CHAPTER 1

PRELIMINARIES

The main aim of this chapter is to provide the basic definitions and theorems that we will use in different chapters of this work. Proofs of the results are not given. Brouwer’s and Schauder’s fixed point theorems are fundamental theorems in the area of fixed point theory and its applications. Here let us begin our discussion with Brouwer’s fixed point theorem.

1.1 Brouwer’s Theorem and Banach’s contraction principle

Definition (1.1.1): [41]

Let $X$ be any normed linear space and $F$ a map of $X$ or a subset of $X$ into $X$. A point $u \in X$ such that $F(u) = u$ is called a fixed point of $F$.

Fixed points and theorems about them play an important role in many areas of pure and applied Mathematics. Within topology itself there is a large body of work on fixed point results. One of the most well known result is the Brouwer Fixed point theorem. This theorem is named after the Dutch Mathematician L.E.S.Brouwer (1881-1966). The formal definition of Banach spaces is due to Banach himself. But examples like the finite dimensional vector space $R^n$ was studied prior to Banach's formal definition of Banach spaces. In 1912, Brouwer proved the following:

Theorem (1.1.1): [13,58]

Let $B$ be the closed unit ball in $R^n$ and $F$ is continuous mapping of $B$ into
itself. Then $F$ has a fixed point.

This theorem has a long history. The ideas used in its proof were known to Poincare as early as 1886. In 1909, Brouwer proved the theorem when $n = 3$ and in 1910 Hadamard gave the first proof for arbitrary $n$, and Brouwer gave another proof in 1912. All of these are older results than the Banach Contraction Principle. Though in nature the two theorems are different, they bare some similarities. A combination of the two led to the so-called metric fixed point theorem in Banach spaces. Indeed, in Brouwer's theorem the convexity, compactness, and the continuity of $T$ are crucial, while the Lipschitz behavior of the contraction and completeness are crucial in Banach's fixed point theorem. Before considering the Banach’s contraction principle let us consider the following.

**Theorem (1.1.2) : Two Dimensional Brouwer fixed point theorem [13]**

Every continuous function $F : D \rightarrow D$, mapping the disk to itself, has a fixed point.

Now let us consider a practical example of Brouwer’s fixed point theorem.

Imagine taking two pieces of the same sized paper and laying one piece on top of the other. Every point on the top sheet of the paper is associated with some point right below it on the other sheet (Fig 1).
Now take the top sheet of the paper and crumple it up into a ball without ripping it. Place the crumpled ball back on top of the bottom sheet of paper. Somewhere on the crumpled ball of the paper there is a point that is sitting directly above the same point on the bottom sheet of the paper that it sat above before crumpling took place (Fig 2).

(Fig 2)

Suppose we take a map of India and place it on the ground anywhere within India. We assume that India is topologically equivalent to a disk, and we refer to it as $D$. Let $F$ be the function assigning to each point in India the point corresponding to it. We can view $F$ as a continuous function from $D$ to itself. Therefore $F$ must have a fixed point, from which it follows that there must be a point on the ground directly beneath it (Fig 3).

(Fig 3)
In 1932, John Neumann (1903-1957) gave a seminar at Princeton entitled “On a system Economic equations and a generalisations of the Brouwer fixed point theorem”. In it, he outlined how fixed point theory could be utilized to prove the existence of equilibria in economic models. Generalisations and applications of this concept have resulted in Nobel Prizes in Economics for the Mathematician John Nash in 1994.

In 1941, Shizuo Kakutani (1911-2004) proved a generalization of the Brouwer fixed point theorem that has had powerful applications since. Instead of applying to functions from n-ball $B^n$ to itself Kakutani’s fixed point theorem applies to so called set valued functions. Usually, functions associate a point $x$ in a domain $X$ to a point $y$ in the range $Y$. But Kakutani considered functions that take a point $x$ in the domain and send it to a non empty subset $A$ of the range $Y$.

