</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>14</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>15</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>17</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
<tr>
<td>19</td>
<td>0.3986</td>
<td>0.3986</td>
<td>0.3986</td>
</tr>
</tbody>
</table>
4.4. A COST FUNCTION AND NUMERICAL ILLUSTRATIONS

### Table 2. Variation in expected number of failed components (θ_0).

<table>
<thead>
<tr>
<th>L</th>
<th>M = 10</th>
<th>M = 15</th>
<th>M = 20</th>
<th>M = 10</th>
<th>M = 15</th>
<th>M = 20</th>
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<td>3</td>
<td>0.0346</td>
<td>0.0346</td>
<td>0.0346</td>
<td>0.8469</td>
<td>0.8469</td>
<td>0.8469</td>
<td>2.2362</td>
<td>2.2408</td>
<td>2.2412</td>
</tr>
<tr>
<td>5</td>
<td>0.0346</td>
<td>0.0346</td>
<td>0.0346</td>
<td>0.9943</td>
<td>0.9944</td>
<td>0.9944</td>
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<td>2.9887</td>
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<td>0.0346</td>
<td>0.0346</td>
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<td>1.0693</td>
<td>1.0693</td>
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<td>3.8058</td>
<td>3.8100</td>
</tr>
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<td>0.0346</td>
<td>0.0346</td>
<td>1.1017</td>
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<td>1.1023</td>
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<td>0.0346</td>
<td>0.0346</td>
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<td>1.1191</td>
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</tr>
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<td>0.0346</td>
<td>0.0346</td>
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<td>1.1224</td>
<td>1.1224</td>
<td>6.4277</td>
<td>6.6551</td>
<td></td>
</tr>
<tr>
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<td>0.0346</td>
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<td></td>
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<td></td>
<td></td>
</tr>
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<td>0.0346</td>
<td>0.0346</td>
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<td></td>
<td></td>
<td>8.2534</td>
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<td></td>
</tr>
</tbody>
</table>

### Table 3. Variation in expected number of pooled customers.

<table>
<thead>
<tr>
<th>L</th>
<th>M = 10</th>
<th>M = 15</th>
<th>M = 20</th>
<th>M = 10</th>
<th>M = 15</th>
<th>M = 20</th>
<th>M = 10</th>
<th>M = 15</th>
<th>M = 20</th>
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<tbody>
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<td>3</td>
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<td>0.3236</td>
<td>0.3236</td>
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<td>2.0186</td>
<td>2.0186</td>
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<td>0.3235</td>
<td>0.3235</td>
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<td>1.8193</td>
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<td>41.7485</td>
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<td>0.3235</td>
<td>0.3235</td>
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</table>

### Table 4. Variation in Traffic intensity(ρ)

<table>
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<th>L</th>
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<th>M = 20</th>
<th>M = 10</th>
<th>M = 15</th>
<th>M = 20</th>
<th>M = 10</th>
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<tr>
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<td>0.408</td>
<td>0.408</td>
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<td>0.408</td>
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<td>0.6081</td>
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### Table 5. Variation of the cost function

<table>
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<th>$L$</th>
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<th>$M = 20$</th>
<th>$M = 10$</th>
<th>$M = 15$</th>
<th>$M = 20$</th>
<th>$M = 10$</th>
<th>$M = 15$</th>
<th>$M = 20$</th>
</tr>
</thead>
<tbody>
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<td>1.4998</td>
<td>1.4998</td>
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<td>-25.190</td>
<td>-25.189</td>
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<td>1.4998</td>
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<td>-27.419</td>
<td>200.107</td>
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</table>

### Table 6. Comparison with no retrial case $n = 35, k = 10, \gamma = 0.7, p = 0.5$ other parameters are same as for other tables

