Chapter 6

Forced oscillation of solutions of a nonlinear fractional partial differential equation

6.1 Introduction

The forced oscillation of solutions of partial differential equations has been studied by several authors [30, 46, 47, 58, 75, 80]. Shoukaku [76] studied the forced oscillation of nonlinear hyperbolic equations with functional arguments and Li et al. [43] obtained results on the forced oscillation of solutions for systems of impulsive parabolic differential equations with several delays. Grace et al. [34] proved the oscillation of fractional differential equations. In [15], Chen et al. considered the forced oscillation of certain fractional differential equation. Feng et al. [26] discussed the oscillation of solutions to nonlinear forced fractional differential equations. In this chapter, we establish the sufficient conditions for the forced oscillation of solutions of nonlinear fractional partial differential equations.
Now, we consider the nonlinear fractional partial differential equation with forced term of the form

$$D_{\alpha,t}^\alpha \left( r(t) D_{\alpha,t}^\alpha u(x,t) \right) + q(x,t) f(u(x,t)) = a(t) \Delta u(x,t) + g(x,t), \quad (x,t) \in G$$  \hspace{1cm} (6.1.1)

with the Neumann boundary condition

$$\frac{\partial u(x,t)}{\partial N} = 0, \quad (x,t) \in \partial \Omega \times \mathbb{R}_+,$$  \hspace{1cm} (6.1.2)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with piecewise smooth boundary \( \partial \Omega \), \( \alpha \in (0,1) \) is a constant, \( G = \Omega \times \mathbb{R}_+ \), \( \mathbb{R}_+ = (0, \infty) \), \( D_{\alpha,t}^\alpha u \) is the Riemann-Liouville fractional derivative of order \( \alpha \) of \( u \) with respect of \( t \), \( \Delta \) is the Laplacian operator and \( N \) is the unit exterior normal vector to \( \partial \Omega \).

### 6.2 Preliminaries

To establish our main theorems we need the following assumptions.

\((A_1)\) \( r \in C^\alpha((0, \infty); \mathbb{R}_+) \) such that \( \int_0^\infty \frac{1}{r(t)} dt = \infty \) and \( a \in C((0, \infty); \mathbb{R}_+) \);

\((A_2)\) \( q \in C(\overline{G}; \mathbb{R}_+) \) and \( \min_{x \in \Omega} q(x,t) = Q(t) \);

\((A_3)\) \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function such that \( xf(x) > 0 \) for all \( x \neq 0 \).

\((A_4)\) \( g \in C(\overline{G}; \mathbb{R}) \) such that \( \int_\Omega g(x,t) dx \leq 0 \).

By a solution of (6.1.1) we mean a function \( u \in C^{2 \alpha}(\overline{\Omega} \times [0, \infty)) \) and satisfies (6.1.1) on \( \overline{G} \).

A solution \( u \) of (6.1.1) is said to be oscillatory in \( G \) if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (6.1.1) is said to be oscillatory if all its solutions are oscillatory.
The following notations will be used for our convenience

\[ v(t) = \int_\Omega u(x, t) \, dx, \quad \xi = \frac{t^\alpha}{\Gamma(1 + \alpha)}, \quad \bar{c}(\xi) = c(t), \quad \bar{r}(\xi) = r(t) \]

\[ \tilde{\sigma}(\xi) = \sigma(t), \quad \tilde{Q}(\xi) = Q(t), \quad \xi_0 = \frac{t_0^\alpha}{\Gamma(1 + \alpha)}, \quad \xi_1 = \frac{t_1^\alpha}{\Gamma(1 + \alpha)}. \]

**Lemma 6.2.1.** Let

\[ F(t) := \int_0^t (t - \nu)^{-\alpha} y(\nu) d\nu \quad \text{for} \quad \alpha \in (0, 1) \quad \text{and} \quad t > 0. \]

Then \( F'(t) = \Gamma(1 - \alpha)(D_+^\alpha y)(t) \).

### 6.3 Main results

**Theorem 6.3.1.** If the fractional differential inequality

\[ D_+^\alpha \left[ r(t) D_+^\alpha v(t) \right] + Q(t) f(v(t)) \leq 0 \quad (6.3.1) \]

has no eventually positive solution, then every solution of (6.1.1) and (6.1.2) is oscillatory in \( G \).