**Definition (1.1.2)** [71]

Let $X$ be a normed linear space. A map $T : X \to X$ is said to be Lipschitzian if there exists a constant $\alpha \geq 0$ such that,

$$\|Tx - Ty\| \leq \alpha \|x - y\| \quad \forall x, y \in X$$

The smallest value of $\alpha$ for which the above inequality holds is said to be Lipschitz constant for $T$ and it is denoted by $L$. If $L < 1$, we say that $T$ is a contraction and if $L = 1$, we say that $T$ is nonexpansive.

**Example (1.1.1):**

Let $X$ be a Banach Space and let $B_r = \{x : \|x\| \leq r\}$. Consider the mapping
$R_r$ defined as $R_r(x) = \begin{cases} x & \text{if } x \in B_r, \\ r x / \|x\| & \text{if } x \notin B_r. \end{cases}$ The mapping $R_r$ is Lipschitzian with Lipschitz constant 2.

Note that a Lipschitzian map is necessarily continuous. However the converse is not true.

**Example (1.1.2):**

Consider $F(x, y) = y^{2/3}$ on $[-1,1] \times [-1,1]$. Then it is continuous. If possible suppose that $F$ is Lipschitzian map. Then there exist $M \geq 0$ such that,

$$|F(x, y_1) - F(x, y_2)| \leq M |y_1 - y_2|, \forall y_1, y_2 \in [-1,1].$$

In particular if $y_2 = 0$,

$$\left|y_1^{2/3} - 0\right| \leq M |y_1 - 0|$$

$$\Rightarrow |y_1|^{\frac{1}{3}} \leq M \forall y_1 \in [-1,1]$$

Which is a contradiction to the fact that $|y_1|^{\frac{1}{3}} \to \infty$ as $y_1 \to 0$

Hence $F$ is not Lipschitzian.

Now let us state Banach’s contraction principle.

**Theorem (1.1.3):**[41,47,71]

Let $F$ be a contraction mapping of a complete metric space $X$ into itself.

Then there is a unique point $u \in X$ such that $F(u) = u$. 

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Perhaps, the above theorem proved by Stefan Banach is the most celebrated theorem in the theory of fixed points. The following theorem gives an interesting result since the mapping $T$ is not even assumed to be continuous.

**Theorem (1.1.4):**

Suppose $(M,d)$ is a complete metric space and suppose $T : M \rightarrow M$ is a mapping for which $T^N$ is a contraction for some positive integer $N$. Then $T$ has a unique fixed point.

A probable question that may arise is whether it is possible to weaken the contraction assumption and still obtain the existence of fixed points. The functions naturally arises when we try to extend the contraction mapping are non expansive and contractive mappings.

**Definition (1.1.3):**

Let $(X,d)$ be a metric space. $T : X \rightarrow X$ is said to be a contractive map if $d(Tx,Ty) < d(x,y)$, for each $x, y \in M$ with $x \neq y$.

But we can see that see that Banach Contraction Principle does not even extend to the class of nonexpansive or contractive mapping.

**Example (1.1.3):**

1. The function $F(x) = x + 1$ for $x \in R$ is a nonexpansive map which has no fixed point in $R$. 

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2. The function \( f(x) = \ln(1 + e^x) \) for \( x \in \mathbb{R} \) is a contractive map which has no fixed point in \( \mathbb{R} \).

3. Consider the space \( X = \mathbb{R}^+ \) with the metric \( d(x, y) = |x - y| \) and let \( F \) be defined as \( F(x) = \left( x^2 + 1 \right)^{\frac{1}{2}} \). Then \( F \) is a continuous mapping satisfying the relation \( d(Fx, Fy) < d(x, y) \) . But \( F \) has no fixed point.

4. Let \( X = [0,1] \) with the induced metric \( R \) and consider \( F \) defined as \( F(x) = \frac{x}{2} \).

Then \( F \) has no fixed point. Note that \( F \) is a contraction. This shows that the contraction mappings defined on incomplete metric spaces fails to have fixed points.