<table>
<thead>
<tr>
<th>Case 1</th>
<th>$\lambda = 0.1$</th>
<th>$\lambda = 0.5$</th>
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<th>$\lambda = 1.5$</th>
<th>$\lambda = 2.0$</th>
<th>$\lambda = 2.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{s\text{\text{na}}}$</td>
<td>$10^{-13}$</td>
<td>$10^{-13}$</td>
<td>$10^{-13}$</td>
<td>$3.6 \times 10^{-11}$</td>
<td>$3.301 \times 10^{-8}$</td>
<td>$5.91 \times 10^{-5}$</td>
</tr>
<tr>
<td>$L = 20, M = 22$</td>
<td>$&lt; 10^{-13}$</td>
<td>$&lt; 10^{-13}$</td>
<td>$-3 \times 10^{-12}$</td>
<td>$0.7493 \times 10^{-8}$</td>
<td>$0.7952 \times 10^{-5}$</td>
<td>$0.9996 \times 10^{-3}$</td>
</tr>
<tr>
<td>$L = 20, M = 25$</td>
<td>$&lt; 10^{-13}$</td>
<td>$&lt; 10^{-13}$</td>
<td>$-3 \times 10^{-12}$</td>
<td>$0.7494 \times 10^{-8}$</td>
<td>$0.7961 \times 10^{-5}$</td>
<td>$1.018 \times 10^{-2}$</td>
</tr>
<tr>
<td>$L = 10, M = 25$</td>
<td>$&lt; 10^{-13}$</td>
<td>$&lt; 10^{-13}$</td>
<td>$&lt; 10^{-13}$</td>
<td>$0.1013 \times 10^{-9}$</td>
<td>$0.8911 \times 10^{-7}$</td>
<td>$0.1298 \times 10^{-4}$</td>
</tr>
<tr>
<td>Case 2</td>
<td>$P_{s\text{\text{na}}}$</td>
<td>$10^{-13}$</td>
<td>$10^{-13}$</td>
<td>$10^{-13}$</td>
<td>$10^{-13}$</td>
<td>$10^{-13}$</td>
</tr>
<tr>
<td>$L = 20, M = 22$</td>
<td>$0.02296$</td>
<td>$0.1148$</td>
<td>$0.2296$</td>
<td>$0.3444$</td>
<td>$0.4592$</td>
<td>$0.5741$</td>
</tr>
<tr>
<td>$L = 20, M = 25$</td>
<td>$0.4216$</td>
<td>$0.5134$</td>
<td>$0.6282$</td>
<td>$0.7431$</td>
<td>$0.8578$</td>
<td>$0.9701$</td>
</tr>
<tr>
<td>$L = 10, M = 25$</td>
<td>$0.4216$</td>
<td>$0.5134$</td>
<td>$0.6282$</td>
<td>$0.7431$</td>
<td>$0.8579$</td>
<td>$0.9718$</td>
</tr>
<tr>
<td>$C_{11}$</td>
<td>$C_{12}$</td>
<td>$\lambda$</td>
<td>$L = 20, M = 22$</td>
<td>$L = 20, M = 25$</td>
<td>$L = 10, M = 25$</td>
<td>$L = 20, M = 25$</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>----------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>0.1</td>
<td>-0.2296</td>
<td>-1.1480</td>
<td>-2.2960</td>
<td>-3.4440</td>
</tr>
<tr>
<td>1000</td>
<td>10</td>
<td>0.5</td>
<td>-5.1340</td>
<td>-6.2820</td>
<td>-7.4310</td>
<td>-8.5772</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1.5</td>
<td>-5.1340</td>
<td>-6.2820</td>
<td>-7.4310</td>
<td>-8.5772</td>
</tr>
</tbody>
</table>
4.5. Comparison of Models in chapters 2, 3 and 4

In tables 8 and 9 we compare the three ways of providing service to external customers which are introduced in Chapters 2, 3 and 4 with the case where no external customers are allowed.

Let I denotes the case of a \( k \)-out-of-\( n \) system where external customers are not allowed, and let II, III and IV denotes the models in chapters 2, 3 and 4 respectively.

The following parameters are common for I, II, III and IV

\[
\begin{align*}
    n &= 11, \ k = 4 \\
    D_0 &= \begin{bmatrix} -5.5 & 3.5 \\ 1.0 & -3.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.5 \end{bmatrix} \\
    S_1 &= \begin{bmatrix} -7.5 & 2.0 \\ 2.1 & -7.7 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -5.06 & 2.06 \\ 4.0 & -6.5 \end{bmatrix} \\
    S_1^0 &= \begin{bmatrix} 5.5 \\ 5.6 \end{bmatrix}, \quad S_2^0 = \begin{bmatrix} 3.0 \\ 2.5 \end{bmatrix} \\
    \alpha &= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}.
\end{align*}
\]

The remaining parameters for II are \( \theta = 10.0 \)

The remaining parameters for III are

\( \theta = 10.0, \ \gamma = 0.7, \ \delta = 0.7, \ N = 4, \ M = 4 \)

The remaining parameters for IV are \( \gamma = 0.7, \ L = 3, \ M = 5, \ p = 0.5 \)