**Proof.** Suppose that \( u \) is a nonoscillatory solution of (6.1.1) and (6.1.2). Without loss of generality we may assume that \( u(x, t) > 0 \) in \( \Omega \times [t_0, \infty) \) for some \( t_0 > 0 \). Integrating (6.1.1) over \( \Omega \), we obtain

\[ \int_\Omega D_+^\alpha \left[ r(t) D_+^\alpha u \right] \, dx + \int_\Omega q(x, t) f(u) \, dx = a(t) \int_\Omega \Delta u \, dx + \int_\Omega g(x, t) \, dx. \quad (6.3.2) \]

Using Green’s formula, it is obvious that

\[ \int_\Omega \Delta u(x, t) \, dx = 0, \quad t \geq t_1. \quad (6.3.3) \]
By using Jensen’s inequality and $(A_2)$, we have
\[ \int_{\Omega} q(x,t)f(u(x,t))dx \geq Q(t)f(\int_{\Omega} u(x,t)dx) = Q(t)f(v(t)). \] (6.3.4)

Combining (6.3.2)-(6.3.4) and using $(A_4)$, we have
\[ D^\alpha_+ \left[ r(t) D^\alpha_+ v(t) \right] + Q(t)f(v(t)) \leq 0. \]

Therefore $v(t)$ is an eventually positive solution of (6.3.1). This contradicts the hypothesis and completes the proof. \[\square\]

**Theorem 6.3.2.** Suppose that the conditions $(A_1) - (A_3)$ hold and assume that $f'(v)$ exists such that $f'(v) \geq \mu$ for some $\mu > 0$ and for all $v \neq 0$. Furthermore assume that there exists a positive function $c \in C^1[t_0, \infty)$ such that
\[ \limsup_{\xi \to \infty} \int_{\xi_1}^\xi \left[ \tilde{c}(s) \tilde{Q}(s) - \frac{\tilde{r}(s)(\tilde{c}'(s))^2}{4\mu \tilde{c}(s)} \right] ds = \infty. \] (6.3.5)

Then every solution of (6.3.1) is oscillatory.

**Proof.** Suppose that $v(t)$ is a nonoscillatory solution of (6.3.1). Without loss of generality we may assume that $v$ is an eventually positive solution of (6.3.1). Then there exists $t_1 \geq t_0$ such that $v(t) > 0$ and $F(t) > 0$ for $t \geq t_1$. Then it is obvious that
\[ D^\alpha_+ [r(t) D^\alpha_+ v(t)] \leq -Q(t)f(v(t)) < 0, \quad t \geq t_0. \] (6.3.6)

Thus $D^\alpha_+ v(t) \geq 0$ or $D^\alpha_+ v(t) < 0$, $t \geq t_1$ for some $t_1 \geq t_0$. We now claim that $(D^\alpha_+ v(t)) \geq 0$ for $t \geq t_1$. Suppose not, then $(D^\alpha_+ v(t)) < 0$ and there exists $T \geq t_1$ such that $(D^\alpha_+ v(T)) < 0$. Since $D^\alpha_+ [r(t)(D^\alpha_+ v(t))] < 0$ for $t \geq t_1$, it is clear that $r(t)(D^\alpha_+ v(t)) \leq r(T)(D^\alpha_+ v(T))$ for $t \geq T$. Therefore, from Lemma 6.2.1, we have
\[ \frac{F'(t)}{\Gamma(1 - \alpha)} = (D^\alpha_+ v(t)) \leq \frac{r(T)(D^\alpha_+ v(T))}{r(t)}. \]
Integrating the above inequality from $T$ to $t$, we have

$$F(t) = F(T) + \Gamma(1 - \alpha)r(T)(D_+^\alpha v(T)) \int_T^t \frac{1}{r(s)} ds.$$ 

Letting $t \to \infty$, we get $\lim_{t \to \infty} F(t) \leq -\infty$ which is a contradiction. Hence $(D_+^\alpha v(t)) \geq 0$ for $t \geq t_1$ holds.

