Chapter 2

Fuzzy Graph Structures - Basic Concepts

In this chapter, we introduce the concept of fuzzy graph structures as an extension to that of graph structures of E. Sampathkumar [61] and investigate some of its basic properties.

We introduce concepts like $\rho_i$-path, $\rho_i$-connectedness, $\rho_i$-tree, fuzzy $\rho_i$-tree, $\rho_i$-bridge, $\rho_i$-cutvertex etc. of a fuzzy graph structure analogous to the concepts of path, connectedness tree, bridge, cutvertex etc. of fuzzy graphs discussed in [45] by Mordeson & Nair. Corresponding results are proved and generalised.

2.1 Introduction

In this section, we recall the concept of graph structure and generalise to fuzzy concept.

Definition 2.1.1. [61] $G = (V, R_1, R_2, ..., R_k)$ is a graph structure if $V$ is a nonempty

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1Some results of this chapter are included in the papers
3. Fuzzy Graph Structures - a Generalised Approach, Communicated.
set and $R_1, R_2, \ldots, R_k$ are relations on $V$ which are mutually disjoint such that each $R_i, i = 1, 2, 3, \ldots, k,$ is symmetric and irreflexive.

**Definition 2.1.2.** Let $G = (V, R_1, R_2, \ldots, R_k)$ be a graph structure and $\mu, \rho_1, \rho_2, \ldots, \rho_k$ be fuzzy subsets of $V, R_1, R_2, \ldots, R_k$ respectively such that $\rho_i(x, y) \leq \mu(x) \land \mu(y) \forall x, y \in V$ and $i = 1, 2, \ldots, k$. Then $\tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_k)$ is a fuzzy graph structure of $G$.

*Convention:* As is the convention in usual Graph Theory, if $(x_r, x_s), (x_s, x_r) \in R_i$, we consider them as only one $R_i$-edge and denote it by either $(x_r, x_s)$ or $(x_s, x_r)$.

**Definition 2.1.3.** Let $\tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_k)$ be a fuzzy graph structure of a graph structure $G = (V, R_1, R_2, \ldots, R_k)$. $\tilde{F} = (\mu, \tau_1, \tau_2, \ldots, \tau_k)$ is a partial fuzzy spanning subgraph structure of $\tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_k)$ if $\tau_r \subseteq \rho_r$ for $r = 1, 2, \ldots, k$.

**Example 1**

Let $V = \{x_0, x_1, x_2, x_3, x_4, x_5\}$,

$R_1 = \{(x_0, x_1), (x_0, x_2), (x_3, x_4)\}$,

$R_2 = \{(x_1, x_2), (x_4, x_5)\}$,

$R_3 = \{(x_2, x_3)\}$

and then $G = (V, R_1, R_2, R_3)$ is a graph structure. Let $\tilde{G} = (\mu, \rho_1, \rho_2, \rho_3)$ where

$\mu(x_0) = 0.8, \mu(x_1) = 0.9, \mu(x_2) = 0.6, \mu(x_3) = 0.5, \mu(x_4) = 0.6, \mu(x_5) = 0.7$,

$\rho_1(x_0, x_1) = 0.8, \rho_1(x_0, x_2) = 0.5, \rho_1(x_3, x_4) = 0.4$,

$\rho_2(x_1, x_2) = 0.6, \rho_2(x_4, x_5) = 0.5$,

$\rho_3(x_2, x_3) = 0.3, \rho_3(x_0, x_5) = 0.5$

*Note:* In cases where the $\rho_r$-values are zeroes, we do not write them explicitly.

Here notice that $\rho_1(x_0, x_1) \leq \mu(x_0) \land \mu(x_1)$ and so on.

Therefore, $\tilde{G}$ is a fuzzy graph structure of $G$. 
Now we move on to define some basic notions of fuzzy graph structures. In all these $i \in \{1, 2, ..., k\}$ and $\tilde{G}$ is the fuzzy graph structure of $G$ as described in Definition 2.1.1.

**Convention:** Throughout this chapter, unless otherwise specified, $G$ and $\tilde{G}$ represent the graph structure $G = (V, R_1, R_2, ..., R_k)$ and its fuzzy graph structure $\tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k)$.

**Definition 2.1.4.** Let $G$ be a graph structure and $\tilde{G}$ be a fuzzy graph structure of $G$. If $(x, y) \in \text{supp}(\rho_i)$, then $(x, y)$ is said to be a $\rho_i$-edge of $\tilde{G}$.

In Example 1, $(x_1, x_0), (x_0, x_2), (x_3, x_4)$ are $\rho_1$-edges, $(x_1, x_2), (x_4, x_5)$ are $\rho_2$-edges and $(x_2, x_3), (x_0, x_5)$ are $\rho_3$-edges.

**Definition 2.1.5.** A $\rho_i$-path of a fuzzy graph structure $\tilde{G}$ is a sequence of vertices, $x_0, x_1, ..., x_n$ which are distinct (except possibly $x_0 = x_n$) such that $(x_{j-1}, x_j)$ is a $\rho_i$-edge for all $j = 1, 2, ..., n$.

In Example 1, $x_1, x_0, x_2$ is a $\rho_1$-path.

**Definition 2.1.6.** Two vertices of a fuzzy graph structure $\tilde{G}$, joined by a $\rho_i$-path are said to be $\rho_i$-connected.

In Example 1, $x_1$ and $x_2$ are $\rho_1$-connected and $x_0$ and $x_5$ are $\rho_3$-connected.

**Definition 2.1.7.** The strength of a $\rho_i$-path $x_0, x_1, ..., x_n$ of a fuzzy graph structure $\tilde{G}$ is $\bigwedge_{j=1}^{n} \rho_i(x_{j-1}, x_j)$ for $i = 1, 2, ..., k$.

In Example 1, strength of the $\rho_1$-path $x_1, x_0, x_2$ is 0.5.
Definition 2.1.8. In any fuzzy graph structure \( \tilde{G} \),

\[ \rho_i^2(x, y) = \rho_i \circ \rho_i(x, y) = \bigvee_z \{ \rho_i(x, z) \wedge \rho_i(z, y) \} \]

and \( \rho_i^j(x, y) = (\rho_i^{j-1} \circ \rho_i)(x, y), j = 2, 3, ..., m \) for any \( m \geq 2 \). Also

\[ \rho_i^\infty(x, y) = \bigvee \{ \rho_i^j(x, y) : j = 1, 2, ... \} \]

We recall the concept of \( R_i \)-cycle introduced by Sampathkumar [61].

Definition 2.1.9. [61] An \( R_i \)-cycle is an alternating sequence of vertices and edges

\( v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n = v_0 \) consisting only of \( R_i \)-edges.

In the fuzzy context, we define the following analogue.

Definition 2.1.10. \( \tilde{G} \) is a \( \rho_i \)-cycle iff \( (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k)) \) is an \( R_i \)-cycle.

Example 2

Let \( V = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \),

\( R_1 = \{(x_0, x_1), (x_1, x_2), (x_2, x_3), (x_0, x_3)\} \), \( R_2 = \{(x_0, x_4)\} \),

\( R_3 = \{(x_3, x_4), (x_4, x_5), (x_5, x_0), (x_6, x_7)\} \) and \( G = (V, R_1, R_2, R_3) \) be a graph structure.

Let \( \tilde{G} = (\mu, \rho_1, \rho_2, \rho_3) \) where \( \mu(x_0) = 0.8, \mu(x_1) = 0.9, \mu(x_2) = 0.6, \mu(x_3) = 0.5, \)

\( \mu(x_4) = 0.6, \mu(x_5) = 0.7, \mu(x_6) = 0.6, \mu(x_7) = 0.5 \)

\( \rho_1(x_0, x_1) = 0.8, \rho_1(x_1, x_2) = 0.5, \rho_1(x_2, x_3) = 0.5, \rho_1(x_0, x_3) = 0.5, \)

\( \rho_2(x_0, x_4) = 0.6, \)

\( \rho_3(x_3, x_4) = 0.5, \rho_3(x_4, x_5) = 0.6, \rho_3(x_5, x_0) = 0.4, \rho_3(x_6, x_7) = 0.5 \)

In Example 2, \( (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_0, x_3) \) is a \( \rho_1 \)-cycle.

Definition 2.1.11. \( \tilde{G} \) is a fuzzy \( \rho_i \)-cycle iff \( (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k)) \)
is an $R_i$-cycle and there exists no unique $(x, y)$ in $\text{supp}(\rho_i)$ such that 
\[ \rho_i(x, y) = \wedge \{ \rho_i(u, v) | (u, v) \in \text{supp}(\rho_i) \}. \]

In Example 2, $(x_0, x_1), (x_1, x_2), (x_2, x_3), (x_0, x_3)$ is a fuzzy $\rho_1$-cycle.

Note: Sampathkumar has defined a graph structure to be an $R_i$-tree if the subgraph structure induced by $R_i$-edges is a tree. $R_i$-forest is defined similarly as follows.

**Definition 2.1.12.** A graph structure is an $R_i$-forest if the subgraph structure induced by $R_i$-edges is a forest, i.e., if it has no $R_i$-cycles.

**Definition 2.1.13.** $\tilde{G}$ is a $\rho_i$-forest if its $\rho_i$-edges form an $R_i$-forest.

In Example 2, $(x_3, x_4), (x_4, x_5), (x_5, x_0), (x_6, x_7)$ is a $\rho_3$-forest.

**Definition 2.1.14.** $\tilde{G}$ is $\rho_i$-connected if there is a $\rho_i$-path joining every pair of vertices. Also $\tilde{G}$ is a $\rho_i$-tree if it is a $\rho_i$-connected $\rho_i$-forest.

In Example 2, $(x_3, x_4), (x_4, x_5), (x_5, x_0)$ is a $\rho_2$-tree.

**Definition 2.1.15.** $\tilde{G}$ is a fuzzy $\rho_i$-forest if it has a partial fuzzy spanning subgraph structure $\tilde{F}_i = (\mu, \tau_1, \tau_2, \ldots, \tau_k)$ which is a $\tau_i$-forest where for all $\rho_i$-edges not in $\tilde{F}_i$, $\rho_i(x, y) < \tau_i^\infty(x, y)$.

**Definition 2.1.16.** $\tilde{G}$ is a fuzzy $\rho_i$-tree if it has a partial fuzzy spanning subgraph structure $\tilde{F}_i = (\mu, \tau_1, \tau_2, \ldots, \tau_k)$ which is a $\tau_i$-tree where for all $\rho_i$-edges not in $\tilde{F}_i$, $\rho_i(x, y) < \tau_i^\infty(x, y)$.

**Example 3**

Let $V = \{x_0, x_1, x_2, x_3, x_4, x_5\}$,

$R_1 = \{(x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0), (x_4, x_5)\}$, $R_2 = \{(x_0, x_4)\}$ and $G = (V, R_1, R_2)$
be a graph structure. Let \( \tilde{G} = (\mu, \rho_1, \rho_2) \) where \( \mu(x_0) = 0.5, \mu(x_1) = 0.6, \mu(x_2) = 0.7, \mu(x_3) = 0.8, \mu(x_4) = 0.9, \mu(x_5) = 0.8 \)

\[
\begin{align*}
\rho_1(x_0, x_1) &= 0.4, \rho_1(x_1, x_2) = 0.5, \rho_1(x_2, x_3) = 0.6, \rho_1(x_0, x_3) = 0.3, \rho_1(x_4, x_5) = 0.8, \\
\rho_2(x_0, x_4) &= 0.5
\end{align*}
\]

\( \tilde{G} \) is a fuzzy graph structure of \( G \). Let \( \tilde{F}_1 = (\mu, \tau_1, \tau_2) \) be a partial fuzzy spanning subgraph structure of \( \tilde{G} \) defined by

\[
\begin{align*}
\tau_1(x_0, x_1) &= 0.4, \tau_1(x_1, x_2) = 0.5, \tau_1(x_2, x_3) = 0.6, \tau_1(x_4, x_5) = 0.8, \\
\tau_2(x_0, x_4) &= 0.5
\end{align*}
\]

and \( \tau_1 = 0, \tau_2 = 0 \) for all other \( \rho_1 \) and \( \rho_2 \) edges. Then \( \tilde{F}_1 = (\mu, \tau_1, \tau_2) \) is a \( \tau_1 \)-forest with \( \rho_1(x, y) < \tau_1^\infty(x, y) \)

for all \( \rho_1 \)-edges not in \( \tilde{F}_1 \). Hence \( \tilde{G} \) is a fuzzy \( \rho_1 \)-forest.

In the above example, \((x_0, x_1), (x_1, x_2), (x_2, x_3)\) form a fuzzy \( \rho_1 \)-tree.

### 2.2 Some results on fuzzy \( \rho_i \)-trees and fuzzy \( \rho_i \)-forests

We use the definitions in the previous section to prove certain results on fuzzy \( \rho_i \)-forests and fuzzy \( \rho_i \)-trees (Here \( i \) can be any element in \( \{1, 2, ..., k\} \)).

**Theorem 2.2.1.** \( \tilde{G} \) is a fuzzy \( \rho_i \)-forest iff in any \( \rho_i \)-cycle, there exists a \( \rho_i \)-edge 

\( (x, y) \) such that \( \rho_i(x, y) < \rho_i^\infty(x, y) \) where \( (\mu, \rho_1', \rho_2', ..., \rho_k') \) is the partial fuzzy spanning subgraph structure obtained by deleting \( (x, y) \) from \( \tilde{G} \).

**Proof.** Let \( \tilde{G} \) have the property that in any \( \rho_i \)-cycle, there exists \( (x, y) \) such that 

\( \rho_i(x, y) < \rho_i^\infty(x, y) \).
If $\tilde{G}$ does not contain any $\rho_i$-cycle, then it is a fuzzy $\rho_i$-forest and there is nothing to prove.

Let $\tilde{G}$ contain a $\rho_i$-cycle. Consider a $\rho_i$-edge $(x, y)$ of that $\rho_i$-cycle with $\rho_i(x, y) < \rho_i^\infty(x, y)$ in such a way that $\rho_i(x, y)$ is the smallest among all $\rho_i$-edges of that $\rho_i$-cycle and having the above property.