Next theorem gives a class of spaces under which contractive mappings has fixed point.

**Theorem (1.1.5):**[47]

Let \( (M, d) \) be a compact metric space and let \( T : M \to M \) be a contractive mapping. Then \( T \) has a unique fixed point \( x_0 \) and moreover, for each \( x \in M \), \( \lim_{n \to \infty} T^n(x) = x_0 \).

Then the question arises, Can we find fixed points for nonexpansive mapping? The answer is yes. Several authors proved existence of fixed points for non expansive mapping under different situations.
**Theorem (1.1.6): Schauder’s theorem**[71]

Let $C$ be a non empty, closed, convex subset of a normed linear space $X$ with $F : C \rightarrow C$ nonexpansive and $F(C)$ a subset of a compact subset of $C$. Then $F$ has a fixed point.

**Theorem (1.1.7):**[71]

Let $C$ be a non empty, closed, bounded, convex set in a real Hilbert space $H$. Then each nonexpansive map $F : C \rightarrow C$ has a fixed point.

This theorem was proved independently by Browder, Gohde and Kirk in 1965.

**Example (1.1.4):**

Notice that uniqueness need not hold as the example $F(x) = x, \ x \in C = [0,1]$ shows.

**Definition (1.1.4):**[75]

A Banach space $X$ has the fixed point property (F.P.P) if given any nonempty closed bounded convex subset $C$ of $X$, every nonexpansive mapping from $C$ to $C$ has a fixed point.

Over the last fifty years, many authors have given generalizations of Banach’s contraction principle. The first such generalization to receive significant attention is the following result of Rakotch.
Theorem (1.1.8): [47]

Let $M$ be a complete metric space and suppose $T : M \to M$ satisfies,

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad \text{for each } x, y \in M,$$

where $\alpha : [0, +\infty) \to [0, 1)$ is monotonically decreasing. Then $T$ has a unique fixed point $\bar{x}$ and $\{T^n x\}$ converges to $\bar{x}$ for each $x \in M$.

Theorem (1.1.9): [71]

Let $(X, d)$ be a complete metric space and let $F : X \to X$ be such that,

$$d(Fx, Fy) \leq \phi(d(x, y)) \quad \forall x, y \in X,$$

where $\phi : [0, \infty) \to [0, \infty)$ is any monotonic, non-decreasing (not necessarily continuous) function with $\lim_{n \to \infty} \phi^n(t) = 0$ for any fixed $t > 0$. Then $F$ has a unique fixed point $u \in X$, with $\lim_{n \to \infty} F^n x = u$ for each $x \in X$.

In 1969 and 1971, R.Kannan [44,45] proved some fixed point theorems for operator $T$ mapping a Banach space $X$ into itself which satisfies the condition

$$\|Tx - Ty\| \leq \alpha \left( \|x - Tx\| + \|y - Ty\| \right) \quad \forall x, y \in X$$

where $0 < \alpha < \frac{1}{2}$. The most interesting fact about Kannan mappings are it need not be even continuous.

In 1972, Chatterge [53] proved that an operator $T$ on a complete metric space satisfying the condition, $d(Tx, Ty) \leq \alpha \left[ d(x, Ty) + d(y, Tx) \right] \quad \forall x, y \in X, \alpha \in \left[ 0, \frac{1}{2} \right)$ has a fixed point.
In 1971, Reich[53] considered operators $T$ on a complete metric space $X$ satisfying 

$$d(Tx, Ty) \leq a_1d(x, y) + a_2d(x, Tx) + a_3d(y, Ty) \quad \forall x, y \in X \text{ and } a_1 + a_2 + a_3 < 1$$

and proved that such mappings has fixed point.

In 1973 G.Hardy and T.Rogers[36] generalized Kannan fixed point theorem. They proved that mappings $T$ of a complete metric space $X$ into itself that satisfy

$$d(Tx, Ty) \leq a_1d(x, y) + a_2d(x, Tx) + a_3d(y, Ty) + a_4d(x, Ty) + a_5d(y, Tx)$$

$$\forall x, y \in X, a_i \geq 0 \text{ and } a_1 + a_2 + a_3 + a_4 + a_5 < 1$$

has a fixed point.