Define the function $w$ by the generalized Riccati substitution

$$w(t) = c(t) r(t) (D_+^\alpha c(t)) f(v(t)) \quad \text{for } t \geq t_1 \quad (6.3.7)$$

Then we have $w(t) > 0$ for $t \geq t_1$. From (6.2.1), (6.3.1) and (6.3.7), it follows that

$$D_+^\alpha w(t) \leq \frac{D_+^\alpha c(t)}{c(t)} w(t) - \frac{c(t)Q(t)f(v(t))}{f(v(t))} - \frac{f'(v(t))w^2(t)}{c(t)r(t)}$$

$$= -c(t)Q(t) \left( \sqrt{\frac{\mu}{c(t)r(t)}} \frac{w(t)}{2} \sqrt{r(t) D_+^\alpha c(t)} \right)^2 + \frac{r(t)(D_+^\alpha c(t))^2}{4\mu c(t)} \leq -c(t)Q(t) + \frac{r(t)(D_+^\alpha c(t))^2}{4\mu c(t)}. \quad (6.3.8)$$

Let $w(t) = \tilde{w}(\xi)$. Then $D_+^\alpha w(t) = \tilde{w}'(\xi)$ and $D_+^\alpha c(t) = \tilde{c}'(\xi)$. So, (6.3.8) is transformed into

$$\tilde{w}'(\xi) \leq -\tilde{c}(\xi)\tilde{Q}(\xi) + \frac{\tilde{r}(\xi)(\tilde{c}'(\xi))^2}{4\mu \tilde{c}(\xi)}$$

Integrating both sides from $\xi_1$ to $\xi$, we have

$$\tilde{w}(\xi) \leq \tilde{w}(\xi_1) - \int_{\xi_1}^\xi \left[ \tilde{c}(s)\tilde{Q}(s) - \frac{\tilde{r}(s)(\tilde{c}'(s))^2}{4\mu \tilde{c}(s)} \right] ds.$$ 

Letting $\xi \to \infty$, we get $\lim_{\xi \to \infty} \tilde{w}(\xi) \leq -\infty$ which contradicts (6.3.5) and completes the proof. \qed
Corollary 6.3.3. Let assumption (6.3.5) in Theorem 6.3.2 be replaced by
\[
\limsup_{\xi \to \infty} \int_{\xi_1}^{\xi} \tilde{c}(s) \tilde{Q}(s) ds = \infty
\]
and
\[
\limsup_{\xi \to \infty} \int_{\xi_1}^{\xi} \frac{\tilde{r}(s)(\tilde{c}'(s))^2}{\tilde{c}(s)} ds < \infty.
\]
Then every solution of (6.3.1) oscillates.

For the following theorem, we introduce a class of functions $\mathcal{R}$. Let
\[
\mathcal{D}_0 = \{(t,s) : t > s \geq t_0\}, \quad \mathcal{D} = \{(t,s) : t \geq s \geq t_0\}.
\]
The function $H \in C(\mathcal{D}, \mathbb{R})$ is said to belong to the class $\mathcal{R}$, if

(i) $H(t,t) = 0$, for $t \geq t_0$, $H(t,s) > 0$, for $(t,s) \in \mathcal{D}_0$;

(ii) $H$ has a continuous and non-positive partial derivative $\frac{\partial H(t,s)}{\partial s}$ on $\mathcal{D}_0$ with respect to $s$.

Theorem 6.3.4. Suppose that the conditions $(A_1) - (A_3)$ hold and assume that $f'(v)$ exists such that $f'(v) \geq \mu$ for some $\mu > 0$ and for all $v \neq 0$. Furthermore assume that there exists $H \in \mathcal{R}$ such that
\[
\limsup_{\xi \to \infty} \frac{1}{H(\xi,\xi_1)} \int_{\xi_1}^{\xi} \left[ \tilde{c}(s) \tilde{Q}(s) H(\xi, s) - \frac{1}{4} \frac{\tilde{c}(s) \tilde{r}(s) h^2(\xi, s)}{\mu H(\xi, s)} \right] ds = \infty, \quad (6.3.9)
\]
where $h(\xi, s) = H'(\xi, s) + H(\xi, s) \frac{\tilde{c}'(s)}{\tilde{c}(s)}$. Then every solution of (6.3.1) is oscillatory.