Remove the $\rho_i$-edge $(x, y)$. The resultant fuzzy graph structure still may contain $\rho_i$-cycles which can be removed by repetition of the above process.

It may be noted that the strength of deleted $\rho_i$-edges in a $\rho_i$-cycle increases in each step. When the fuzzy graph structure is cleared of all $\rho_i$-cycles, the resultant partial fuzzy spanning subgraph structure is a $\rho_i$-forest, say $\tilde{F}_i$.

If $(x, y) \notin \tilde{F}_i$, $(x, y)$ was deleted. So there exists a $\rho_i$-path from $x$ to $y$ stronger than $(x, y)$. Even if some of its $\rho_i$-edges were deleted, there will be stronger $\rho_i$-paths for diverting around.

Repeating the process, we get a $\rho_i$-path consisting only of $\rho_i$-edges of $\tilde{F}_i$.

Therefore $\tilde{G}$ is a fuzzy $\rho_i$-forest.

Conversely, let $\tilde{G}$ be a fuzzy $\rho_i$-forest.

Consider a $\rho_i$-cycle $C_i$ of $\tilde{G}$. Some $\rho_i$-edge $(x, y)$ of $C_i$ is not in the partial fuzzy spanning subgraph structure $\tilde{F}_i = (\mu, \tau_1, \tau_2, ..., \tau_k)$ which is a $\tau_i$-forest and $\rho_i(x, y) < \tau_i^\infty(x, y)$.

But $\tau_i^\infty(x, y) < \rho_i^\infty(x, y)$ where $(\mu, \rho_i', \rho_2', ..., \rho_k')$ is the partial fuzzy spanning subgraph structure obtained by deleting $(x, y)$ from $\tilde{G}$ since $(x, y)$ is not in $\tilde{F}_i$.

Therefore $\rho_i(x, y) < \rho_i^\infty(x, y)$.

\begin{Theorem} \label{thm:2.2.2}
Let $\tilde{G}$ be a fuzzy graph structure. If there is at most one strongest $\rho_i$-path between any two vertices, then $\tilde{G}$ must be a fuzzy $\rho_i$-forest.
\end{Theorem}
Proof. Suppose there exists at most one strongest $\rho_i$-path between any two vertices of $\tilde{G}$.

If possible, let $\tilde{G}$ be not a fuzzy $\rho_i$-forest. Then there exists a $\rho_i$-cycle, say $C_i$, in $\tilde{G}$ such that $\rho_i(x, y) \geq \rho'_i(x, y) \forall (x, y)$ in $C_i$ where $(\mu, \rho'_1, \rho'_2, \ldots, \rho'_k)$ is a partial fuzzy spanning subgraph structure obtained by the deletion of $(x, y)$ by Theorem 2.2.1. ie., $(x, y)$ is the strongest $\rho_i$-path from $x$ to $y$.

Strength of a $\rho_i$-path is the strength of the weakest $\rho_i$-edge of that $\rho_i$-path. Thus $(x, y)$ cannot be a weakest $\rho_i$-edge of $C_i$ since in that case the remaining $\rho_i$-edges of $C_i$ form a strongest $\rho_i$-path which is a contradiction to our assumption.

Therefore $\tilde{G}$ is a fuzzy $\rho_i$-forest.

\begin{proof}

Let $\tilde{G}$ be a fuzzy $\rho_i$-tree and $\tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), \ldots, \text{supp}(\rho_k))$ be not a $\rho_i$-tree. Then there exists at least one $\rho_i$-edge $(u, v)$ in $\text{supp}(\rho_i)$ for which $\rho_i(u, v) < \rho_i^\infty(u, v)$.

Let $\tilde{G}$ be a fuzzy $\rho_i$-tree. Then there exists a partial fuzzy spanning subgraph structure $\tilde{F}_i = (\mu, \tau_1, \tau_2, \ldots, \tau_k)$ which is a $\tau_i$-tree and $\rho_i(u, v) < \tau_i^\infty(u, v)$ for all $(u, v)$ not in $\tilde{F}_i$.

Clearly $\tau_i^\infty(u, v) \leq \rho_i^\infty(u, v)$. Therefore $\rho_i(u, v) < \rho_i^\infty(u, v) \forall (u, v)$ not in $\tilde{F}_i$.

$\tilde{G}$ is a fuzzy $\rho_i$-tree and $\tilde{G}^*$ is not a $\rho_i$-tree. Hence there exists at least one $\rho_i$-edge $(u, v)$ not in $\tilde{F}_i$. ie., there exists at least one $\rho_i$-edge $(u, v)$ in $\text{supp}(\rho_i)$ with $\rho_i(u, v) < \rho_i^\infty(u, v)$.

Now we move on to prove a lemma on a special type of fuzzy graph structures.

\begin{lemma}

Let $\tilde{G}$ be a fuzzy graph structure with $\rho_i(u, v) = \mu(u) \land \mu(v)$ for some $i$ and $\forall (u, v) \in \text{supp}(\rho_i)$ where $\text{supp}(\rho_i) \neq \emptyset$. Then $\rho_i^\infty(u, v) = \rho_i(u, v)$ for that $i$.
\end{lemma}
Proof. Let $\rho_i^{\infty}(u,v) = \mu(u) \land \mu(v)$ for some $i$ and $\forall (u,v) \in \text{supp}(\rho_i)$.

\[ \rho_i^{\infty}(u,v) = \bigvee \{\rho_j^i(u,v), j = 1, 2, \ldots\} \]

\[ = \mu(u) \land \mu(v) \text{ since } \rho_j^i(u,v) \leq \mu(u) \land \mu(v) \forall j \text{ and } \rho_1^i(u,v) = \mu(u) \land \mu(v) \]

\[ = \rho_i(u,v). \]

Using the above result, we can prove the following property of fuzzy $\rho_i$-tree.

**Theorem 2.2.5.** Let $\tilde{G}$ be a fuzzy $\rho_i$-tree. Then $\rho_i(u,v) < \mu(u) \land \mu(v)$ for some $(u,v) \in \text{supp}(\rho_i)$.

**Proof.** If possible, let $\rho_i(u,v) = \mu(u) \land \mu(v)$ and $\forall (u,v) \in \text{supp}(\rho_i)$.

Then by Lemma 2.2.4, $\rho_i^{\infty}(u,v) = \rho_i(u,v)$.

Let $\tilde{G}$ be a fuzzy $\rho_i$-tree. Then $\tilde{G}$ has a partial fuzzy spanning subgraph structure $\tilde{F}_i = (\mu, \tau_1, \tau_2, \ldots, \tau_k)$ which is a $\tau_i$-tree with

\[ \rho_i(u,v) < \tau_i^{\infty}(u,v) \forall (u,v) \text{ not in } \tilde{F}_i. \]

i.e., $\rho_i^{\infty}(u,v) < \tau_i^{\infty}(u,v)$.

This is not possible.

Thus $\rho_i(u,v) < \mu(u) \land \mu(v)$ for some $(u,v) \in \text{supp}(\rho_i)$. \qed

**Theorem 2.2.6.** Let $\tilde{G}$ be a $\rho_i$-cycle. $\tilde{G}$ is a fuzzy $\rho_i$-cycle iff $\tilde{G}$ is not a fuzzy $\rho_i$-tree.

**Proof.** Let $\tilde{G}$ be a fuzzy $\rho_i$-cycle.

If possible, let $\tilde{G}$ be a fuzzy $\rho_i$-tree. Then it has a partial fuzzy spanning subgraph structure $\tilde{F}_i = (\mu, \tau_1, \tau_2, \ldots, \tau_k)$ which is a $\tau_i$-tree.

Then $\text{supp}(\rho_i) - \text{supp}(\tau_i) = \{(u,v)\}$ for some $u,v \in V$ since $\tilde{G}$ is a $\rho_i$-cycle.
By definition of a fuzzy $\rho_i$-cycle, there does not exist unique $\rho_i$-edge $(x, y)$ with $\rho_i(x, y) = \wedge \{ \rho_i(u, v) | (u, v) \in \text{supp}(\rho_i) \}$.  

So there exists no $\tau_i$-path in $\tilde{F}_i$ from $u$ to $v$ having greater strength than $\rho_i(u, v)$. Otherwise, $\tilde{G}$ will not be a fuzzy $\rho_i$-cycle.  

So by definition of fuzzy $\rho_i$-tree, $\tilde{G}$ is not a fuzzy $\rho_i$-tree.  

Conversely, let $\tilde{G}$ be not a fuzzy $\rho_i$-tree. Then it has no partial fuzzy spanning subgraph structure $\tilde{F}_i$ which is a $\tau_i$-tree. By assumption, $\tilde{G}$ is a $\rho_i$-cycle.  

Let $(\mu, \tau_1, \tau_2, ..., \tau_k)$ be a partial fuzzy spanning subgraph structure of $\tilde{G}$, which is a $\tau_i$-tree. Then $\tau_i^{\infty}(u, v) \leq \rho_i(u, v) \forall (u, v) \in \text{supp}(\rho_i)$ and $\tau_i(u, v) = 0$.  

Thus $\rho_i$ does not attain $\wedge \{ \rho_i(x, y) | (x, y) \in \text{supp}(\rho_i) \}$ uniquely. Therefore $\tilde{G} = (\mu, \tau_1, \tau_2, ..., \tau_k)$ is a fuzzy $\rho_i$-cycle. 

\[ \Box \]

2.3 Generalisation

First we introduce some new terms like $R_{i_1 i_2 ... i_r}$-cycle, $R_{i_1 i_2 ... i_r}$-forest, $R_{i_1 i_2 ... i_r}$-tree etc. by generalising the above concepts.

First we define $R_{i_1 i_2 ... i_r}$-path, $1 \leq r \leq k$, as follows.

**Definition 2.3.1.** An alternating sequence of vertices and $R_i$-edges for some $i \in \{i_1, i_2, ..., i_r\}$, $1 \leq r \leq k$, of a graph structure $G = (V, R_1, R_2, ..., R_k)$ is an $R_{i_1 i_2 ... i_r}$-path where $R_{i_1}, R_{i_2}, ..., R_{i_r}$ are some among $R_1, R_2, ..., R_k$ which are represented in it.

**Definition 2.3.2.** An alternating sequence of vertices and $R_i$-edges
$v_0, e_1, v_1, e_2, \ldots, v_{n−1}, e_n, v_n = v_0$ of a fuzzy graph structure $\tilde{G}$ consisting only of $R_i$-edges for some $i \in \{1, 2, \ldots, r\}$, (ie., $e_k$ is an $R_i$-edge for some $i \in \{i_1, i_2, \ldots, i_r\}$, $k \in \{1, 2, \ldots, n − 1\}$) is said to be an $R_{i_1i_2\ldots i_r}$-cycle if $R_{i_1}, R_{i_2}, \ldots, R_{i_r}$ are some among $R_1, R_2, \ldots R_k$ which are represented in it by $R_i$-edges, $i = 1, 2, \ldots, k$.

Note that $R_i$-cycle of Definition 2.1.9 is an $R_{i_1i_2\ldots i_r}$-cycle with $r = 1$ and $i_1 = i$.

**Definition 2.3.3.** A graph structure which does not contain $R_{i_1i_2\ldots i_r}$-cycles for $i_1, i_2, \ldots i_r \in \{1, 2, \ldots, k\}$ which need not be distinct, is an $R_{i_1i_2\ldots i_r}$-forest.

**Definition 2.3.4.** An $R_{i_1i_2\ldots i_r}$-forest is an $R_{i_1i_2\ldots i_r}$-tree if it is connected by an $R_{i_1i_2\ldots i_r}$-path.

**Definition 2.3.5.** Let $x_0, x_1, \ldots, x_n$ be a sequence of distinct vertices of $\tilde{G}$. Let $\rho_{i_p}(x_{j−1}, x_j) > 0 \forall j = 1, 2, \ldots, n$ for some $p \in \{1, 2, \ldots, r\}$ where $\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_r}$ are some among $\rho_1, \rho_2, \ldots, \rho_k$. Then $x_0, x_1, \ldots x_n$ is a $\rho_{i_1i_2\ldots i_r}$-path.

In Example 1, $x_0, x_1, x_2$ is a $\rho_{12}$-path and $x_0, x_5, x_4, x_3$ is a $\rho_{123}$-path.

**Definition 2.3.6.** Two vertices of a fuzzy graph structure $\tilde{G}$ joined by a $\rho_{i_1i_2\ldots i_r}$-path are said to be $\rho_{i_1i_2\ldots i_r}$-connected.

**Definition 2.3.7.** The strength of a $\rho_{i_1i_2\ldots i_r}$-path $x_0, x_1, \ldots x_n$ of a fuzzy graph structure $\tilde{G}$ is

$$\bigwedge_{j=1}^{n} \bigvee_{q=1}^{r} \rho_{i_q}(x_{j−1}, x_j).$$

We may denote the strength of a $\rho_{i_1i_2\ldots i_r}$-path from $x$ to $y$ as $\rho_{i_1i_2\ldots i_r}(x, y)$ and the strength of a strongest $\rho_{i_1i_2\ldots i_r}$-path from $x$ to $y$ as $\rho_{i_{\infty}}(x, y)$.

We now define $\rho_{i_1i_2\ldots i_r}$-cycle, $\rho_{i_1i_2\ldots i_r}$-tree, $\rho_{i_1i_2\ldots i_r}$-forest etc.
Definition 2.3.8. \( \tilde{G} \) is a \( \rho_{i_1 i_2 ... i_r} \)-cycle iff \((\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k))\) is an \( R_{i_1 i_2 ... i_r} \)-cycle (\( \rho_{i_1}, \rho_{i_2}, ..., \rho_{i_r} \) are some among \( \rho_1, \rho_2, ..., \rho_k \) which correspond to \( R_{i_1}, R_{i_2}, ..., R_{i_r} \)).