**Table 8**

<table>
<thead>
<tr>
<th>( \mathcal{P}_{down} )</th>
<th>( \lambda = 0.1 )</th>
<th>( \lambda = 0.5 )</th>
<th>( \lambda = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I ( &lt; 10^{-13} )</td>
<td>( .3956 \times 10^{-8} )</td>
<td>( .4014 \times 10^{-6} )</td>
<td></td>
</tr>
<tr>
<td>II ( &lt; 10^{-13} )</td>
<td>( .1321 \times 10^{-7} )</td>
<td>( .1133 \times 10^{-5} )</td>
<td></td>
</tr>
<tr>
<td>III ( .1801 \times 10^{-7} )</td>
<td>( .3289 \times 10^{-4} )</td>
<td>( .3909 \times 10^{-3} )</td>
<td></td>
</tr>
<tr>
<td>IV ( .112 \times 10^{-10} )</td>
<td>( .1437 \times 10^{-5} )</td>
<td>( .6186 \times 10^{-4} )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \mathcal{P}_{buy} )</th>
<th>I 0.0180</th>
<th>0.0901</th>
<th>0.1802</th>
</tr>
</thead>
<tbody>
<tr>
<td>II 0.4408</td>
<td>0.5129</td>
<td>0.5850</td>
<td></td>
</tr>
<tr>
<td>III 0.7500</td>
<td>0.7941</td>
<td>0.8341</td>
<td></td>
</tr>
<tr>
<td>IV 0.8565</td>
<td>0.9285</td>
<td>0.9994</td>
<td></td>
</tr>
</tbody>
</table>
Table 8 shows the effect of providing service to external customers in a \( k \)-out-of-\( n \) system as described in Chapters 2, 3, 4. To make these effects more clear, we construct a cost function as

\[
\text{ID}_{\text{cost}} = C_{11} P_{\text{down}} - C_{12} P_{\text{busy}},
\]

where \( C_{11} \) is the cost per unit time due to the system becoming non operational and \( C_{12} \) is the profit per unit time due to the server becoming busy, whose variation according to Table 8 is given in Table 9.

<table>
<thead>
<tr>
<th>ID</th>
<th>( \lambda = 0.1 )</th>
<th>( \lambda = 0.5 )</th>
<th>( \lambda = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>--0.1800</td>
<td>--0.9010</td>
<td>--1.8020</td>
</tr>
<tr>
<td>II</td>
<td>--4.4080</td>
<td>--5.1290</td>
<td>--5.8499</td>
</tr>
<tr>
<td>III</td>
<td>--7.5000</td>
<td>--7.9377</td>
<td>--8.3019</td>
</tr>
<tr>
<td>IV</td>
<td>--8.5650</td>
<td>--9.2849</td>
<td>--9.9878</td>
</tr>
<tr>
<td>I</td>
<td>--0.1800</td>
<td>--0.9010</td>
<td>--1.8016</td>
</tr>
<tr>
<td>II</td>
<td>--4.4080</td>
<td>--5.1290</td>
<td>--5.8489</td>
</tr>
<tr>
<td>III</td>
<td>--7.5000</td>
<td>--7.9081</td>
<td>--7.9501</td>
</tr>
<tr>
<td>IV</td>
<td>--8.5650</td>
<td>--9.2836</td>
<td>--9.9321</td>
</tr>
<tr>
<td>I</td>
<td>--0.1800</td>
<td>--0.9010</td>
<td>--1.7980</td>
</tr>
<tr>
<td>II</td>
<td>--4.4080</td>
<td>--5.1289</td>
<td>--5.8387</td>
</tr>
<tr>
<td>III</td>
<td>--7.4998</td>
<td>--7.6121</td>
<td>--4.4320</td>
</tr>
<tr>
<td>IV</td>
<td>--8.5650</td>
<td>--9.2706</td>
<td>--9.3754</td>
</tr>
</tbody>
</table>

Table 9 shows the cost decreases continuously when we allow external customers as in chapters 2, 3, 4 except when \( C_{11} \) is 1000 times larger than \( C_{12} \) and \( \lambda = 0.9 \), where the cost in chapter 3 model is more than that in chapter 2, but it is less than the cost in the model where no external customers are allowed. It also shows that cost is minimum for the model described in this chapter where the external customers are kept in a pool of postponed work. From Table 8 we see that eventhough \( P_{\text{down}} \) is the least if we consider the model in chapter 2, the server busy probability is the highest for the model described in this chapter which makes that model the best from a server idle time utilization point of view.