Proof. Suppose that $v(t)$ is a nonoscillatory solution of (6.3.1). Without loss of generality we may assume that $v$ is an eventually positive solution of (6.3.1). Then there exists $t_1 \geq t_0$ such that $v(t) > 0$ and $F(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 6.3.2, we obtain
\[
D^2_{+} w(t) \leq \frac{D^0_{+} (c(t))}{c(t)} w(t) - c(t) Q(t) - \frac{\mu w^2(t)}{c(t) r(t)}.
\]
Let \( w(t) = \tilde{w}(\xi) \). Then \( D_+^a w(t) = \tilde{w}'(\xi) \) and \( D_+^a c(t) = \tilde{c}'(\xi) \). So, the above inequality is transformed into

\[
\tilde{w}'(\xi) \leq \frac{\tilde{c}'(\xi)}{\tilde{c}(\xi)} \tilde{w}(\xi) - \tilde{c}(\xi) \tilde{Q}(\xi) + \frac{\mu}{\tilde{c}(\xi) \tilde{r}(\xi)} \tilde{w}^2(\xi). \tag{6.3.10}
\]

Substituting \( \xi \) with \( s \), multiplying both sides of (6.3.10) by \( H(\xi, s) \) and integrating from \( \xi_1 \) to \( \xi \), for \( \xi \geq \xi_1 \), we have

\[
\int_{\xi_1}^{\xi} \tilde{c}(s) \tilde{Q}(s) H(\xi, s) ds \leq - \int_{\xi_1}^{\xi} \tilde{w}'(s) H(\xi, s) ds + \int_{\xi_1}^{\xi} \frac{\tilde{c}'(s)}{\tilde{c}(s)} \tilde{w}(s) H(\xi, s) ds \\
- \mu \int_{\xi_1}^{\xi} \frac{\tilde{w}^2(s)}{\tilde{c}(s) \tilde{r}(s)} H(\xi, s) ds. \tag{6.3.11}
\]

Using the integration by parts, we get

\[
- \int_{\xi_1}^{\xi} H(\xi, s) \tilde{w}'(s) ds < H(\xi, \xi_1) \tilde{w}(\xi_1) + \int_{\xi_1}^{\xi} H'(\xi, s) \tilde{w}(s) ds. \tag{6.3.12}
\]

Substituting (6.3.12) into (6.3.11), we have

\[
\int_{\xi_1}^{\xi} \tilde{c}(s) \tilde{Q}(s) H(\xi, s) ds \leq H(\xi, \xi_1) \tilde{w}(\xi_1) + \int_{\xi_1}^{\xi} \left[ \left( H'(\xi, s) + H(\xi, s) \frac{\tilde{c}'(s)}{\tilde{c}(s)} \right) \tilde{w}(s) \\
- \frac{\mu H(\xi, s)}{\tilde{c}(s) \tilde{r}(s)} \tilde{w}^2(s) \right] ds \\
\leq H(\xi, \xi_1) \tilde{w}(\xi_1) + \int_{\xi_1}^{\xi} \left[ h(\xi, s) \tilde{w}(s) - \frac{\mu H(\xi, s)}{\tilde{c}(s) \tilde{r}(s)} \tilde{w}^2(s) \right] ds \\
\leq H(\xi, \xi_1) \tilde{w}(\xi_1) + \frac{1}{4} \int_{\xi_1}^{\xi} \frac{\tilde{c}(s) \tilde{r}(s) h^2(\xi, s)}{\mu H(\xi, s)} ds,
\]

which yields

\[
\int_{\xi_1}^{\xi} \left[ \tilde{c}(s) \tilde{Q}(s) H(\xi, s) - \frac{1}{4} \frac{\tilde{c}(s) \tilde{r}(s) h^2(\xi, s)}{\mu H(\xi, s)} \right] ds \leq H(\xi, \xi_1) \tilde{w}(\xi_1).
\]