Note that \( \rho_{i_1 i_2 ... i_r} \)-cycle is a \( \rho_{i} \)-cycle of Definition 2.1.10 for \( r = 1 \) and \( i_1 = i \).

Definition 2.3.9. \( \tilde{G} \) is a fuzzy \( \rho_{i_1 i_2 ... i_r} \)-cycle iff
\[(\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k))\] is an \( R_{i_1 i_2 ... i_r} \)-cycle and there exists no unique \((x, y)\) in \( \bigcup_{q=1}^{r} \text{supp}(\rho_q) \) such that
\[\bigvee_{q=1}^{r} \rho_q(x, y) = \bigwedge\{\bigvee_{q=1}^{r} \rho_q(u, v) | (u, v) \in \bigcup_{q=1}^{r} \text{supp}(\rho_q)\} \).

Definition 2.3.10. \( \tilde{G} \) is a \( \rho_{i_1 i_2 ... i_r} \)-forest if its \( \rho_p \)-edges (\( p \in \{1, 2, ..., r\} \)) form an \( R_{i_1 i_2 ... i_r} \)-forest.

Definition 2.3.11. \( \tilde{G} \) is a \( \rho_{i_1 i_2 ... i_r} \)-tree if it is connected by a \( \rho_{i_1 i_2 ... i_r} \)-path and it is a \( \rho_{i_1 i_2 ... i_r} \)-forest.

Definition 2.3.12. \( \tilde{G} \) is a fuzzy \( \rho_{i_1 i_2 ... i_r} \)-forest if it has a partial fuzzy spanning subgraph structure \( \tilde{F} = (\mu, \tau_1, \tau_2, ..., \tau_k) \) which is a \( \tau_{i_1 i_2 ... i_r} \)-forest (\( \tau_{i_1}, \tau_{i_2}, ..., \tau_{i_r} \) are some among \( \tau_1, \tau_2, ..., \tau_k \) which are represented in \( \tilde{F} \)) where for all \( \rho_p \)-edges (\( p \in \{1, 2, ..., r\} \)) not in \( \tilde{F} \), \( \rho_p(x, y) < \tau_{i_\infty}(x, y) \).

Definition 2.3.13. \( \tilde{G} \) is a fuzzy \( \rho_{i_1 i_2 ... i_r} \)-tree if it has a partial fuzzy spanning subgraph structure \( \tilde{F} = (\mu, \tau_1, \tau_2, ..., \tau_k) \) which is a \( \tau_{i_1 i_2 ... i_r} \)-tree where for all \( \rho_p \)-edges (\( p \in \{1, 2, ..., r\} \)) not in \( \tilde{F} \), \( \rho_p(x, y) < \tau_{i_\infty}(x, y) \).
2.4 Some results on fuzzy $\rho_{i_1i_2...i_r}$-trees and fuzzy $\rho_{i_1i_2...i_r}$-forests

We generalise the results discussed earlier to fuzzy $\rho_{i_1i_2...i_r}$-trees and fuzzy $\rho_{i_1i_2...i_r}$-forests.

**Theorem 2.4.1.** $\tilde{G}$ is a fuzzy $\rho_{i_1i_2...i_r}$-forest iff in any $\rho_{i_1i_2...i_r}$-cycle for any $p \in \{1, 2, ..., r\}$, there exists some $\rho_{ip}$-edge $(x, y)$ such that $\rho_{ip}(x, y) < \rho'_{i\infty}(x, y)$ where $(\mu, \rho'_1, \rho'_2, ..., \rho'_k)$ is the partial fuzzy spanning subgraph structure obtained by deleting $(x, y)$ and $\rho'_{i_1}, \rho'_{i_2}, ..., \rho'_{i_r}$ are defined accordingly.

**Proof. Sufficiency**

If $\tilde{G}$ does not contain a $\rho_{i_1i_2...i_r}$-cycle, it is a fuzzy $\rho_{i_1i_2...i_r}$-forest. Then there is nothing to prove.

Let $\tilde{G}$ contains a $\rho_{i_1i_2...i_r}$-cycle. Let $p \in \{1, 2, ..., r\}$. Consider some $\rho_{ip}$-edge $(p \in \{1, 2, ..., r\}) (x, y)$ with $\rho_{ip}(x, y) < \rho'_{i\infty}(x, y)$.

Remove $(x, y)$. Still there may be $\rho_{i_1i_2...i_r}$-cycles. Repeat the process with some $\rho_{iq}$-edge $(q \in \{1, 2, ..., r\}, q$ need not be different from $p)$.

Strength of the deleted $\rho_{ip}$-edges $(p \in \{1, 2, ..., r\})$ increases in each step. When the fuzzy graph structure is cleared of all $\rho_{i_1i_2...i_r}$-cycles, the resultant partial fuzzy spanning subgraph structure is a $\rho_{i_1i_2...i_r}$-forest. Let it be $\tilde{F}$.

If $(x, y) \notin \tilde{F}$, it was deleted. So there exists a $\rho_{i_1i_2...i_r}$-path from $x$ to $y$ in $\tilde{F}$, stronger than $(x, y)$. There will be stronger $\rho_{i_1i_2...i_r}$-paths for diverting around deleted $\rho_{ip}$-edges $(p \in \{1, 2, ..., r\})$. Repeating the process, we get a $\rho_{i_1i_2...i_r}$-path consisting only of $\rho_{ip}$-edges $(p \in \{1, 2, ..., r\})$. Hence $\tilde{G}$ is a fuzzy $\rho_{i_1i_2...i_r}$-forest.
Necessity

Let $\tilde{G}$ be a fuzzy $\rho_{i_1 i_2 \ldots i_r}$-forest. Let $C$ be a $\rho_{i_1 i_2 \ldots i_r}$-cycle. Some $\rho_{i_p}$ - edge
$(p \in \{1, 2, \ldots, r\}) \ (x, y)$ of $C$ is not in $\tilde{F} = (\mu, \tau_1, \tau_2, \ldots, \tau_k)$, the partial fuzzy spanning
subgraph structure which is a $\tau_{i_1 i_2 \ldots i_r}$-forest ($\tau_1, \tau_2, \ldots, \tau_r$ are some among $\tau_1, \tau_2, \ldots, \tau_k$
which are represented in it)and $\rho_{i_p}(x, y) < \tau_{i_p}(x, y)$.

But $\tau_{i_p}(x, y) < \rho'_{i_p}(x, y)$ where $(\mu, \rho'_1, \rho'_2, \ldots, \rho'_k)$ is obtained from $\tilde{G}$ by deleting
$(x, y)$ and $\rho'_1, \rho'_2, \ldots, \rho'_r$ are defined accordingly.

Therefore $\rho_{i_p}(x, y) < \rho'_{i_p}(x, y)$.

\begin{theorem}
Let $\tilde{G}$ be a fuzzy graph structure. If there is at most one strongest $\rho_{i_1 i_2 \ldots i_r}$-path between any two vertices, then $\tilde{G}$ must be a fuzzy $\rho_{i_1 i_2 \ldots i_r}$-forest.
\end{theorem}

\begin{proof}
Suppose there exists at most one strongest $\rho_{i_1 i_2 \ldots i_r}$-path between any two vertices of $\tilde{G}$.

If possible, let $\tilde{G}$ be not a fuzzy $\rho_{i_1 i_2 \ldots i_r}$-forest. Then by definition, there exists a
$\rho_{i_1 i_2 \ldots i_r}$-cycle $C$ such that for every $\rho_{i_p}$-edge $(p \in \{1, 2, \ldots, r\}) \ (x, y)$ in $C$,
$\rho_{i_p}(x, y) \geq \rho'_{i_1 i_2 \ldots i_r}(x, y)$ where $(\mu, \rho'_1, \rho'_2, \ldots, \rho'_k)$ is the partial fuzzy spanning subgraph structure obtained by deleting $(x, y)$ from $\tilde{G}$ and $\rho'_1, \rho'_2, \ldots, \rho'_r$ is the strength of $\rho_{i_1 i_2 \ldots i_r}$-path (there is only one such $\rho_{i_1 i_2 \ldots i_r}$-path) from $x$ to $y$ not involving $(x, y)$.

ie., $(x, y)$ is the strongest $\rho_{i_1 i_2 \ldots i_r}$-path from $x$ to $y$.

If $(x, y)$ is the weakest $\rho_{i_p}$-edge $(p \in \{1, 2, \ldots, r\})$ of $C$, the remaining $\rho_{i_p}$-edges
$(p \in \{1, 2, \ldots, r\})$ of $C$ form a strongest $\rho_{i_1 i_2 \ldots i_r}$-path which is a contradiction.

Therefore $\tilde{G}$ is a fuzzy $\rho_{i_1 i_2 \ldots i_r}$-forest.
\end{proof}

\begin{theorem}
Let $\tilde{G}$ be a fuzzy $\rho_{i_1 i_2 \ldots i_r}$-tree and
$\tilde{G}^* = \langle \text{Supp}(\mu), \text{Supp}(\rho_1), \text{Supp}(\rho_2), \ldots, \text{Supp}(\rho_k) \rangle$ be not a $\rho_{i_1 i_2 \ldots i_r}$-tree. Then for
$p \in \{1, 2, \ldots, r\}$, there exists at least one $\rho_p$-edge $(u, v)$ in $\bigcup_{q=1}^{r} \text{supp}(\rho_q)$ for which $\rho_p(u, v) < \rho_{i_{\infty}}(u, v)$.

**Proof.** Let $\tilde{G}$ be a fuzzy $\rho_{i_{12} \ldots i_r}$-tree. Let $p \in \{1, 2, \ldots, r\}$. Then there exists a partial fuzzy spanning subgraph structure $\tilde{F} = (\mu, \tau_1, \tau_2, \ldots, \tau_k)$ which is a $\tau_{i_{12} \ldots i_r}$-tree and for every $\rho_p$-edge $(u, v)$ not in $\tilde{F}$, $\rho_p(u, v) < \tau_{i_{\infty}}(u, v)$.

Also $\tau_{i_{\infty}}(u, v) \leq \rho_{i_{\infty}}(u, v)$

Therefore $\rho_p(u, v) < \rho_{i_{\infty}}(u, v) \forall \rho_p$-edge $(u, v)$ not in $\tilde{F}$. $\tilde{G}$ is a fuzzy $\rho_{i_{12} \ldots i_r}$-tree and $\tilde{G}^*$ is not a $\rho_{i_{12} \ldots i_r}$-tree. Therefore there exists at least one $\rho_p$-edge $(p \in \{1, 2, \ldots, r\})$ not in $\tilde{F}$. i.e., there exists at least one $\rho_p$-edge $(p \in \{1, 2, \ldots, r\})$ $(u, v)$ in $\bigcup_{q=1}^{r} \text{supp}(\rho_q)$ with $\rho_p(u, v) < \rho_{i_{\infty}}(u, v)$.

**Lemma 2.4.4.** Let $\tilde{G}$ be a fuzzy graph structure with $\rho_p(u, v) = \mu(u) \land \mu(v)$ for some $p \in \{1, 2, \ldots, r\}$ and for every $\rho_p$-edge $(u, v) \in \bigcup_{q=1}^{r} \text{supp}(\rho_q)$ where $\bigcup_{q=1}^{r} \text{supp}(\rho_q) \neq \phi$.

Then $\rho_{i_{\infty}}(u, v) = \rho_p(u, v)$ for that $p$.

**Proof.** $\rho_p(u, v) = \mu(u) \land \mu(v)$ for every $\rho_p$-edge $(u, v)$ for some $p \in \{1, 2, \ldots, r\}$,

$(u, v) \in \bigcup_{q=1}^{r} \text{supp}(\rho_q)$

$\rho_{i_{\infty}}(u, v)$ = strength of the strongest $\rho_{i_{12} \ldots i_r}$-path from $u$ to $v$.

ie., $\rho_{i_{\infty}}(u, v) = \mu(u) \land \mu(v) = \rho_p(u, v)$ for that $p$.

**Theorem 2.4.5.** Let $\tilde{G}$ be a fuzzy $\rho_{i_{12} \ldots i_r}$-tree. Let $p \in \{1, 2, \ldots, r\}$. Then $\rho_p(u, v) < \mu(u) \land \mu(v)$ for some $\rho_p$-edge $(u, v)$ in $\bigcup_{q=1}^{r} \text{supp}(\rho_q)$.  

**Proof.** If possible, let $\rho_p(u, v) = \mu(u) \land \mu(v) \forall (u, v) \in \bigcup_{q=1}^{r} \text{supp}(\rho_q)$

Then by Lemma 2.4.4, where $\rho'_j = \rho_j$ for $j \neq i$ $\rho_p(u, v) = \rho_{i_{\infty}}(u, v)$ for that $p$ for which $\rho_p(u, v) = \mu(u) \land \mu(v)$.  

Let $\tilde{G}$ be a fuzzy $\rho_{i_1 i_2 ... i_r}$-tree. Then it has a partial fuzzy spanning subgraph structure $\tilde{F} = (\mu, \tau_1, \tau_2, ..., \tau_k)$ which is a $\tau_{i_1 i_2 ... i_r}$-tree with

$$\rho_p(u, v) < \tau_{i_\infty}(u, v) \forall \rho_p$$-edge $(u, v)$ not in $\tilde{F}$.

Therefore $\rho_{i_\infty}(u, v) < \tau_{i_\infty}(u, v)$ which is a contradiction.

Hence $\rho_p(u, v) < \mu(u) \land \mu(v)$ for some $\rho_p$-edge $(u, v)$.

**Theorem 2.4.6.** Let $\tilde{G}$ be a $\rho_{i_1 i_2 ... i_r}$-cycle. $\tilde{G}$ is a fuzzy $\rho_{i_1 i_2 ... i_r}$-cycle iff $\tilde{G}$ is not a fuzzy $\rho_{i_1 i_2 ... i_r}$-tree.