In 1986 Lucimar Nova[53] proved that if $K$ is a closed subset of a Banach space $X$, $T$ a mapping from $K$ to $K$ satisfying

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + \|y - Ty\|, \quad \forall x, y \in K, 0 \leq a, b \leq 1, \text{ then } \; T \text{ has a unique fixed point.}$$

Now let us consider the following examples.

**Example (1.1.5):**

i. Consider $T : [0,1] \to \mathbb{R}$ by $Tx = \begin{cases} \frac{x}{4}, & x \in \left[0, \frac{1}{2}\right) \\ \frac{x}{5}, & x \in \left[\frac{1}{2}, 1\right] \end{cases}$. Then $T$ is discontinuous and $T$ is not a contraction. But it is Kannan mapping.
ii. Consider \( T : [0,1] \rightarrow [0,1] \) defined by \( Tx = \frac{x}{3} \). Then \( T \) is contraction but not a Kannan mapping.

iii. Consider \( T : [0,1] \rightarrow [0,1] \) defined by

\[
Tx = \begin{cases} 
\frac{7}{20} x, & x \in \left[ 0, \frac{1}{2} \right) \\
\frac{3}{10} x, & x \in \left[ \frac{1}{2}, 1 \right]
\end{cases}
\]

Then \( T \) is discontinuous, but it is not a contraction mapping and not a Kannan mapping.

iv. Consider \( T : R \rightarrow R \) defined by \( Tx = -\frac{x}{2} \). It is Kannan mapping but not a Chatterje mapping.

v. Consider \( T : [0,1] \rightarrow [0,1] \) by

\[
Tx = \begin{cases} 
\frac{x}{2}, & x \in [0,1) \\
0, & x = 1
\end{cases}
\]

Then \( T \) is a Chatterje operator but not a Kannan operator.

A number of results were proved by several authors by extending the above operators. In chapter 4 we will prove a common fixed point theorem for non expansive type mappings.

1.2 Normal Structure

W.A.Kirk is the first mathematician who identified the property of ‘normal structure’ and the role played by weak compactness. A detailed study about normal structure can be seen in [28] and [70].
**Definition (1.2.1):** [28]

Let $X$ be a Banach space and $K$ be a closed convex subset of $X$. Then we say that $K$ has a normal structure if any bounded convex subset $H$ of $K$ which contains more than one point contains a nondiametral point. That is, there exist a point $x_0 \in H$ such that,

$$\sup \{ \|x_0 - x\| : x \in H \} < \text{diam}(H) = \sup \{ \|x - y\| : x, y \in H \} = \infty.$$

For $D \subset X$ let

$$r_x(D) = \sup \{ \|x - v\| : v \in D \}$$

$$r(D) = \inf \{ r_x(D) : x \in D \}$$

If $X$ is reflexive and if $D$ is bounded closed and convex, then weak compactness of closed balls in $X$ yields the fact that the set

$$C(D) = \{ z \in D : r_z(D) = r(D) \}$$

is a non empty closed convex subset of $D$. The number $r(D)$ is called Chebyshev radius and the set $C(D)$ is called Chebyshev centre of $D$.

Now let us discuss following theorems on normal structure which provides examples for spaces having normal structure and which gives conditions for existence of fixed points for nonexpansive mappings.
Theorem (1.2.1): Kirk's theorem [70]

Let $X$ be a Banach space and suppose that $C$ is a nonempty weakly compact convex subset of $X$ which has the normal structure. Then any nonexpansive mapping $T : C \to C$ has a fixed point.

Theorem (1.2.2): [28]

Let $X$ be a reflexive Banach space and suppose $K$ is a bounded closed convex subset of $X$ which has normal structure. Then any nonexpansive mapping $T : K \to K$ has a fixed point. In particular if $X$ has normal structure then $X$ has F.P.P.

