CHAPTER V

CLASSIFICATION OF MODELS IN DISCRETE TIME

5.1 Introduction

The majority of literature on the various criteria for ageing centers around continuous life time models. Recently there is some spurt of activity towards reliability analysis in the discrete time domain. As mentioned earlier there are several instances in which the failure time distribution can be modeled by a discrete random variable. The pioneer work in this area is due to Xekalaki (1983), who pointed out that, limitations of measuring devices and the fact that discrete models provide good approximations to their continuous counterparts, necessitate assessment of reliability in discrete time. Accordingly, elaboration of various concepts analogous to those in the continuous case become necessary to distinguish classes of life distributions based on the notions of ageing.

As in the continuous set up, the ageing behaviour of the system or component usually studied by failure rate function or by MRL function. Various authors have studied classes of life distributions based on different concept of ageing. Langberg et.al(1980) discussed properties of discrete models with decreasing
failure rates. Ebrahimi (1986) provided two parametric families of discrete distributions which are suitable for fitting decreasing and increasing mean residual life models to life test data in discrete time. Guess and Park (1988) developed a general approach to modeling discrete bathtub shaped MRL function. Salvia and Bollinger (1982) have established simple bounds for residual life when the device has a monotonic hazard rate sequence.

As is well known, the monotonicity of failure rate of a life distribution plays a very important role in modeling failure time data. Therefore, the identification of the increasing failure rate (IFR) or decreasing failure rate (DFR) distributions and their properties have been extensively discussed in the literature for the continuous case. However, for the discrete case, the determination of the IFR and DFR models is not straightforward because of the complexity of the failure rate. In this direction, Gupta et. al (1997) developed techniques for the determination of IFR and DFR models for a wide class of discrete distributions.

In the following section, we provide a new method to identify an IFR/DFR model in the generalized Ord family.

**Definition 5.1**

The distribution of $X$ is said to have discrete IFR (DFR) property if $h(x) \leq (\geq) h(x+1)$ for every $x=0,1,2,...$

For the classification of discrete lifetime models through failure rate function, we refer to Abouammoh (1990) and Roy and Gupta (1992). However for many distributions $h(x)$ is not in a
simple form. Now we suggest a method, using $\beta(x)$ to identify an IFR(DFR) model in the generalized Ord family, where $\beta(x)$ is defined as

$$\beta(x) = \frac{p(x) - p(x+1)}{p(x)}.$$

**Theorem 5.1:**

If the inequality

$$\beta(x) \leq (\geq) \beta(x+1)$$

holds for every non-negative integer $x$, then $X$ has IFR (DFR) property.

**Proof**

From the definition of $h(x)$, we have

$$\frac{1}{h(x)} = 1 + \frac{p(x+1)}{p(x)} + \frac{p(x+2)}{p(x)} + \ldots \quad (5.1)$$

If the inequality $\beta(x) \leq \beta(x+1)$ holds, (5.1) becomes $h(x) \leq h(x+1)$. Thus $X$ has IFR property. The proof when $\beta(x) \geq \beta(x+1)$ is similar.

Now we use $\beta(x)$ for the classification of distributions belonging to the generalized Ord family. For the generalized Ord family (4.3),

$$\beta(x+1) - \beta(x) = \frac{kx^2 + mx + n}{(d_0 + d_1x + x^2)(d_0 + d_1(x+1) + d_2(x+1)^2)} \quad (5.2)$$

where

$$k = c_1d_2 - c_2d_1$$

$$m = c_1d_2 - c_2d_1 + 2(c_0d_2 - c_2d_0)$$

and

$$n = c_0(d_2 + d_1) - d_0(c_2 + c_1).$$
When \( d_0 + d_1 x + d_2 x^2 > 0 \) for all \( x = 0, 1, 2, \ldots \)
\[
\beta(x+1) - \beta(x) > 0
\]
according as \( kx^2 + mx + n > 0 \). Thus, the roots of the equation
\[
kx^2 + mx + n = 0
\]
determines the sign of \( \beta(x+1) - \beta(x) \).