**Proof.** Let $\tilde{G}$ be a fuzzy $\rho_{i_1 i_2 ... i_r}$-cycle.

If possible, let it be a fuzzy $\rho_{i_1 i_2 ... i_r}$-tree also. Then it has a partial fuzzy spanning subgraph structure $\tilde{F} = (\mu, \tau_1, \tau_2, ..., \tau_k)$ which is a $\tau_{i_1 i_2 ... i_r}$-tree. Then

$$\left[ \bigcup_{q=1}^{r} \text{supp}(\rho_q) - \bigcup_{q=1}^{r} \text{supp}(\tau_q) \right] = \{(u, v)\} \text{ for some } u, v \in V \text{ since } \tilde{G} \text{ is a } \rho_{i_1 i_2 ... i_r}\text{-cycle.}$$

There does not exist a unique $(x, y)$ in $\bigcup_{q=1}^{r} \text{supp}(\rho_q)$ such that

$$\bigwedge_{q=1}^{r} \rho_q(x, y) = \bigwedge_{q=1}^{r} \left\{ \rho_q(u, v) : (u, v) \in \bigcup_{q=1}^{r} \text{supp}(\rho_q) \right\}.$$ 

Therefore there exists no $\tau_{i_1 i_2 ... i_r}$-path in $\tilde{F}$ from $u$ to $v$ having greater strength than $\rho_{i_1 i_2 ... i_r}(u, v)$. Otherwise $\tilde{G}$ will not be a fuzzy $\rho_{i_1 i_2 ... i_r}$-cycle.

So $\tilde{G}$ is not a fuzzy $\rho_{i_1 i_2 ... i_r}$-tree.

Conversely, let $\tilde{G}$ be not a fuzzy $\rho_{i_1 i_2 ... i_r}$-tree. Then it has no partial fuzzy spanning subgraph structure $\tilde{F}$ which is a $\tau_{i_1 i_2 ... i_r}$-tree.

$\tilde{G}$ is a $\rho_{i_1 i_2 ... i_r}$-cycle by assumption. Let $(\mu, \tau_1, \tau_2, ..., \tau_k)$ be a partial fuzzy spanning subgraph structure of $\tilde{G}$, which is a $\tau_{i_1 i_2 ... i_r}$-tree with $\rho_p(u, v) < \tau_{i_\infty}(u, v)$ for all $\rho_p$-edge $(p \in \{1, 2, ..., r\})$ not in $\tilde{F}$.

$$\tau_{i_\infty}(u, v) \leq \rho_p(u, v) \forall \rho_p$$-edge $(u, v) \in \bigcup_{q=1}^{r} \text{supp}(\rho_q), p \in \{1, 2, ..., r\}$, where

$$\tau_p(u, v) = 0 \forall p = 1, 2, ..., r.$$
\begin{align*}
\tau_p(x, y) &= \rho_p(x, y) \quad \forall (x, y) \in \bigcup_{q=1}^r \text{supp}(\rho_q) - \{(u, v)\} \text{ for } p \in \{1, 2, \ldots, r\}.
\end{align*}

Therefore there does not exist unique \((x, y)\) with \(\bigvee_{q=1}^r \rho_q(x, y) = \bigwedge_{q=1}^r \{\bigvee_{q=1}^r \rho_q(u, v) | (u, v) \in \text{supp}(\rho_q)\}\).

Therefore \(\tilde{G}\) is a fuzzy \(\rho_1\rho_2\ldots\rho_r\)-cycle.

\section*{2.5 \(\rho_i\)-bridges and \(\rho_i\)-cut vertices}

We introduce some more new concepts like \(\rho_i\)-bridges and \(\rho_i\)-cut vertices of a fuzzy graph structure.

**Definition 2.5.1.** Let \((x, y)\) be a \(\rho_i\)-edge of \(\tilde{G}\). Let \((\mu, \rho'_1, \rho'_2, \ldots, \rho'_i, \rho'_{i+1}, \ldots, \rho'_k)\) be a partial fuzzy spanning subgraph structure obtained by deleting \((x, y)\) with \(\rho'_i(x, y) = 0\) and \(\rho'_i(x_1, y_1) = \rho_i(x_1, y_1) \forall \rho_i\)-edge \((x_1, y_1)\) other than \((x, y)\). If \(\rho'_i(u, v) < \rho_i(u, v)\) for some \((u, v) \in \text{supp}(\rho_i)\), then \((u, v)\) is a \(\rho_i\)-bridge.

Now we move on to some results using the concept of \(\rho_i\)-bridges.

**Theorem 2.5.1.** Let \(\tilde{G}\) be a fuzzy graph structure. If \((x, y)\) is a \(\rho_i\)-bridge, then \(\rho'_i(x, y) < \rho_i(x, y)\) where \((\mu, \rho'_1, \rho'_2, \ldots, \rho'_i, \rho'_{i+1}, \ldots, \rho'_k)\) is a partial fuzzy spanning subgraph structure obtained by deleting \((x, y)\), for \(i = 1, 2, \ldots, k\).

**Proof.** If possible, let \(\rho'_i(x, y) \geq \rho_i(x, y)\) for some \(\rho_i\)-bridge \((x, y)\). i.e., there is a \(\rho_i\)-path of strength greater than \(\rho_i(x, y)\) from \(x\) to \(y\) which does not have the \(\rho_i\)-edge \((x, y)\). Thus any \(\rho_i\)-path having the \(\rho_i\)-edge \((x, y)\) as a part of it can be replaced by a \(\rho_i\)-path without the \(\rho_i\)-edge \((x, y)\) not reducing its strength.

This is a contradiction to the fact that \((x, y)\) is a \(\rho_i\)-bridge.

Hence \(\rho'_i(x, y) < \rho_i(x, y)\) for \(i = 1, 2, \ldots, k\).
Remark 2.5.1. Converse of the above result also holds. i.e., if $\rho'_i(x, y) < \rho_i(x, y)$, then $(x, y)$ is a $\rho_i$-bridge.

Proof. If possible, let $(x, y)$ be not a $\rho_i$-bridge. Then 
$\rho''_i(x, y) = \rho''_i(x, y) \geq \rho_i(x, y)$ which is a contradiction to our assumption.

Therefore $(x, y)$ is a $\rho_i$-bridge.

Theorem 2.5.2. Let $\tilde{G}$ be a fuzzy graph structure which is a fuzzy $\rho_i$-forest. Then the $\rho_i$-edges of the partial fuzzy spanning subgraph structure $\tilde{F}_i = (\mu, \tau_1, \tau_2, ..., \tau_k)$ which is a $\tau_i$-forest, are the $\rho_i$-bridges of $\tilde{G}$.

Proof. Case 1: $(x, y)$ is a $\rho_i$-edge not in $\tilde{F}_i$

By definition of a fuzzy $\rho_i$-forest, $\rho_i(x, y) < \tau_i(x, y) \leq \rho''_i(x, y)$ where 
$(\mu, \rho'_1, \rho'_2, ..., \rho'_k)$ is a partial fuzzy spanning subgraph structure obtained by deleting $(x, y)$.

Therefore $(x, y)$ is not a $\rho_i$-bridge by Theorem 2.5.1.

Case 2: $(x, y)$ is a $\tau_i$-edge of $\tilde{F}_i$

If possible, let $(x, y)$ be not a $\rho_i$-bridge.

Then there exists a $\rho_i$-path $P_i$ from $x$ to $y$ not involving $(x, y)$ with strength greater than or equal to $\rho_i(x, y)$. So $P_i$ and $\tilde{F}_i$ form a $\rho_i$-cycle.

But $\tilde{F}_i$ does not contain $\tau_i$-cycles. Therefore, $P_i$ contains $\rho_i$-edges not in $\tilde{F}_i$.

Let $(u, v)$ be such a $\rho_i$-edge of $P_i$.

This can be replaced by a $\tau_i$-path $P_i$ in $\tilde{F}_i$ having strength greater than $\rho_i(u, v)$ by definition of a fuzzy $\rho_i$-forest.

Also $\rho_i(u, v) \geq \rho_i(x, y)$

All $\tau_i$-edges of $P_i$ are stronger than $\rho_i(u, v)$ which is greater than or equal to $\rho_i(x, y)$. 
Therefore $P_i$ does not contain $(x, y)$. If it contains $(x, y)$, its strength will be less than or equal to $\tau_i(x, y) \leq \rho_i(x, y)$.

Thus we have a $\tau_i$-path in $\tilde{F}_i$ from $x$ to $y$ not involving $(x, y)$.

This gives a $\tau_i$-cycle in $\tilde{F}_i$ and hence a $\rho_i$-cycle which is not possible.

Hence $(x, y)$ is a $\rho_i$-bridge.

Thus the $\rho_i$-edges of $\tilde{F}_i$ are the $\rho_i$-bridges of $\tilde{G}$. \hfill \Box

Now, we define a $\rho_i$-cut vertex. For that first we define the partial subgraph structure $(\mu', \rho'_1, \rho'_2, ..., \rho'_k)$.

**Definition 2.5.2.** $\tilde{G}' = (\mu', \rho'_1, \rho'_2, ..., \rho'_k)$ is the partial fuzzy subgraph structure obtained by removing $w$ of $\tilde{G}$. i.e.,

$\mu'(w) = 0$ and $\mu'(u) = \mu(u) \forall u \neq w$

$\rho'_i(w, v) = 0 \forall v \in V$ and $\rho'_i(u, v) = \rho_i(u, v) \forall (u, v) \neq (w, v), i = 1, 2, ..., k.$

**Definition 2.5.3.** A vertex $w$ of $\tilde{G}$ is a $\rho_i$-cut vertex if

$\rho^\infty_i(u, v) < \rho^\infty_i(u, v)$ for some $u, v$ with $u \neq w \neq v$ where $\mu'$ and $\rho'_i$ are as in Definition 2.5.2.

Now we discuss some results on $\rho_i$-bridges and $\rho_i$-cut vertices.

**Theorem 2.5.3.** Let $\tilde{G}$ be a fuzzy graph structure with

$\tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k))$ a fuzzy $\rho_i$-cycle. If a vertex of $\tilde{G}$ is a $\rho_i$-cut vertex of $\tilde{G}$, then it is a common vertex of two $\rho_i$-bridges.

**Proof.** Consider a $\rho_i$-cut vertex $w$ of $\tilde{G}$. By the definition of a $\rho_i$-cut vertex, there exists two vertices $u$ and $v$ different from $w$ such that $w$ is on every strongest $u - v \rho_i$-path.
Given that \(\tilde{G}^*\) is a fuzzy \(\rho_i\)-cycle. Then there exists only one strongest \(\rho_i\)-path \(P_i\) from \(u\) to \(v\) containing \(w\). All \(\rho_i\)-edges of \(P_i\) are \(\rho_i\)-bridges.

So \(w\) is common to two \(\rho_i\)-bridges.

Converse of the above result also holds as is evident from the next theorem.

**Theorem 2.5.4.** Let \(\tilde{G}\) be a fuzzy graph structure. If \(w\) is common to at least two \(\rho_i\)-bridges of \(\tilde{G}\), then \(w\) is a \(\rho_i\)-cut vertex.

**Proof.** Let \((u_1, w)\) and \((w, v_2)\) be two \(\rho_i\)-bridges with \(w\) as the common vertex.

Since \((u_1, w)\) is a \(\rho_i\)-bridge, it is on every strongest \(u - v\) \(\rho_i\)-path for some \(u\) and \(v\).

*Case 1:* \(w \neq u, w \neq v\)

In this case, \(w\) is on every strongest \(u - v\) \(\rho_i\)-path for some \(u\) and \(v\). Then \(w\) is a \(\rho_i\)-cut vertex.

*Case 2:* Either \(w = u\) or \(w = v\)

In this case either \((u_1, w)\) is on every strongest \(u - w\) \(\rho_i\)-path or \((w, v_2)\) is on every strongest \(w - v\) \(\rho_i\)-path.

If possible, let \(w\) be not a \(\rho_i\)-cut vertex.

By definition of \(\rho_i\)-cut vertex, there exists a strongest \(\rho_i\)-path not containing \(w\) between any pair of vertices. Consider such a path \(P_i\) joining \(u_1\) and \(v_2\). Then \(P_i, (u_1, w), (w, v_2)\) form a \(\rho_i\)-cycle.

a) Let \(u_1, w, v_2\) be not a strongest \(\rho_i\)-path.

Then \((u_1, w)\) or \((w, v_2)\) or both become the weakest \(\rho_i\)-edges of the above \(\rho_i\)-cycle consisting of \(P_i, (u_1, w)\) and \((w, v_2)\) since every \(\rho_i\)-edge of \(P_i\) will be stronger than \((u_1, w)\) and \((w, v_2)\).
This is not possible since \((u_1, w)\) and \((w, v_2)\) are \(\rho_i\)-bridges.

b) Let \(u_1, w, v_2\) also be a strongest \(\rho_i\)-path joining \(u_1\) and \(v_2\).

Then \(\rho_i\)(\(u_1, v_2\)) = \(\rho_i\)(\(u_1, w\)) \& \(\rho_i\)(\(w, v_2\)) ie., either \((u_1, w)\) or \((w, v_2)\) or both are the weakest \(\rho_i\)-edges of the above \(\rho_i\)-cycle because \(P_i\) is as strong as \(u_1, w, v_2\).

This is not possible because \(u_1, w, v_2\) is a strongest \(\rho_i\)-path.

Therefore, \(w\) is a \(\rho_i\)-cut vertex.

Now we prove that the internal vertices of a \(\rho_i\)-tree of a fuzzy \(\rho_i\)-tree are the \(\rho_i\)-cut vertices.

**Theorem 2.5.5.** Let \(\tilde{G}\) be a fuzzy \(\rho_i\)-tree for which \(\tilde{F}_i = (\mu, \tau_1, \tau_2, ..., \tau_k)\) is a partial fuzzy spanning subgraph structure which is a \(\tau_i\)-tree and \(\rho_i(x, y) < \tau_i(x, y)\forall(x, y)\) not in \(\tilde{F}_i\). Then the internal vertices of \(\tilde{F}_i\) are precisely the \(\rho_i\)-cut vertices of \(\tilde{G}\).