It has been known for some time that even in reflexive spaces, normal structure is not essential for F.P.P. An example is provided by the spaces $X_\beta, \beta > 0$ defined by

$$X_\beta = \left\{ x \in l_2 : \| x \|_\beta = \max \left\{ \| x \|_2, \| x \|_\infty \right\} \right\}.$$  

$X_\beta$ is reflexive since it is isomorphic to the Hilbert space $l_2$ and it has normal structure iff $\beta < \sqrt{2}$.

Theorem (1.2.3): [70]

Every compact convex subset $C$ of a Banach space $X$ has normal structure.

Theorem (1.2.4): [70]

Every finite dimensional Banach space has normal structure.
Theorem (1.2.5): [70]

Every closed convex bounded subset C of a uniformly convex Banach space X has normal structure.

Theorem (1.2.6): [70]

Every uniformly convex Banach space has normal structure.

The following example gives a space which does not have normal structure.

Example (1.2.1):

The space $C[0,1]$ of continuous real-valued functions with “sup” norm does not have normal structure.

Definition (1.2.2): [70]

A non empty convex subset $C$ of a Banach space is said to have uniformly normal structure if there exists a constant $\alpha \in (0,1)$, independent of $C$, such that each closed convex bounded subset $D$ of $C$ with $\text{diam}(D) > 0$ contains a point $x_0 \in C$ such that $\sup \{ \|x_0 - x\| : x \in D \} \leq \alpha \text{diam}(D)$.

Theorem (1.2.7): [70]

Every uniformly convex Banach space $X$ has uniformly normal structure.

Definition (1.2.3): [70]

Let $X$ be a Banach space. Then the number $N(X)$ is said to be the normal
structure coefficient if \( N(X) = \inf \left\{ \frac{\text{diam}(C)}{r(C)} \right\} \) where the infimum is taken over all closed convex bounded subsets \( C \) of \( X \) with \( \text{diam}(C) > 0 \).

It is clear that \( N(X) \geq 1, N(X) > 1 \) iff \( X \) has uniformly normal structure.

**Example (1.2.2):**

For a Hilbert space \( H \), \( N(H) = \sqrt{2} \).

The concept of asymptotic normal structure was introduced by Baillon and Schoneberg in 1981[3]. A Banach space \( X \) has asymptotic normal structure if each non empty bounded closed and convex subset \( K \) of \( X \) which contains more than one point has the property, if \( \{x_n\} \subset K \) satisfies \( \|x_n - x_{n+1}\| \to 0 \) then there exists \( x \in K \) such that \( \liminf_{n \to \infty} \|x_n - x\| < \text{diam}(K) \).

Baillon and Schoneberg proved the following theorem.

**Theorem (1.2.8):** [3]

In a reflexive Banach space, asymptotic normal structure \( \Rightarrow \) F.P.P.

**1.3 Definitions and examples of different types of mappings**

In this section we consider different types of mappings that we will use in our discussion.
Definition (1.3.1): [41]

Let $X, Y$ be topological space. A map $F : X \to Y$ is called compact if $F(X)$ is contained in a compact subset of $Y$. \[ \square \]

Definition (1.3.2): [41]

Let $X$ and $Y$ be metric spaces and $F : X \to Y$. $F$ is completely continuous if the image of each bounded set in $X$ is contained in a compact subset of $Y$. \[ \square \]

Example (1.3.1):

1. Let $X$ be a Banach space, $U \subset X$ open and $F : U \to X$ a compact map that has a derivative $G = F'(x_0)$ at a point $x_0 \in U$. Then $G$ is a completely continuous linear map from $X$ to $X$. \[ \square \]

2. For some fixed $g \in C \left( [0,1]; R \right)$ define the linear operator $T$ by,

$$ (Tf)(x) = \int_0^x f(t) g(t) \, dt, $$

more generally if $\Omega$ is any domain in $R^n$, and the integral kernel $k : \Omega \times \Omega \to R$ is a Hilbert Schimdt Kernel, the operator $T$ on $L^2(\Omega,R)$ defined by,

$$ (Tf)(x) = \int_{\Omega} k(x,y) f(y) \, dy $$

is a completely continuous operator. \[ \square \]
Now we state non-linear alternative of Leray-Schauder type for compact and completely continuous operators.