Since \( 0 < H(\xi, s) \leq H(\xi, \xi_1) \) for \( \xi > s \geq \xi_1 \), we have \( 0 < \frac{H(\xi, s)}{H(\xi, \xi_1)} \leq 1 \) for \( \xi > s \geq \xi_1 \). Hence we have

\[
\frac{1}{H(\xi, \xi_1)} \int_{\xi_1}^{\xi} \left[ \tilde{c}(s) \tilde{Q}(s) H(\xi, s) - \frac{1}{4} \frac{\tilde{c}(s) \tilde{r}(s) h^2(\xi, s)}{\mu H(\xi, s)} \right] ds \leq \tilde{w}(\xi_1).
\]
Letting $\xi \to \infty$, we have
\[
\limsup_{\xi \to \infty} \frac{1}{H(\xi, \xi_1)} \int_{\xi_1}^{\xi} \left[ \tilde{c}(s)\tilde{Q}(s)H(\xi, s) - \frac{1}{4} \frac{\tilde{e}(s)\tilde{r}(s)h^2(\xi, s)}{\mu H(\xi, s)} \right] ds \leq \tilde{w}(\xi_1)
\]
which contradicts (6.3.9) and completes the proof. \qed

**Theorem 6.3.5.** Suppose that the conditions $(A_1) - (A_3)$ hold and assume that \[ \frac{f(v)}{v} \geq \gamma > 0 \text{ for all } v \neq 0. \] Furthermore assume that there exists a positive function $c \in C^1[t_0, \infty)$ such that
\[
\limsup_{\xi \to \infty} \int_{\xi_1}^{\xi} \left[ \gamma \tilde{c}(s)\tilde{Q}(s) - \frac{\tilde{e}(s)(\tilde{c}'(s))^2}{4\tilde{c}(s)} \right] ds = \infty. \tag{6.3.13}
\]
Then every solution of (6.3.1) is oscillatory.

**Proof.** Suppose that $v(t)$ is a nonoscillatory solution of (6.3.1). Without loss of generality we may assume that $v$ is an eventually positive solution of (6.3.1). Then there exists $t_1 \geq t_0$ such that $v(t) > 0$ and $F(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 6.3.2, we obtain $D_+^av(t) \geq 0$ for $t \geq t_1$. Now we define the Riccati substitution $w$ by
\[
w(t) = c(t) \frac{r(t)(D_+^av(t))}{v(t)} \text{ for } t \geq t_1. \tag{6.3.14}
\]
Then we have $w(t) > 0$ for $t \geq t_1$. From (6.2.1), (6.3.1) and (6.3.14), it follows that
\[
D_+^aw(t) \leq -\gamma c(t)Q(t) + \frac{r(t)(D_+^ac(t))^2}{4c(t)}. \tag{6.3.15}
\]
Let $w(t) = \hat{w}(\xi)$. Then $D_+^aw(t) = \hat{w}'(\xi)$ and $D_+^ac(t) = \hat{c}'(\xi)$. So, (6.3.15) is transformed into
\[
\hat{w}'(\xi) \leq -\gamma \hat{c}(\xi)\hat{Q}(\xi) + \frac{\hat{r}(\xi)(\hat{c}'(\xi))^2}{4\hat{c}(\xi)}. \]
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Integrating both sides from $\xi_1$ to $\xi$, we have

$$\tilde{w}(\xi) \leq \tilde{w}(\xi_1) - \int_{\xi_1}^{\xi} \left[ \gamma \tilde{c}(s) \tilde{Q}(s) - \frac{\tilde{r}(s)(\tilde{c}'(s))^2}{4\tilde{c}(s)} \right] ds.$$

Letting $\xi \to \infty$, we get $\lim_{\xi \to \infty} \tilde{w}(\xi) \leq -\infty$ which contradicts (6.3.13) and completes the proof.

**Theorem 6.3.6.** Suppose that the conditions $(A_1) - (A_3)$ hold and assume that

$$\frac{f(v)}{v} \geq \gamma > 0 \text{ for all } v \neq 0.$$ Furthermore assume that there exists $H \in \mathbb{R}$ such that

$$\limsup_{\xi \to \infty} \frac{1}{H(\xi, \xi_1)} \int_{\xi_1}^{\xi} \left[ \gamma \tilde{c}(s) \tilde{Q}(s)H(\xi,s) - \frac{1}{4} \frac{\tilde{c}(s)\tilde{r}(s)h^2(\xi,s)}{H(\xi,s)} \right] ds = \infty, \quad (6.3.16)$$

where $h(\xi, s) = H'(\xi, s) + H(\xi, s) \frac{\tilde{c}'(s)}{\tilde{c}(s)}$. Then every solution of (6.3.1) is oscillatory.