**Proof.** Consider a vertex \(w\) of \(\tilde{F}_i\).

Case 1: \(w\) is not an end vertex of \(\tilde{F}_i\)

\(w\) is common to two \(\tau_i\)-edges of \(\tilde{F}_i\) at least and by Theorem 2.5.2, they are \(\rho_i\)-bridges of \(\tilde{G}\). Then by Theorem 2.5.4, \(w\) is a \(\rho_i\)-cut vertex.

Case 2: \(w\) is an end vertex of \(\tilde{F}_i\)

If \(w\) is a \(\rho_i\)-cut vertex, it lies on every strongest \(\rho_i\)-path and hence \(\tau_i\)-path joining \(u\) and \(v\) for some \(u\) and \(v\) in \(V\). One of such \(\tau_i\)-paths lies in \(\tilde{F}_i\). But \(w\) is an end vertex of \(\tilde{F}_i\). So this is not possible. So \(w\) is not a \(\rho_i\)-cut vertex. ie., the internal vertices of \(\tilde{F}_i\) are precisely the \(\rho_i\)-cut vertices of \(\tilde{G}\).

The above theorem leads us to the following corollary.
Corollary 2.5.6. A $\rho_i$-cut vertex, of a fuzzy graph structure $\tilde{G}$ which is a fuzzy $\rho_i$-tree, is common to at least two $\rho_i$-bridges.

2.6 $\rho_{i_1i_2...i_r}$-bridges and $\rho_{i_1i_2...i_r}$-cut vertices

We introduce some more new concepts like $\rho_{i_1i_2...i_r}$-bridges and $\rho_{i_1i_2...i_r}$-cut vertices of a fuzzy graph structure.

Definition 2.6.1. Let $i \in \{i_1, i_2, ..., i_r\}$ where $1 \leq r \leq k$. If $\rho_i^\infty(u,v) < \rho_i^\infty(u,v)$ for some $(u,v) \in \bigcup_{i = i_1}^{i_r} \text{supp}(\rho_i)$, then $(u,v)$ is a $\rho_{i_1i_2...i_r}$-bridge.

We move on to some results using the concept of $\rho_{i_1i_2...i_r}$-bridges.

Theorem 2.6.1. Let $\tilde{G}$ be a fuzzy graph structure. If $(x,y)$ is a $\rho_{i_1i_2...i_r}$-bridge, then $\rho_i^\infty(x,y) < \rho_i(x,y)$ where $(\mu, \rho_1', \rho_2', ..., \rho_k')$ is a partial fuzzy spanning subgraph structure obtained by deleting $(x,y)$, for $i = 1, 2, ..., k$.

Proof. If possible, let $\rho_i^\infty(x,y) \geq \rho_i(x,y)$ for some $\rho_{i_1i_2...i_r}$-bridge $(x,y)$. i.e., there is a $\rho_{i_1i_2...i_r}$-path of strength greater than $\rho_i(x,y)$, $i = \text{some } i_1, i_2, ..., i_r$, $1 \leq r \leq k$ from $x$ to $y$ which does not have the $\rho_i$-edge $i \in \{i_1, i_2, ..., i_r\}$ $(x,y)$. Thus any $\rho_{i_1i_2...i_r}$-path having the $\rho_i$-edge $(x,y)$ as a part of it can be replaced by a $\rho_{i_1i_2...i_r}$-path without the $\rho_i$-edge $(x,y)$ not reducing its strength.

This is a contradiction to the fact that $(x,y)$ is a $\rho_{i_1i_2...i_r}$-bridge.

Hence $\rho_i^\infty(x,y) < \rho_i(x,y)$. \qed

Remark 2.6.1. Converse of the above result also holds. i.e., if $\rho_i^\infty(x,y) < \rho_i(x,y)$ for some $i \in \{i_1, i_2, ..., i_r\}$, $(\rho_i > 0)$ then $(x,y)$ is a $\rho_{i_1i_2...i_r}$-bridge.
Proof. If possible, let \((x, y)\) be not a \(\rho_{i_1,i_2,...,i_r}\)-bridge. Then
\[
\rho_i^\infty(x, y) = \rho_i^\infty(x, y) \geq \rho_i(x, y),
\]
for that \(i \in \{i_1, i_2, ..., i_r\}\) for which \(\rho_i^\infty(x, y) < \rho_i(x, y)\) which is a contradiction to our assumption.

Therefore \((x, y)\) is a \(\rho_{i_1,i_2,...,i_r}\)-bridge.

\[\square\]

**Theorem 2.6.2.** Let \(\tilde{G}\) be a fuzzy graph structure which is a fuzzy \(\rho_{i_1,i_2,...,i_r}\)-forest. Then the \(\rho_i\)-edges, \(i = i_1, i_2, ..., i_r\) of the partial fuzzy spanning subgraph structure \(\tilde{F}_i = (\mu, \tau_1, \tau_2, ..., \tau_k)\) which is a \(\tau_{i_1,i_2,...,i_r}\)-forest, are the \(\rho_{i_1,i_2,...,i_r}\)-bridges of \(\tilde{G}\).

Proof. Case 1: \((x, y)\) is a \(\rho_i\)-edge for some \(i \in \{i_1, i_2, ..., i_r\}\), not in \(\tilde{F}_i\)

By definition of a fuzzy \(\rho_{i_1,i_2,...,i_r}\)-forest, \(\rho_i(x, y) < \tau_i^\infty(x, y) \leq \rho_i^\infty(x, y), \) for that \(i \in \{i_1, i_2, ..., i_r\}\) for which \(\rho_i(x, y) > 0\).

Therefore \((x, y)\) is not a \(\rho_{i_1,i_2,...,i_r}\)-bridge by Theorem 2.6.1.

Case 2: \((x, y)\) is a \(\tau_i\)-edge, \(i \in \{i_1, i_2, ..., i_r\}, 1 \leq r \leq k\) of \(\tilde{F}_i\)

If possible, let \((x, y)\) be not a \(\rho_{i_1,i_2,...,i_r}\)-bridge.

Then there exists a \(\rho_{i_1,i_2,...,i_r}\)-path \(P_i\) from \(x\) to \(y\) not involving \((x, y)\) with strength greater than or equal to \(\rho_i(x, y), i = i_1, i_2, ..., i_r\). So \(P_i\) and \(\tilde{F}_i\) form a \(\rho_{i_1,i_2,...,i_r}\)-cycle.

But \(\tilde{F}_i\) does not contain \(\tau_{i_1,i_2,...,i_r}\)-cycles. Therefore, \(P_i\) contains \(\rho_i\)-edges, for some \(i \in \{i_1, i_2, ..., i_r\}\) not in \(\tilde{F}_i\).

Let \((u, v)\) be such a \(\rho_i\)-edge of \(P_i\) for some \(i \in \{i_1, i_2, ..., i_r\}\).

This can be replaced by a \(\tau_{i_1,i_2,...,i_r}\)-path \(P_i\) in \(\tilde{F}_i\) having strength greater than \(\rho_i(u, v), i = i_1, i_2, ..., i_r\) by definition of a fuzzy \(\rho_{i_1,i_2,...,i_r}\)-forest.

Also \(\rho_i(u, v) \geq \rho_i(x, y)\), \(i \in \{i_1, i_2, ..., i_r\}\)

All \(\tau_i\)-edges, \(i = i_1, i_2, ..., i_r\), of \(P_i\) are stronger than \(\rho_i(u, v), i \in \{i_1, i_2, ..., i_r\}\), which is greater than or equal to \(\rho_i(x, y), i \in \{i_1, i_2, ..., i_r\}\).

Therefore \(P_i\) does not contain \((x, y)\). If it contains \((x, y)\), its strength will be less
than or equal to \( \tau_i(x, y) \) for some \( i \in \{i_1, i_2, ..., i_r\} \), which is less than or equal to \( \rho_i(x, y) \).

Thus we have a \( \tau_{i_1, i_2, ..., i_r} \)-path in \( \tilde{F}_i \) from \( x \) to \( y \) not involving \( (x, y) \).

This gives a \( \tau_{i_1, i_2, ..., i_r} \)-cycle in \( \tilde{F}_i \) and hence a \( \rho_{i_1, i_2, ..., i_r} \)-cycle which is not possible. Hence \( (x, y) \) is a \( \rho_{i_1, i_2, ..., i_r} \)-bridge.

Thus the \( \rho_{i_1, i_2, ..., i_r} \)-edges, \( i \in \{i_1, i_2, ..., i_r\} \) of \( \tilde{F}_i \) are the \( \rho_{i_1, i_2, ..., i_r} \)-bridges of \( \tilde{G} \).

Now, we define a \( \rho_{i_1, i_2, ..., i_r} \)-cut vertex. For that first we define the partial subgraph strucure(\( \mu', \rho'_1, \rho'_2, ..., \rho'_k \)).

**Definition 2.6.2.** A vertex \( w \) of \( \tilde{G} \) is a \( \rho_{i_1, i_2, ..., i_r} \)-cut vertex if
\[
\rho'_i(u, v) < \rho_i(u, v) \quad \text{for some } u, v \text{ for some } i \in \{i_1, i_2, ..., i_r\}, \text{ with } u \neq w \neq v \text{ where } \mu' \text{ and } \rho'_i \text{ are as defined earlier.}
\]

Now we discuss some results on \( \rho_{i_1, i_2, ..., i_r} \)-bridges and \( \rho_{i_1, i_2, ..., i_r} \)-cut vertices.

**Theorem 2.6.3.** Let \( \tilde{G} \) be a fuzzy graph structure with
\[
\tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k)) \text{ a fuzzy } \rho_{i_1, i_2, ..., i_r} \text{-cycle. If a vertex of } \tilde{G} \text{ is a } \rho_{i_1, i_2, ..., i_r} \text{-cut vertex of } \tilde{G}, \text{ then it is a common vertex of two } \rho_{i_1, i_2, ..., i_r} \text{-bridges.}
\]

**Proof.** Consider a \( \rho_{i_1, i_2, ..., i_r} \)-cut vertex \( w \) of \( \tilde{G} \). By the definition of a \( \rho_{i_1, i_2, ..., i_r} \)-cut vertex, there exist two vertices \( u \) and \( v \) different from \( w \) such that \( w \) is on every strongest \( u - v \rho_{i_1, i_2, ..., i_r} \)-path.

Given that \( \tilde{G}^* \) is a fuzzy \( \rho_{i_1, i_2, ..., i_r} \)-cycle. Then there exists only one strongest \( \rho_{i_1, i_2, ..., i_r} \)-path \( P_i \) from \( u \) to \( v \) containing \( w \). All \( \rho_i \)-edges, \( i \in \{i_1, i_2, ..., i_r\} \) of \( P_i \) are \( \rho_{i_1, i_2, ..., i_r} \)-bridges.

So \( w \) is common to two \( \rho_{i_1, i_2, ..., i_r} \)-bridges.

Converse of the above result also holds as is evident from the next theorem.
Theorem 2.6.4. Let $\tilde{G}$ be a fuzzy graph structure. If $w$ is common to at least two $\rho_{i_1,i_2,\ldots,i_r}$-bridges of $\tilde{G}$, then $w$ is a $\rho_{i_1,i_2,\ldots,i_r}$-cut vertex.

Proof. Let $(u_1, w)$ and $(w, v_2)$ be two $\rho_{i_1,i_2,\ldots,i_r}$-bridges with $w$ as the common vertex.

Since $(u_1, w)$ is a $\rho_{i_1,i_2,\ldots,i_r}$-bridge, it is on every strongest $u - v\rho_{i_1,i_2,\ldots,i_r}$-path for some $u$ and $v$.

Case 1: $w \neq u, w \neq v$

In this case, $w$ is on every strongest $u - v\rho_{i_1,i_2,\ldots,i_r}$-path for some $u$ and $v$. Then $w$ is a $\rho_{i_1,i_2,\ldots,i_r}$-cut vertex.

Case 2: Either $w = u$ or $w = v$

In this case either $(u_1, w)$ is on every strongest $u - w\rho_{i_1,i_2,\ldots,i_r}$-path or $(w, v_2)$ is on every strongest $w - v\rho_{i_1,i_2,\ldots,i_r}$-path.

If possible, let $w$ be not a $\rho_{i_1,i_2,\ldots,i_r}$-cut vertex.

By definition of $\rho_{i_1,i_2,\ldots,i_r}$-cut vertex, there exists a strongest $\rho_{i_1,i_2,\ldots,i_r}$-path not containing $w$ between any pair of vertices. Consider such a path $P_i$ joining $u_1$ and $v_2$. Then $P_i, (u_1, w), (w, v_2)$ form a $\rho_{i_1,i_2,\ldots,i_r}$-cycle.

a) Let $u_1, w, v_2$ be not a strongest $\rho_{i_1,i_2,\ldots,i_r}$-path.

Then $(u_1, w)$ or $(w, v_2)$ or both become the weakest $\rho_i$-edges, $i \in \{i_1, i_2, \ldots, i_r\}$, of the above $\rho_{i_1,i_2,\ldots,i_r}$-cycle consisting of $P_i, (u_1, w)$ and $(w, v_2)$ since every $\rho_i$-edge, $i = i_1, i_2, \ldots, i_r$, of $P_i$ will be stronger than $(u_1, w)$ and $(w, v_2)$.

This is not possible since $(u_1, w)$ and $(w, v_2)$ are $\rho_{i_1,i_2,\ldots,i_r}$-bridges.

b) Let $u_1, w, v_2$ also be a strongest $\rho_{i_1,i_2,\ldots,i_r}$-path joining $u_1$ and $v_2$.

Then $\rho_i^\infty(u_1, v_2) = \rho_i(u_1, w) \wedge \rho_i(w, v_2)$ for some $i \in \{i_1, i_2, \ldots, i_r\}$, i.e., either $(u_1, w)$ or $(w, v_2)$ or both are the weakest $\rho_i$-edges, $i \in \{i_1, i_2, \ldots, i_r\}$, of the above $\rho_{i_1,i_2,\ldots,i_r}$-cycle because $P_i$ is as strong as $u_1, w, v_2$. 
This is not possible because $u_1, w, v_2$ is a strongest $\rho_{i_1,i_2,...,i_r}$-path.