**Theorem (1.3.1):** [41]

Let $X$ be a normed linear space, $C \subset E$ be a convex set and let $U$ be open in $C$ such that $0 \in U$. Then each compact map $F: \overline{U} \to C$ has at least one of the following two properties,

(a) $F$ has a fixed point,

(b) there exists $x \in \partial U$ and $\lambda \in (0,1)$ such that $x = \lambda Fx$ where $\partial U$ denote boundary of $U$.

Many fixed point theorems can be derived from the nonlinear alternative by imposing conditions that prevent occurrence of the second property. The nonlinear alternative, applied to completely continuous operators yields the following theorem.

**Theorem (1.3.2):** [41]

Let $C$ be a convex subset of a normed linear space $X$ and assume $0 \in C$. Let $F: C \to C$ be a completely continuous operator and let,

$$\varepsilon(F) = \{ x \in C : x = \lambda F(x) \text{ for some } 0 < \lambda < 1 \}.$$ 

Then either $\varepsilon(F)$ is unbounded or $F$ has a fixed point.

**Definition (1.3.3):** [58]

Let $X$ and $Y$ be normed spaces and $F: X \to Y$. $F$ is weakly continuous at
\( x_0 \in X \) if, for any sequence \( \{ x_n \} \) which converges weakly to \( x_0 \), the sequence \( \{ Fx_n \} \) converges weakly to \( Fx_0 \).

**Theorem (1.3.3):** \([58]\)

Let \( X \) be a reflexive Banach space, \( K \) a closed convex subset of \( X \) and \( F \) a weakly continuous mapping of \( K \) into a bounded subset of \( K \). Then \( F \) has a fixed point in \( K \).

Now let us recall the notion of quasi-bounded operator defined by A. Granas in 1962 and used in the fixed point theory.

**Definition (1.3.4):** \([38]\)

Let \( X \) be a normed space and \( F : X \to X \). \( F \) is quasi bounded if

\[
\limsup_{\| x \| \to \infty} \frac{\| Fx \|}{\| x \|} < \infty \quad \text{and it is strictly quasi bounded if} \quad \limsup_{\| x \| \to \infty} \frac{\| Fx \|}{\| x \|} < 1.
\]

**Example (1.3.2):**

1. If \( X \) is a Banach space and \( F : X \to X \) is linear continuous then we have

\[
\limsup_{\| x \| \to \infty} \frac{\| Fx \|}{\| x \|} = \| F \| < \infty.
\]

2. We say that a mapping \( F : X \to X \) satisfies condition (BN) (Brezis-Nirenberg) if

\[
\lim_{\| x \| \to \infty} \frac{\| Fx \|}{\| x \|} = 0.
\]

Any operator satisfying this condition (Example: constant function on \( \mathbb{R} \)) is a quasi bounded operator.
The following theorem gives a necessary and sufficient condition for an operator to become quasi bounded.

**Theorem (1.3.4): [85]**

A mapping $F : X \to X$ is quasi bounded iff there exist $\rho > 0$ and two positive constants $M_1$ and $M_2$ such that $\|F(x)\| \leq M_1 \|x\| + M_2$ for any $x \in X$ with $\|x\| > \rho$. □

**Theorem (1.3.5): [85]**

If there exists a mapping $\varphi : R_+ \to R$ such that $\lim_{t \to \infty} \frac{\varphi(t)}{t} = 0$ then any mapping $F : X \to X$ such that for any $x \in X$, $\|F(x)\| \leq M \|x\| + \varphi(\|x\|)$ with $M > 0$ is a quasi bounded mapping. □