**Proof.** Suppose that $v(t)$ is a nonoscillatory solution of (6.3.1). Without loss of generality we may assume that $v$ is an eventually positive solution of (6.3.1). Then there exists $t_1 \geq t_0$ such that $v(t) > 0$ and $F(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 6.3.5, we obtain

$$D^+_\alpha w(t) \leq \frac{D^+_\alpha(c(t))}{c(t)} w(t) - \gamma c(t)Q(t) - \frac{w^2(t)}{c(t)r(t)}.$$

Let $w(t) = \tilde{w}(\xi)$. Then $D^+_\alpha w(t) = \tilde{w}'(\xi)$ and $D^+_\alpha c(t) = \tilde{c}'(\xi)$. So, the above inequality is transformed into

$$\tilde{w}'(\xi) \leq \frac{\tilde{c}'(\xi)}{\tilde{c}(\xi)} \tilde{w}(\xi) - \gamma \tilde{c}(\xi) \tilde{Q}(\xi) + \tilde{w}^2(\xi) \frac{\tilde{c}(\xi)}{\tilde{c}(\xi)\tilde{r}(\xi)} \quad (6.3.17)$$

$$\frac{1}{H(\xi, \xi_1)} \int_{\xi_1}^{\xi} \left[ \gamma \tilde{c}(s) \tilde{Q}(s)H(\xi,s) - \frac{1}{4} \frac{\tilde{c}(s)\tilde{r}(s)h^2(\xi,s)}{H(\xi,s)} \right] ds \leq \tilde{w}(\xi_1).$$
Letting $\xi \to \infty$, we have
\[
\limsup_{\xi \to \infty} \frac{1}{H(\xi, \xi_1)} \int_{\xi_1}^{\xi} \left[ \gamma \tilde{c}(s) \tilde{Q}(s) H(\xi, s) - \frac{1}{4} \frac{\tilde{c}(s) r(s) h^2(\xi, s)}{H(\xi, s)} \right] ds \leq \bar{w}(\xi_1)
\]
which contradicts (6.3.16) and completes the proof. 

6.4 Examples

Example 6.4.1. Consider the nonlinear fractional partial differential equation with forced term
\[
D^\alpha_{+, t} \left[ \left( \frac{t^\alpha}{\Gamma(1 + \alpha)} \right)^2 D^\alpha_{+, t} u(x, t) \right] + e^x u(x, t) = e^t \Delta u(x, t) + \cos x \sin t, \quad (x, t) \in (0, \pi) \times \mathbb{R}_+,
\]
with the boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$, where $\alpha \in (0, 1)$, In (6.4.1), $r(t) = \left( \frac{t^\alpha}{\Gamma(1 + \alpha)} \right)^2$, $Q(t) = \min_{x \in \Omega} q(x, t) = \min_{x \in (0, \pi)} e^x = 1$, $a(t) = e^t$, $f(u) = u$ and $g(x, t) = \cos x \sin t$. Take $t_0 > 0$ and $\gamma = 1$. Thus all the conditions of the Theorem 6.3.4 hold. Therefore every solution of (6.4.1) is oscillatory.

Example 6.4.2. Consider the nonlinear fractional partial differential equation with forced term
\[
D^\alpha_{+, t} \left( D^\alpha_{+, t} u(x, t) \right) + \frac{u(x, t)(2 + u^2(x, t))}{1 + u^2(x, t)} = \Delta u(x, t) + \cos x \sin t, \quad (x, t) \in (0, \pi) \times \mathbb{R}_+,
\]
with the boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$, where $\alpha \in (0, 1)$, In (6.4.2), $r(t) = 1$, $Q(t) = \min_{x \in \Omega} q(x, t) = 1$, $a(t) = 1$, $f(u) = \frac{u(2 + u^2)}{1 + u^2}$ and $g(x, t) = \cos x \sin t$. Take $t_0 > 0$ and $\gamma = 1$. Thus all the conditions of the Theorem 6.3.6 hold. Therefore every solution of (6.4.2) is oscillatory.