Therefore, $w$ is a $\rho_{i_1,i_2,...,i_r}$-cut vertex. \hfill \Box

Now we prove that the internal vertices of a $\rho_{i_1,i_2,...,i_r}$-tree of a fuzzy $\rho_{i_1,i_2,...,i_r}$-tree are the $\rho_{i_1,i_2,...,i_r}$-cut vertices.

**Theorem 2.6.5.** Let $\tilde{G}$ be a fuzzy $\rho_{i_1,i_2,...,i_r}$-tree for which $\tilde{F}_i = (\mu, \tau_{i_1}, \tau_{i_2}, ..., \tau_k)$ is a partial fuzzy spanning subgraph structure which is a $\tau_{i_1,i_2,...,i_r}$-tree and $\rho_i(x,y) < \tau_i^{\infty}(x,y)$ for some $i \in \{i_1,i_2,...,i_r\}$ $\forall (x,y)$ not in $\tilde{F}_i$. Then the internal vertices of $\tilde{F}_i$ are precisely the $\rho_{i_1,i_2,...,i_r}$-cut vertices of $\tilde{G}$.

**Proof.** Consider a vertex $w$ of $\tilde{F}_i$.

**Case 1:** $w$ is not an end vertex of $\tilde{F}_i$

$w$ is common to two $\tau_i$-edges, $i \in \{i_1,i_2,...,i_r\}$, of $\tilde{F}_i$ at least and by Theorem 2.6.2, they are $\rho_{i_1,i_2,...,i_r}$-bridges of $\tilde{G}$. Then by Theorem 2.6.4, $w$ is a $\rho_{i_1,i_2,...,i_r}$-cut vertex.

**Case 2:** $w$ is an end vertex of $\tilde{F}_i$

If $w$ is a $\rho_{i_1,i_2,...,i_r}$-cut vertex, it lies on every strongest $\rho_{i_1,i_2,...,i_r}$-path and hence $\tau_{i_1,i_2,...,i_r}$-path joining $u$ and $v$ for some $u$ and $v$ in $V$. One of such $\tau_{i_1,i_2,...,i_r}$-paths lies in $\tilde{F}_i$. But $w$ is an end vertex of $\tilde{F}_i$. So this is not possible. So $w$ is not a $\rho_{i_1,i_2,...,i_r}$-cut vertex. ie., the internal vertices of $\tilde{F}_i$ are precisely the $\rho_{i_1,i_2,...,i_r}$-cut vertices of $\tilde{G}$. \hfill \Box

The above theorem leads us to the following corollary.

**Corollary 2.6.6.** A $\rho_{i_1,i_2,...,i_r}$-cut vertex, of a fuzzy graph structure $\tilde{G}$ which is a fuzzy $\rho_{i_1,i_2,...,i_r}$-tree, is common to at least two $\rho_{i_1,i_2,...,i_r}$-bridges.
2.7 Regularity in fuzzy graph structures

We recall some concepts introduced by Sampathkumar[61].

**Definition 2.7.1.** [61] The degree of a vertex \( v \) is the number of \( R_i \)-edges incident at \( v \) for various \( i \).

The \( R_i \)-degree of \( v \) is the number of \( R_i \)-edges incident at \( v \).

We define \( R_{i_1i_2...i_r} \)-degree as follows.

**Definition 2.7.2.** The number of \( R_{i_1}−R_{i_2}−...R_{i_r} \)-edges, \( 1 < r < k \), incident at \( v \) is the \( R_{i_1i_2...i_r} \)-degree of \( v \).

We extend the concepts of degree, order and size of a fuzzy graph introduced in [47] by Nagoor Gani and Basheer Ahamed to fuzzy graph structures.

**Definition 2.7.3.** Let \( G \) be a graph structure and \( \tilde{G} \) be a fuzzy graph structure of \( G \).

\( \rho_i \)-degree of a vertex \( u \) is \( d_{\rho_i}(u) = \sum_{(u,v) \in R_i} \rho_i(u,v) \). Note that this is equal to \( \sum_{(u,v) \in \text{supp}(\rho_i)} \rho_i(u,v) \).

Maximum \( \rho_i \)-degree of \( \tilde{G} \) is \( \Delta_{\rho_i}(\tilde{G}) = \bigvee \{d_{\rho_i}(v) : v \in V \} \).

Minimum \( \rho_i \)-degree of \( \tilde{G} \) is \( \delta_{\rho_i}(\tilde{G}) = \bigwedge \{d_{\rho_i}(v) : v \in V \} \).

**Example 4**

Let \( V = \{x_0, x_1, x_2, x_3, x_4, x_5\} \), \( R_1 = \{(x_0, x_1), (x_0, x_2), (x_3, x_4)\} \), \( R_2 = \{(x_1, x_2), (x_4, x_5)\} \), \( R_3 = \{(x_2, x_3)\} \)

and \( G = (V, R_1, R_2, R_3) \) be a graph structure. Let \( \tilde{G} = (\mu, \rho_1, \rho_2, \rho_3) \) where \( \mu(x_0) = 0.8, \mu(x_1) = 0.9, \mu(x_2) = 0.6, \mu(x_3) = 0.5, \mu(x_4) = 0.6, \mu(x_5) = 0.7 \)
\[ \begin{align*}
\rho_1(x_0, x_1) &= 0.8, \rho_1(x_0, x_2) = 0.5, \rho_1(x_3, x_4) = 0.4, \\
\rho_2(x_1, x_2) &= 0.6, \rho_2(x_4, x_5) = 0.6, \\
\rho_3(x_2, x_3) &= 0.3, \rho_3(x_0, x_5) = 0.3
\end{align*} \]

\[ \begin{align*}
d_{\rho_1}(x_0) &= 1.3, d_{\rho_1}(x_1) = 0.8, d_{\rho_1}(x_2) = 0.5, d_{\rho_1}(x_3) = 0.4, d_{\rho_1}(x_4) = 0.4, d_{\rho_1}(x_5) = 0, \\
d_{\rho_2}(x_0) &= 0, d_{\rho_2}(x_1) = 0.6, d_{\rho_2}(x_2) = 0.6, d_{\rho_2}(x_3) = 0, d_{\rho_2}(x_4) = 0.6, d_{\rho_2}(x_5) = 0.6, \\
d_{\rho_3}(x_0) &= 0.3, d_{\rho_3}(x_1) = 0, d_{\rho_3}(x_2) = 0.3, d_{\rho_3}(x_3) = 0.3, d_{\rho_3}(x_4) = 0, d_{\rho_3}(x_5) = 0.3
\end{align*} \]

\[ \begin{align*}
\Delta_{\rho_1}(\tilde{G}) &= 1.3, \Delta_{\rho_2}(\tilde{G}) = 0.6, \Delta_{\rho_3}(\tilde{G}) = 0.3, \\
\delta_{\rho_1}(\tilde{G}) &= 0.4, \delta_{\rho_2}(\tilde{G}) = 0.6, \delta_{\rho_3}(\tilde{G}) = 0.3.
\end{align*} \]

**Definition 2.7.4.** Let \( G \) be a graph structure and \( \tilde{G} \) be a fuzzy graph structure of \( G \).

- \( \rho_{i_1i_2...i_r} \)-degree of a vertex \( u \) is \( d_{\rho_{i_1i_2...i_r}}(u) = \sum_{q=i_1:(u,v) \in \bigcup_{i=1}^{i_r} R_q} \rho_q(u,v) \).
- Maximum \( \rho_{i_1i_2...i_r} \)-degree of \( \tilde{G} \) is \( \Delta_{\rho_{i_1i_2...i_r}}(\tilde{G}) = \bigvee \{d_{\rho_{i_1i_2...i_r}}(v) : v \in V \} \).
- Minimum \( \rho_{i_1i_2...i_r} \)-degree of \( \tilde{G} \) is \( \delta_{\rho_{i_1i_2...i_r}}(\tilde{G}) = \bigwedge \{d_{\rho_{i_1i_2...i_r}}(v) : v \in V \} \).

In the above example, \( d_{\rho_{12}}(x_0) = 1.3, d_{\rho_{12}}(x_1) = 1.4, d_{\rho_{12}}(x_2) = 1.1, d_{\rho_{12}}(x_3) = 0.4, d_{\rho_{12}}(x_4) = 1, d_{\rho_{12}}(x_5) = 0.6. \Delta_{\rho_{12}}(\tilde{G}) = 1.4, \delta_{\rho_{12}}(\tilde{G}) = 0.4. \)

**Definition 2.7.5.** \( \rho_i \)-size of \( \tilde{G} \) is \( S_{\rho_i}(\tilde{G}) = \sum_{(u,v) \in R_i} \rho_i(u,v) \).

- \( \rho_{i_1i_2...i_r} \)-size of \( \tilde{G} \) is \( S_{i_1i_2...i_r}(\tilde{G}) = \sum_{q=i_1:(u,v) \in \bigcup_{i=1}^{i_r} R_q} \rho_q(u,v) \).

Order of \( \tilde{G} \) is \( O(\tilde{G}) = \sum_{u \in V} \mu(u) \).
In the above example, \( S_{\rho_1}(\tilde{G}) = 1.7, S_{\rho_2}(\tilde{G}) = 1.2, S_{\rho_3}(\tilde{G}) = 0.6, \)
\( O(\tilde{G}) = 4.1. \)

Now we extend the concept of regularity in fuzzy graphs discussed by Nagoor Gani and Radha in [48] to fuzzy graph structures.

**Definition 2.7.6.** Let \( G \) be a graph structure and \( \tilde{G} \) be a fuzzy graph structure of \( G \). If \( d_{\rho_i}(v) = p \forall v \in V, \tilde{G} \) is said to be \( p - \rho_i\)-regular.
If \( d_{\rho_{12...r}}(v) = p \forall v \in V, \tilde{G} \) is said to be \( p - \rho_{12...r}\)-regular.

In the above example, \( \tilde{G} \) is 0.3 - \( \rho_3\)-regular and 0.6 - \( \rho_2\)-regular.

Now we prove some results on \( \rho_i\)-regularity and \( \rho_{12...r}\)-regularity.

**Result** \( \tilde{G} \) is \( p - \rho_i\)-regular iff \( \delta_{\rho_i}(\tilde{G}) = \Delta_{\rho_i}(\tilde{G}) = p. \)
\( \tilde{G} \) is \( p - \rho_{12...r}\)-regular iff \( \delta_{\rho_{12...r}}(\tilde{G}) = \Delta_{\rho_{12...r}}(\tilde{G}) = p. \)

**Theorem 2.7.1.** Let \( \tilde{G} \) be a fuzzy graph structure of \( G \) and \( \tilde{G}^* = (V, R_1, R_2, ..., R_k) \) be an odd \( \rho_i\)-cycle. \( \tilde{G} \) is \( \rho_i\)-regular iff \( \rho_i \) is a constant for all \( \rho_i\)-edges in \( R_i. \)

**Proof.** Let \( \rho_i \) be a constant function say \( \rho_i((u, v)) = c_i \forall (u, v) \in R_i. \)
\( \text{ie., } d_{\rho_i}(v) = 2c_i \forall v \in V. \) Therefore \( \tilde{G} \) is \( 2c_i - \rho_i\)-regular.

Conversely, let \( \tilde{G} \) be a \( k - \rho_i\)-regular. Let \( e_1, e_2, ..., e_{2n+1} \) be \( \rho_i\)-edges in \( R_i \) and let
\( \rho_i(e_1) = k_1. \) Then \( \rho_i(e_2) = k - k_1, \rho_i(e_3) = k - (k - k_1) = k_1, ... \)
\( \text{ie., } \rho_i(e_s) = k_1 \text{ if } s \text{ is odd and } \rho_i(e_s) = k - k_1 \text{ if } s \text{ is even.} \)
If \( e_1 \) and \( e_{2n+1} \) are incident with \( u, \) \( d_{\rho_i}(u) = k_1 + k_1 = 2k_1, \)
\( d_{\rho_i}(u) = k \) and so \( k_1 = k/2. \) \( \text{ie., } d_{\rho_i}(e_s) = k/2 \forall s \text{ or } \rho_i \text{ is a constant in } R_i. \)

Note that the above result is true if \( \tilde{G}^* \) is replaced by
\( (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k)). \)
Theorem 2.7.2. Let $\tilde{G}$ be a fuzzy graph structure of $G$ and $\tilde{G}^* = (V, R_1, R_2, \ldots, R_k)$ be an odd $\rho_{i_1 \ldots i_r}$-cycle. $\tilde{G}$ is $\rho_{i_1 \ldots i_r}$-regular iff $\rho_i$ is a constant for all $\rho_i$-edges $(i \in \{i_1, i_2, \ldots, i_r\})$ in $\bigcup_{i=i_1}^{i_r} R_i$.

Proof. Let $\rho_i$ be a constant function say $\rho_i((u, v)) = c\forall(u, v) \in \bigcup_{i=i_1}^{i_r} R_i$.

ie., $d_{\rho_{i_1 \ldots i_r}}(v) = 2c\forall v \in V$. Therefore $\tilde{G}$ is $2c - \rho_{i_1 i_2 \ldots i_r}$-regular.

Conversely, let $\tilde{G}$ be a $k - \rho_{i_1 \ldots i_r}$-regular. Let $e_1, e_2, \ldots, e_{2n+1}$ be $\rho_i$-edges in $\bigcup_{i=i_1}^{i_r} R_i$.

Let $\rho_i(e_1) = k_1$ for some $i \in \{i_1, i_2, \ldots, i_r\}$. Then $\rho_i(e_2) = k - k_1$ for some $i \in \{i_1, i_2, \ldots, i_r\}.

$e_1$ and $e_{2n+1}$ are incident with $u$, $d_{\rho_{i_1 \ldots i_r}}(u) = k_1 + k_1 = 2k_1$.