Another result obtained by applying the non linear alternative to quasi bounded operators is,

**Theorem (1.3.6):[41]**

Let $X$ be a normed linear space and $F : X \to X$ be a quasi bounded completely continuous operator. Then for each real $|\lambda| < \frac{1}{\|F\|}$ (and for all real $\lambda$ whenever $\|F\| = 0$) the operator $\lambda F$ has at least one fixed point. More generally for each $y \in E$ and $|\lambda| < \frac{1}{\|F\|}$ the equation $y = x - \lambda F(x)$ has at least one solution. □

In 1965, Browder in fact discovered something truly amazing. Let $C$ be a bounded closed convex subset of a uniformly convex Banach space $X$ and $T : C \to C$
be nonexpansive. If \( \{x_n\} \subset C \) converges weakly to \( x \) and \( \{x_n - Tx_n\} \) converges strongly (with respect to the norm) to 0, then we have \( x - Tx = 0 \). This is known as the Demiclosedness principle.

**Definition (1.3.5):** [40]

Let \( X \) be a normed linear space and let \( T : X \to X \). Then \( T \) is demi-closed if for any sequence \( \{x_n\} \) weakly convergent to an element \( x_* \) with \( \{T(x_n)\} \) norm-convergent to an element \( y_* \), then \( T(x_*) = y_* \). \( \blacksquare \)

From the Browder’s discovery it is clear that if \( K \) is a closed convex subset of a uniformly convex Banach space \( X \) and \( T : K \to X \) a nonexpansive map then \( I - T \) is demiclosed.

Another definition that we require in our discussion is of ‘eigenvalues’. The term “eigenvalue” is a partial translation of the German “eigenvort”. A complete translation would be something like “own value” or “characteristic value”, but they are rarely used.

**Definition (1.3.6):** [4,40]

Let \( X \) be a normed linear space. A scalar \( \lambda \) is said to be an eigenvalue of \( T : K \to X \) if there exist an element \( 0 \neq x \in X \) such that \( Tx = \lambda x \). \( \blacksquare \)

### 1.4 \( F \)-norm and \( D \)-metric

A linear topological space \( X \) is metrizable if and only if it has a countable base of neighbourhoods of zero. The topology of a metrizable topological vector
space can be defined by a real-valued function called F-norm which can be defined as,

**Definition (1.4.1):** [1]

\[ F-\text{norm is a real valued function } \| \| : X \to R \text{ such that } \forall x, y \in X, \]

\begin{align*}
1. \quad & \|x\| \geq 0 \\
2. \quad & \|x\| = 0 \Rightarrow x = 0 \\
3. \quad & \|x + y\| \leq \|x\| + \|y\| \\
4. \quad & \|\lambda x\| \leq \|x\| \quad \forall \lambda \in K \text{ with } |\lambda| \leq 1 \\
5. \quad & \text{If } \lambda_n \to 0 \text{ and } \lambda_n \in K, \text{ then } \|\lambda_n x\| \to 0. \quad \blacksquare
\end{align*}

In all our discussion in section (4.1) \( F \) will denote an \( F \)-norm unless otherwise specified. If \( C \) is a closed convex subset of a Banach space \( X \) and if \( T : C \to C \) is a linear mapping such that \( \|Tx - Ty\| \leq a\|x - y\| + b\|Tx - x\| + c\|Ty - y\| \forall x, y \in C \) where \( 0 < a < 1, b \geq 0, c \geq 0 \) and \( a + b + c = 1 \), Gregus [35] proved that \( T \) has a unique fixed point. J.O. Olaleru and H. Akewe [66] generalized the result to a complete metrizable topological vector space using the condition that,

\[ F(Tx - Ty) \leq aF(x - y) + bF(x - Tx) + cF(y - Ty) + eF(y - Tx) + fF(x - Ty), \forall x, y \in C, \]
where \(0 < a < 1,b \geq 0,c \geq 0,e \geq 0,f \geq 0\) and \(a+b+c+e+f=1\). In (4.1) we derive common fixed points for Gregus type mapping which generalizes the result proved in [66].