If \( \Delta = m^2 - 4kn \) is the discriminant of the expression \( kx^2 + mx + n = 0 \),
then from the elementary algebra we have the following theorems.

**Theorem 5.2**

Suppose that the p.m.f of \( X \) belongs to the family of (4.3) with
\[
d_0 + d_1 x + d_2 x^2 > 0,
\]
then

(A) \( \beta(x+1) - \beta(x) > 0 \) if (i) \( k > 0 \) and either

(a) \( \Delta = 0 \) and \( x \neq \frac{-m}{2k} \) or

(b) \( \Delta < 0 \) or

(c) \( \Delta > 0 \) and \( x \in (\alpha, \beta) \) or

(ii) \( k < 0, \Delta > 0 \) and \( x \in (\alpha, \beta) \).

(B) \( \beta(x+1) - \beta(x) < 0 \) if (i) \( k < 0 \) and either

(a) \( \Delta = 0 \) and \( x \neq \frac{-m}{2k} \) or

(b) \( \Delta < 0 \) or

(c) \( \Delta > 0 \) and \( x \in (\alpha, \beta) \) or

(ii) \( k > 0, \Delta > 0 \) and \( x \in (\alpha, \beta) \).

**Theorem 5.3**

A distribution belonging to the generalized Ord family (4.3) has IFR property in a region if condition (A) of Theorem 5.2 holds
in that region and has DFR property in a region if condition (B) of Theorem 5.2 holds in that region.

**Remark 5.1**

When $k=0$, $\beta(x+1)-\beta(x)>(<)0$ according as $mx+n>(<)0$.

**Corollary 5.1**

When $c_2=0$ in (4.3), Theorem 5.2 reduces to the result of Ord family of distributions given by Sankaran and Sindu (2001).

**Corollary 5.2**

For the verification of the theorem, we consider the Borel-Tanner distribution with p.m.f. (4.8), then we have

$$\beta(x)=\frac{x^2+x(nq-n+1)-n}{nqx}.$$  

Since

$$\beta(x+1)-\beta(x)=\frac{x^2+x+n}{nx+nqx^2}>0$$

the distribution (4.8) is IFR.

**Remark 5.2**

Table 5.1 gives the region where the distribution possesses the IFR (DFR) property based on $\beta(x)$ for some popular models belonging to the family (4.3).
Table 5.1
The region where the distribution possesses the IFR (DFR) property

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Distributions with p.m.f</th>
<th>$\beta(x)$</th>
<th>Region</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Binomial $p^x(1-p)^{n-x}$, $x=0,1,...$</td>
<td>$\frac{x-[np-(1-p)]}{(1-p)(x+1)}$</td>
<td>IFR</td>
</tr>
<tr>
<td>2</td>
<td>Poisson $e^{-\lambda} \frac{\lambda^x}{x!}$, $x=0,1,...$</td>
<td>$\frac{x+1-\lambda}{x+1}$</td>
<td>IFR</td>
</tr>
<tr>
<td>3</td>
<td>Negative Binomial $\binom{x+y-1}{x-1} p^x(1-p)^y$, $x=0,1,...$</td>
<td>$\frac{x-py}{1-p} \frac{1}{x/(1-p)}$</td>
<td>IFR</td>
</tr>
<tr>
<td>4</td>
<td>Hypergeometric $\binom{x+y-1}{x} \binom{n-D}{n-x}$, Max(0, D-N+n) ≤ x ≤ Min(D, n)</td>
<td>$\frac{x+n(D+1)-[N+1+D]}{(x+1)(N-D-n+x+1)}$</td>
<td>IFR</td>
</tr>
<tr>
<td>5</td>
<td>Waring $\frac{(a-b)(b)x}{(a)x}$, $x=0,1,...$</td>
<td>$\frac{x(1+a-b)}{x(1+a+x)}$</td>
<td>DFR</td>
</tr>
<tr>
<td>6</td>
<td>Beta Pascal $\frac{\frac{A}{A+K}(\frac{x-1}{x})^{A-1}}{\binom{x+y-1}{A-1}(\frac{A+K}{A})^A}$, $x=1,2,...$</td>
<td>$\frac{x(A+1)-[KB-(K+A+B)]}{x^2+(K+A+B+1)x+K+A+B}$</td>
<td>DFR</td>
</tr>
<tr>
<td>7</td>
<td>Borel-Tanner $e^{-nq}(nq)^{x} \frac{x}{n(n-x)!}$, $x=1,2,...$</td>
<td>$\frac{x^2+x(nq-n+1)-n}{nqx}$</td>
<td>IFR</td>
</tr>
</tbody>
</table>
5.2 Characterization using Mean Residual Life