Therefore $d_{\rho_{i_1 \ldots i_r}}(e_s) = k/2\forall s$ ie., $\rho_i$ is a constant in $\bigcup_{i=i_1}^{i_r} R_i$. \hfill $\Box$

Note that the above result is true if $\tilde{G}^*$ is replaced by $(\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), \ldots, \text{supp}(\rho_k))$.

Theorem 2.7.3. Let $\tilde{G}$ be a fuzzy graph structure of $G$ and $\tilde{G}^* = (V, R_1, R_2, \ldots, R_k)$ be an even $\rho_i$-cycle. Then $\tilde{G}$ is $\rho_i$-regular iff either $\rho_i$ is a constant for all $\rho_i$-edges in $R_i$ or alternate $\rho_i$-edges in $R_i$ have the same membership value.

Proof. Let $\rho_i$ be a constant in $R_i$. Then $d_{\rho_i}(v)$ is a constant for all $v \in V$.

If alternate $\rho_i$-edges in $R_i$ have same membership values, $d_{\rho_i}(v)$ is a constant $\forall v \in V$.

So $\tilde{G}$ is $\rho_i$-regular.

Let $\tilde{G}$ be $k - \rho_i$-regular. Let $e_1, e_2, \ldots, e_k$ be the $\rho_i$-edges of $R_i$. Then $\rho_i(e_j) = k$ if $j$ is odd and $\rho_i(e_j) = k - k_1$ if $j$ is even. If $k_1 = k - k_1$, $\rho_i$ is a constant in $R_i$. If not, alternate $\rho_i$-edges in $R_i$ have same membership value. \hfill $\Box$
Note that the above result is true if $\tilde{G}^*$ is replaced by
$(\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k))$.

**Theorem 2.7.4.** Let $\tilde{G}$ be a fuzzy graph structure of $G$ and $\tilde{G}^* = (V, R_1, R_2, ..., R_k)$ be an even $\rho_{i_1i_2...i_r}$-cycle. Then $\tilde{G}$ is $\rho_{i_1i_2...i_r}$-regular iff either $\rho_i$ is a constant for all $\rho_i$-edge in $\bigcup_{i=1}^{i_r} R_i$ or alternate $\rho_i$-edges in $\bigcup_{i=1}^{i_r} R_i$ have the same membership value.

**Proof.** Let $\rho_{i_1i_2...i_r}$ be a constant in $\bigcup_{i=1}^{i_r} R_i$. Then $d_{\rho_{i_1i_2...i_r}}(v)$ is a constant for all $v \in V$. If alternate $\rho_i$-edges in $\bigcup_{i=1}^{i_r} R_i$ have same membership values, $d_{\rho_{i_1i_2...i_r}}(v)$ is a constant $\forall v \in V$. So $\tilde{G}$ is $\rho_{i_1i_2...i_r}$-regular.

Let $\tilde{G}$ be $k - \rho_{i_1i_2...i_r}$-regular. Let $e_1, e_2, ..., e_k$ be the $\rho_i$-edges $(i \in \{i_1, i_2, ..., i_r\})$ of $\tilde{G}^*$. Then $\rho_i(e_j) = k$ if $j$ is odd and $\rho_i(e_j) = k - k_1$ if $j$ is even $(i \in \{i_1, i_2, ..., i_r\})$. If $k_1 = k - k_1$, $\rho_i$ is a constant for all $\rho_i$-edges in $\bigcup_{i=1}^{i_r} R_i$. If not, alternate $\rho_i$-edges in $\bigcup_{i=1}^{i_r} R_i$ have same membership value. $\square$

**Theorem 2.7.5.** The $\rho_i$-size of a $p - \rho_i$-regular fuzzy graph structure $\tilde{G}$ of $G$ on $G^* = (V, R_1, R_2, ..., R_k)$ is $np/2$ where $n$ is the number of vertices in $V$.

**Proof.** The $\rho_i$-size of $\tilde{G}$ is $S_{\rho_i}(\tilde{G}) = \sum_{(u,v) \in R_i} \rho_i(u,v)$. $d_{\rho_i}(v) = p \forall v \in V$. So $\sum_{v \in V} d_{\rho_i}(v) = 2 \sum_{(u,v) \in R_i} \rho_i(u,v)$. ie., $2S_{\rho_i}(\tilde{G}) = \sum_{v \in V} d_{\rho_i}(v) = \sum_{v \in V} p = np$ where $n$ is the number of vertices in $V$. Therefore $S_{\rho_i}(\tilde{G}) = np/2$. $\square$

Note that the above result is true if $\tilde{G}^*$ is replaced by $(\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k))$. 
Theorem 2.7.6. The $\rho_{i_1i_2...i_r}$-size of a $p-\rho_{i_1i_2...i_r}$-regular fuzzy graph structure $\tilde{G}$ of $G$ on $\tilde{G}^* = (V, R_1, R_2, ..., R_k)$ is $np/2$ where $n$ is the number of vertices in $V$.

Proof. The $\rho_{i_1i_2...i_r}$-size of $\tilde{G}$ is $S_{\rho_{i_1i_2...i_r}}(\tilde{G}) = \sum_{(u,v) \in \bigcup_{q=i_1}^{i_r} R_q} \rho_q(u,v)$.

d_{\rho_{i_1i_2...i_r}}(v) = k \forall v \in V$. So $\sum_{v \in V} d_{\rho_{i_1i_2...i_r}}(v) = 2 \sum_{(u,v) \in \bigcup_{q=i_1}^{i_r} R_q} \rho_q(u,v)$.

ie., $2S_{\rho_{i_1i_2...i_r}}(\tilde{G}) = \sum_{v \in V} d_{\rho_{i_1i_2...i_r}}(v) = \sum_{q=i_1}^{i_r} \rho_q(u,v)$.

Therefore $S_{\rho_{i_1i_2...i_r}}(\tilde{G}) = np/2$.

Note that the above result is true if $\tilde{G}^*$ is replaced by $(\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k))$.

Theorem 2.7.7. Let $\tilde{G}$ be a $\rho_i$-regular fuzzy graph structure of $G$ where $\tilde{G}^*$ is a $\rho_i$-cycle. Then $\tilde{G}$ is a fuzzy $\rho_i$-cycle. It cannot be a fuzzy $\rho_i$-tree.

Proof. Let $\tilde{G}$ be $\rho_i$-regular on $\rho_i$-cycle $\tilde{G}^*$. Then $\rho_i$ is either a constant for all $\rho_i$-edges in $R_i$ or alternate $\rho_i$-edges in $R_i$ have the same membership values.

Therefore there does not exist unique $\rho_i$-edge $(x, y)$ in $R_i$ such that $\rho_i(x, y) = \land \rho_i(u, v)$.

ie., $\tilde{G}$ is a fuzzy $\rho_i$-cycle. So it cannot be a fuzzy $\rho_i$-tree.

Note that the above result is true if $\tilde{G}^*$ is replaced by $(\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k))$.

Theorem 2.7.8. Let $\tilde{G}$ be a $\rho_{i_1i_2...i_r}$-regular fuzzy graph structure of $G$ where $\tilde{G}^*$ is a $\rho_{i_1i_2...i_r}$-cycle. Then $\tilde{G}$ is a fuzzy $\rho_{i_1i_2...i_r}$-cycle. It cannot be a fuzzy $\rho_{i_1i_2...i_r}$-tree.
Proof. Let \( \tilde{G} \) be \( \rho_{i_{1}i_{2}...i_{r}} \)-regular on \( \rho_{i_{1}i_{2}...i_{r}} \)-cycle \( \tilde{G}^{*} \). Then either \( \rho_{i_{2}...r} \) is either a constant for all \( \rho_{i} \)-edges in \( \bigcup_{i=i_{1}}^{i_{r}} R_{i} \) or alternate \( \rho_{i} \)-edges in \( \bigcup_{i=i_{1}}^{i_{r}} R_{i} \) have the same membership values.

Therefore there does not exist unique \( \rho_{i} \)-edge \((i \in \{i_{1}, i_{2}, ..., i_{r}\}) (x, y)\) such that \( \rho_{i}(x, y) = \land\{\rho_{i}(u, v)|\rho_{i}(u, v) > 0\} (i \in \{i_{1}, i_{2}, ..., i_{r}\})\). i.e., \( \tilde{G} \) is a fuzzy \( \rho_{i_{1}i_{2}...i_{r}} \)-cycle. So it cannot be a fuzzy \( \rho_{i_{1}i_{2}...i_{r}} \)-tree.

Note that the above result is true if \( \tilde{G}^{*} \) is replaced by \((\text{supp}(\mu), \text{supp}(\rho_{1}), \text{supp}(\rho_{2}), ..., \text{supp}(\rho_{k}))\).

**Theorem 2.7.9.** A \( \rho_{i} \)-regular fuzzy graph structure \( \tilde{G} \) of \( G \) on an odd \( \rho_{i} \)-cycle does not have a fuzzy \( \rho_{i} \)-bridge. Hence it does not have a fuzzy \( \rho_{i} \)-cut vertex.

**Proof.** Let \( \tilde{G} \) be \( \rho_{i} \)-regular on an odd \( \rho_{i} \)-cycle. Then \( \rho_{i} \) is a constant for all \( \rho_{i} \)-edges in \( R_{i} \) by Theorem 2.7.1. Hence removal of a \( \rho_{i} \)-bridge does not reduce the strength of \( \rho_{i} \)-connectedness between any pair of vertices. Hence \( \tilde{G} \) does not have a fuzzy \( \rho_{i} \)-bridge. A vertex is a \( \rho_{i} \)-cut vertex iff it is common vertex of two \( \rho_{i} \)-bridges by Theorem 2.5.3 and 2.5.4. Therefore \( \tilde{G} \) has no \( \rho_{i} \)-cut vertex. \( \square \)

**Theorem 2.7.10.** A \( \rho_{i_{1}i_{2}...i_{r}} \)-regular fuzzy graph structure \( \tilde{G} \) of \( G \) on an odd \( \rho_{i_{1}i_{2}...i_{r}} \)-cycle does not have a fuzzy \( \rho_{i_{1}i_{2}...i_{r}} \)-bridge. Hence it does not have a fuzzy \( \rho_{i_{1}i_{2}...i_{r}} \)-cut vertex.

**Proof.** Let \( \tilde{G} \) be \( \rho_{i_{1}i_{2}...i_{r}} \)-regular on an odd \( \rho_{i_{1}i_{2}...i_{r}} \)-cycle. Then \( \rho_{i} \) is a constant for all \( \rho_{i} \)-edges in \( \bigcup_{i=i_{1}}^{i_{r}} R_{i} \) by Theorem 2.7.2. Hence removal of a \( \rho_{i_{1}i_{2}...i_{r}} \)-bridge \((i \in \{i_{1}, i_{2}, ..., i_{r}\}) \) does not reduce the strength of \( \rho_{i_{1}i_{2}...i_{r}} \)-connectedness between any pair of vertices. Hence \( \tilde{G} \) does not have a fuzzy \( \rho_{i_{1}i_{2}...i_{r}} \)-bridge \((i \in \{i_{1}, i_{2}, ..., i_{r}\}) \). A vertex is a \( \rho_{i_{1}i_{2}...i_{r}} \)-cut vertex \((i \in \{i_{1}, i_{2}, ..., i_{r}\}) \) iff it is common vertex of two \( \rho_{i_{1}i_{2}...i_{r}} \)-bridges.
(i \in \{i_1, i_2, ..., i_r\}) by Theorem 2.6.3 and 2.6.4.

Therefore \( \tilde{G} \) has no \( \rho_{i_1 i_2 ... i_r} \)-cut vertex. \( \square \)

**Theorem 2.7.11.** Let \( \tilde{G} \) be a \( \rho_i \)-regular fuzzy graph structure of \( G \) on an even \( \rho_i \)-cycle \( \tilde{G}^* = (V, R_1, R_2, ..., R_k) \). Then either \( \tilde{G} \) does not have a fuzzy \( \rho_i \)-bridge or it has \( q_i / 2 \) fuzzy \( \rho_i \)-bridges where \( q_i = |R_i| \).

*Proof.* Let \( \tilde{G} \) be \( \rho_i \)-regular on an even \( \rho_i \)-cycle. Then either \( \rho_i \) is a constant for all \( \rho_i \)-edges in \( R_i \) or alternate \( \rho_i \)-edges in \( R_i \) have the same membership by Theorem 2.7.3.

**Case 1:** \( \rho_i \) is a constant for all \( \rho_i \)-edges in \( R_i \)

Removal of a \( \rho_i \)-edge does not reduce the strength of the \( \rho_i \)-path between any two vertices. Hence \( \tilde{G} \) does not have a \( \rho_i \)-bridge and so does not have a \( \rho_i \)-cut vertex by Theorem 2.5.3.

**Case 2:** Alternate \( \rho_i \)-edges in \( R_i \) have same membership

\( \tilde{G}^* \) is a \( \rho_i \)-cycle. \( \rho_i \)-edges with greater membership are the \( \rho_i \)-bridges of \( \tilde{G}^* \). There are \( q_i / 2 \) such \( \rho_i \)-edges. No vertex is common to two \( \rho_i \)-bridges. Hence it does not have a \( \rho_i \)-cut vertex by Theorem 2.5.3. \( \square \)

**Theorem 2.7.12.** Let \( \tilde{G} \) be a \( \rho_{i_1 i_2 ... i_r} \)-regular fuzzy graph structure of \( G \) on an even \( \rho_{i_1 i_2 ... i_r} \)-cycle \( \tilde{G}^* = (V, R_1, R_2, ..., R_k) \). Then either \( \tilde{G} \) does not have a fuzzy \( \rho_{i_1 i_2 ... i_r} \)-bridge or it has \( q_{i_1 i_2 ... i_r} / 2 \) fuzzy \( \rho_{i_1 i_2 ... i_r} \)-bridges where \( q_{i_1 i_2 ... i_r} = | \bigcup_{i=i_1}^{i_r} R_i | \).