In 1984, Dhage[9] introduced a generalization of metric space which is called generalized metric space or \(D\)-metric space, and proved the existence of unique fixed point of a self map satisfying a contractive condition. In 1996 Rhoades[72] generalized Dhage’s contractive condition and obtained some fixed point theorems.

Now let us state about the concept of \(D\)-metric.

**Definition (1.4.2):** [9]

Let \(X\) be a non empty set. \(D\)-metric is a function, \(D : X \times X \times X \rightarrow \mathbb{R}\) satisfying the properties,

1. \(D(x, y, z) \geq 0\) \(\forall x, y, z \in X\) and equality holds iff \(x = y = z\).

2. \(D(x, y, z) = D(x, z, y) = D(y, x, z) = D(z, x, y) = D(z, y, x)\) \(\forall x, y, z \in X\)

3. \(D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)\) \(\forall x, y, z, a \in X\).

The non empty set \(X\) together with the metric \(D\) is called a \(D\)-metric space.

Geometrically, in plane, \(D\)-metric represents perimeter of the triangle. Like ordinary metric several \(D\)-metrics can be defined on a single non empty set, thus obtaining different \(D\)-metric spaces.

**Example (1.4.1):**

1. Let \(X\) be a metric space with the metric \(D\) defined by,

\[
D(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\} \quad \forall x, y, z \in X .
\]

2. Let \(X\) be a metric space with the metric \(D\) defined by,
\[ D(x, y, z) = d(x, y) + d(y, z) + d(x, z) \quad \forall x, y, z \in X. \]

3. Let \( X \) be an arbitrary non empty set. Define \( D \) by

\[ D(x, y, z) = \begin{cases} 
0 & , x = y = z \\
1 & , x \neq y \neq z 
\end{cases} \quad \text{whenever } x, y, z \in X. \]

This example shows that every nonempty set can be considered as \( D \)-metric space as is the case of ordinary metric on \( X \).

4. Given a \( D \)-metric \( D \) on \( X \) we define another \( D \) metric \( D' \) on \( X \) using \( D \) as is the case of ordinary metric on \( X \) by,

\[ D'(x, y, z) = \frac{D(x, y, z)}{1 + D(x, y, z)}. \]

**Definition (1.4.3):** [62]

A sequence \( (x_n) \) in \( X \) is called a \( D \)-Cauchy sequence if for each \( \varepsilon > 0 \), there exist a positive integer \( n_0 \) such that \( \forall m > n, p \geq n_0, D(x_m, x_n, x_p) < \varepsilon \).

**Definition (1.4.4):** [62]

A sequence \( \{x_n\} \) in \( X \) is said to be \( D \)-convergent to a point \( x \in X \) if \( \forall \varepsilon > 0 \), there exist a positive integer \( n_0 \) such that \( \forall m, n \geq n_0, D(x_m, x_n, x) < \varepsilon \).

Dhage himself and several other authors have proved fixed point theorems on \( D \)-metric spaces.

**Theorem (1.4.1):**[18]

Let \( (X,D) \) be a complete bounded \( D \)-metric space and \( T \) be a self map of \( X \) satisfying the following condition:

there exists a \( k \in [0, 1) \) such that for all \( x, y, z \in X \),

\[ D(Tx, Ty, Tz) \leq k \max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)\} \]

Then \( T \) has a unique fixed point \( p \) in \( X \) and \( T \) is continuous at \( p \).
Theorem (1.4.2): [72]

Let \((X,D)\) be a compact \(D\)-metric space and \(T\) be a continuous self map of \(X\) satisfying for all \(x, y, z \in X\) with \(D(x, y, z) \neq 0\),

\[
D(Tx, Ty, Tz) < \max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)\}
\]

Then \(T\) has a unique fixed point \(p\) in \(X\). □