*Proof.* Let \( \tilde{G} \) be \( \rho_{i_1 i_2 ... i_r} \)-regular on an even \( \rho_{i_1 i_2 ... i_r} \)-cycle. Then either \( \rho_i \) is a constant for all \( \rho_i \)-edges in \( \bigcup_{i=i_1}^{i_r} R_i \) or alternate \( \rho_i \)-edges in \( \bigcup_{i=i_1}^{i_r} R_i \) have the same membership by Theorem 2.7.4.
Case 1: \( \rho_{i_1i_2...i_r} \) is a constant for all \( \rho_i \)-edges in \( \bigcup_{i=1}^{i_r} R_i \)

Removal of a \( \rho_i \)-edge \( (i \in \{i_1, i_2, ..., i_r\}) \) does not reduce the strength of the \( \rho_{i_1i_2...i_r} \)-path between any two vertices. Hence \( \tilde{G} \) does not have a \( \rho_{i_1i_2...i_r} \)-bridge and so does not have a \( \rho_{i_1i_2...i_r} \)-cut vertex by Theorem 2.6.3

Case 2: Alternate \( \rho_i \)-edges in \( \bigcup_{i=1}^{i_r} R_i \) have same membership

\( \tilde{G}^* \) is a \( \rho_{i_1i_2...i_r} \)-cycle. \( \rho_i \)-edges \( (i \in \{i_1, i_2, ..., i_r\}) \) with greater membership are the \( \rho_{i_1i_2...i_r} \)-bridges of \( \tilde{G}^* \). There are \( q_{i_1i_2...i_r}/2 \) such \( \rho_i \)-edges \( (i \in \{i_1, i_2, ..., i_r\}) \). No vertex is common to two \( \rho_{i_1i_2...i_r} \)-bridges. Hence it does not have a \( \rho_{i_1i_2...i_r} \)-cut vertex by Theorem 2.6.3.

Note that the above result is true if \( \tilde{G}^* \) is replaced by

\( (\text{supp}(\mu), \text{supp}(\rho_1), \text{supp}(\rho_2), ..., \text{supp}(\rho_k)) \).

**Theorem 2.7.13.** A \( \rho_i \)-connected \( p-\rho_i \)-regular fuzzy graph structure \( \tilde{G} \) of \( G \) where \( p > 0 \) with number of vertices greater than or equal to 3, cannot have an end vertex of \( \rho_i \)-paths.

**Proof.** \( d_{\rho_i}(v) > 0 \forall v \in V \).

Therefore each vertex is adjacent to at least one vertex by a \( \rho_i \)-edge. If possible, let \( u \) be an end vertex of a \( \rho_i \)-path. Let \( (u, v) \in R_i \). \( d_{\rho_i}(u) = p = \rho_i((u, v)) \). \( \tilde{G} \) is \( \rho_i \)-connected and number of vertices greater than or equal to 3. Therefore \( v \) is adjacent to some vertex \( w \neq u \) by a \( \rho_i \)-edge. \( \text{ie.}, d_{\rho_i}(v) \geq \rho_i((u, v)) + \rho_i((v, w)) > \rho_i((u, v)) \).

Therefore \( d_{\rho_i} > p \) which is a contradiction. Therefore \( \tilde{G} \) cannot have an end vertex of \( \rho_i \)-paths.

**Theorem 2.7.14.** A \( \rho_{i_1i_2...i_r} \)-connected \( p-\rho_{i_1i_2...i_r} \)-regular fuzzy graph structure where
\[ p > 0 \text{ with number of vertices greater than or equal to 3, cannot have an end vertex of } \rho_{i_1i_2...i_r}\text{-paths.} \]

**Proof.** \[ d_{\rho_{i_1i_2...i_r}}(v) > 0 \forall v \in V. \]

Therefore each vertex is adjacent to at least one vertex by a \( \rho_i\)-edge \((i \in \{i_1, i_2, ..., i_r\})\).

If possible, let \( u \) be an end vertex of a \( \rho_{i_1i_2...i_r}\)-path. Let \((u, v) \in \bigcup_{q=1}^{i_r} R_q\).

\[ d_{\rho_{i_1i_2...i_r}}(u) = p = \rho_i((u, v)) \text{ for some } i \in \{i_1, i_2, ..., i_r\}. \]

\( \tilde{G} \) is \( \rho_{i_1i_2...i_r}\)-connected and number of vertices greater than or equal to 3. Therefore \( v \) is adjacent to some vertex \( w \neq u \) by a \( \rho_i\)-edge \((i \in \{i_1, i_2, ..., i_r\})\). i.e., \[ d_{\rho_{i_1i_2...i_r}}(v) \geq \rho_i((u, v)) + \rho_i((v, w)) > \rho_i((u, v)) \text{ for some } i \in \{i_1, i_2, ..., i_r\}. \]

Therefore \( \tilde{G} \) cannot have an end vertex of \( \rho_{i_1i_2...i_r}\)-paths.

\[ \square \]

### 2.8 Homomorphism and isomorphism

Isomorphism between two graph structures is introduced by Sampathkumar [61] as follows.

**Definition 2.8.1.** [61] Let \( G = (V, R_1, R_2, ..., R_m) \) and \( H = (V', R'_1, R'_2, ..., R'_n) \) be graph structures. Then \( G \) and \( H \) are isomorphic if \( m = n \) and there exists a bijection \( f : V \to V' \) and a bijection \( \psi : \{R_1, R_2, ..., R_m\} \to \{R'_1, R'_2, ..., R'_n\} \) such that for all \((u, v) \in V, uv \in R_i \) implies \((f(u), f(v)) \in \psi(R_i), i = 1, 2, ..., n. \)

**Definition 2.8.2.** [61] Two graph structures \( G = (V, R_1, R_2, ..., R_n) \) and \( H = (V, R_1, R_2, ..., R_n) \) on the same vertex set \( V \) are identical if there exists a bijection \( f : V \to V \) such that for all \( u \) and \( v \) in \( V, (u, v) \) is an \( R_i\)-edge in \( G \) implies \((f(u), f(v)) \) is an \( R_i\)-edge in \( H, 1 \leq i \leq n \) and their edge sets are equal.
Mordeson & Nair [45] defines isomorphism between fuzzy graphs as follows.

**Definition 2.8.3.** [45] Let \((\mu, \rho)\) and \((\mu', \rho')\) be fuzzy graphs of \(G\) and \(G'\) respectively. Let \(f\) be a one to one function of \(V\) onto \(V'\). Then

1. \(f\) is called a vertex-isomorphism of \((\mu, \rho)\) onto \((\mu', \rho')\) iff \(\forall v \in V, \mu(v) = \mu'(f(v))\).
2. \(f\) is called a line-isomorphism of \((\mu, \rho)\) onto \((\mu', \rho')\) iff \(\forall (u, v) \in X, \rho(u, v) = \rho'(f(u), f(v))\).

If \(f\) is a vertex-isomorphism and a line-isomorphism of \((\mu, \rho)\) onto \((\mu', \rho')\), then \(f\) is called an isomorphism of \((\mu, \rho)\) onto \((\mu', \rho')\).

We define isomorphism between two fuzzy graph structures as follows.

**Definition 2.8.4.** A fuzzy graph structure \(\tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_m)\) of 
\(G = (V, R_1, R_2, \ldots, R_m)\) is \(\rho_i\)-isomorphic to \(\tilde{G}' = (\mu', \rho'_1, \rho'_2, \ldots, \rho'_n)\) of 
\(G' = (V', R'_1, R'_2, \ldots, R'_n)\) iff \(m = n\) and there exists \(f_i : V \to V'\), a 
bijection such that \(\mu(v) = \mu'(f_i(v)) \forall v \in V\) and \(\rho_i(u, v) = \rho'_i((f_i(u), f_i(v)))\).

In particular, if \(V = V'\) and \(\rho'_i = \rho_i\), we say that the above two \(\rho_i\)-isomorphic fuzzy graph structures are \(\rho_i\)-identical.

**Definition 2.8.5.** A fuzzy graph structure \(\tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_m)\) of 
\(G = (V, R_1, R_2, \ldots, R_m)\) is \(\rho_{i_1 \ldots i_r}\)-isomorphic to \(\tilde{G}' = (\mu', \rho'_1, \rho'_2, \ldots, \rho'_n)\) of 
\(G' = (V', R'_1, R'_2, \ldots, R'_n)\) iff \(m = n\) and there exists \(f_{i_1 \ldots i_r} : V \to V'\), a 
bijection such that \(\mu(v) = \mu'(f_{i_1 \ldots i_r}(v)) \forall v \in V\) and \(\rho_i(u, v) = \rho'_i((f_{i_1 \ldots i_r}(u), f_{i_1 \ldots i_r}(v)))\), \(i_1 \leq i \leq i_r\).

In particular, if \(V = V'\) and \(\rho'_{i} = \rho_i, i_1 \leq i \leq i_r\), we say that the above two \(\rho_{i_1 \ldots i_r}\)-isomorphic fuzzy graph structures are \(\rho_{i_1 \ldots i_r}\)-identical.
Definition 2.8.6. A fuzzy graph structure \( \tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_m) \) of \( G = (V, R_1, R_2, ..., R_m) \) is isomorphic to \( \tilde{G}' = (\mu', \rho'_1, \rho'_2, ..., \rho'_n) \) of \( G' = (V', R'_1, R'_2, ..., R'_n) \) iff \( m = n \) and there exists \( f : V \to V' \), a bijection such that \( \mu(v) = \mu'(f(v)) \forall v \in V \) and \( \rho_i(u, v) = \rho'_i((f(u), f(v))) \), \( i = 1, 2, ..., m = n \).

In particular, if \( V = V' \) and \( \rho'_i = \rho_i \forall i = 1, 2, ..., m = n \), we say that the above two isomorphic fuzzy graph structures are identical.

Example 5
Let \( \tilde{G} = (\mu, \rho_1, \rho_2) \) with \( V = \{x_0, x_1, x_2, x_3, x_4, x_5\} \). Let \( \mu(x_0) = 0.8, \mu(x_1) = 0.9, \mu(x_2) = 0.6, \mu(x_3) = 0.5, \mu(x_4) = 0.6, \mu(x_5) = 0.7 \)
\( \rho_1(x_0, x_1) = 0.8, \rho_1(x_0, x_2) = 0.5, \rho_1(x_3, x_4) = 0.4, \)
\( \rho_2(x_1, x_2) = 0.6, \rho_2(x_4, x_5) = 0.5. \)

Let \( \tilde{G}' = (\mu', \rho'_1, \rho'_2) \) with \( V' = \{x'_0, x'_1, x'_2, x'_3, x'_4, x'_5\} \). Let \( \mu'(x'_0) = 0.8, \mu'(x'_1) = 0.9, \mu'(x'_2) = 0.6, \mu'(x'_3) = 0.5, \mu'(x'_4) = 0.6, \mu'(x'_5) = 0.7 \)
\( \rho'_1(x'_0, x'_1) = 0.8, \rho'_1(x'_0, x'_2) = 0.5, \rho'_1(x'_3, x'_4) = 0.4, \)
\( \rho'_2(x'_1, x'_2) = 0.6, \rho'_2(x'_4, x'_5) = 0.5. \)

Define \( f : V \to V' \) as \( f(x_0) = x'_0, ..., f(x_5) = x'_5 \).

Here \( f \) is an isomorphism from \( \tilde{G} \) to \( \tilde{G}' \). Now we introduce homomorphism between two fuzzy graph structures in line with the homomorphism on fuzzy graphs introduced by Bhutani, K.R. in [9].

Definition 2.8.7. A \( \rho_i \)-homomorphism of fuzzy graph structures \( \tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k) \) and \( \tilde{G}' = (\mu', \rho'_1, \rho'_2, ..., \rho'_k) \) of graph structures \( G = (V, R_1, R_2, ..., R_k) \) and \( G' = (V', R'_1, R'_2, ..., R'_k) \) is a map \( f_i : V \to V' \) satisfying \( \mu(v) \leq \mu'(f_i(v)) \forall v \in V \) and \( \rho_i(u, v) \leq \rho'_i((f_i(u), f_i(v))) \forall u, v \in V \).
Definition 2.8.8. A $\rho_{i_1i_2...i_r}$-homomorphism of fuzzy graph structures

$\tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k)$ and $\tilde{G}' = (\mu', \rho'_1, \rho'_2, ..., \rho'_k)$ of graph structures

$G = (V, R_1, R_2, ..., R_k)$ and $G' = (V', R'_1, R'_2, ..., R'_k)$ is a map

$f_{i_1i_2...i_r} : V \to V'$ satisfying $\mu(v) \leq \mu'(f_{i_1i_2...i_r}(v)) \forall v \in V$ and

$\rho_i(u, v) \leq \rho'_i((f_{i_1i_2...i_r}(u), f_{i_1i_2...i_r}(v))) \forall u, v \in V(\bigcup_{i=1}^r \text{supp}(\rho_i)), i \in \{i_1, i_2, ..., i_r\},

1 \leq r \leq k.$

Definition 2.8.9. A homomorphism of fuzzy graph structures

$\tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k)$ and $\tilde{G}' = (\mu', \rho'_1, \rho'_2, ..., \rho'_k)$ of graph structures

$G = (V, R_1, R_2, ..., R_k)$ and $G' = (V', R'_1, R'_2, ..., R'_k)$ is a map $f : V \to V'$ satisfying

$\mu(v) \leq \mu'(f(v)) \forall v \in V$ and $\rho_i(u, v) \leq \rho'_i((f(u), f(v))) \forall u, v \in V$ for $i = 1, 2, ..., k$

We have so far investigated $\rho_i$-trees, $\rho_i$-forests, $\rho_i$-bridges, $\rho_i$-cutvertices, $\rho_i$-regularity, $\rho_i$-homomorphism and $\rho_i$-isomorphism. The analogue of a number of other basic concepts like fuzzy line graphs, domination, co-cycle space etc. remain to be studied. This we are not dealing with at present. We move on to study operations on fuzzy graph structures in the next chapter.