Mean Residual Life (MRL) function is widely used in the analysis of lifetime data. Muth (1977) pointed out that the MRL function is superior to the failure rate function in many practical situations. It is well known that an increasing (decreasing) failure rate class of distributions is a subclass of a decreasing (increasing) MRL class of distributions. In the following, we establish a characterization result for IFR(DFR) class of distributions in the generalized Ord family, using MRL function.

Definition 5.2

A non-negative random variable \( X \) has increasing mean residual life (IMRL) if

\[
r(x) = \frac{1}{R(x+1)} \sum_{t=1}^{\infty} R(t)
\]

is non-decreasing in \( x=0,1,2... \) and decreasing mean residual life (DMRL) if

\[
r(x) = \frac{1}{R(x+1)} \sum_{t=1}^{\infty} R(t)
\]

is non-increasing in \( x=0,1,2... \).

Theorem 5.4

Let the distribution of \( X \) belongs to the generalized Ord family (4.3). If \( d_0 + d_1 x + d_2 x^2 > 0 \) and \( c_2 [V(x) - V(x-1)] > 0 \), then \( X \) is said to have discrete IFR(DFR) property if and only if

\[
d_1 + 2d_3 x - d_2 + r(x)[c_1 + 2d_2 + c_3 (r(x) + r(x-1)) + c_3 (2x-1)] \leq (z)0. (5.3)
\]
Proof

When the distribution of $X$ belongs to the generalized Ord family (4.3), we have the identity (4.25).

From (1.14),

$$m_2(x) = V(x) + 2xm_1(x) - x^2 + r^2(x)$$

where $V(x)$ is the variance residual life, (4.25) can be written as

$$c_2[V(x)+x^2+2xr(x)+r^2(x)] + c_1+2d_2(x+r(x))+(c_0+d_1-d_2)$$

$$+(d_0+d_1x+d_2x^2) h(x+1) = 0. \quad (5.4)$$

Changing $x$ to $x-1$ in (5.4) and subtracting the resulting expression from (5.4), we get

$$c_2 [(V(x)- V(x-1))+2x(r(x)-r(x-1)) + (r^2(x)- r^2(x-1))+2x-1+2r(x-1)]$$

$$+(c_1+2d_2)(r(x)- r(x-1)+1) +(d_1+2d_2x-d_2)h(x)$$

$$+(d_0+d_1x+d_2x^2) [h(x+1)- h(x)] = 0. \quad (5.5)$$

Substituting the relationship between the failure rate and the MRL in the discrete case given by

$$1- h(x) = \frac{r(x-1)-1}{r(x)} \quad (5.6)$$

(5.5) provides,

$$h(x)[d_1+2d_2x-d_2+(c_1+2d_2)r(x) +c_2 r(x)[ r(x)+ r(x-1)+2x-1]$$

$$+c_2[V(x)- V(x-1)] +(d_0+d_1x+d_2x^2) [h(x+1)- h(x)] = 0. \quad (5.7)$$

If $d_0+d_1x+d_2x^2>0$ and $c_2[V(x)- V(x-1)]>0$, then from (5.7),

$$h(x+1)-h(x) \leq (\leq) 0$$

if and only if,

$$d_1+2d_2x-d_2+r(x)[ c_1+2d_2 +c_2(r(x)+r(x-1)) +c_2(2x-1)] \leq (\leq) 0$$

which completes the proof.
Corollary 5.3

When $c_2=0$, Theorem 5.4 reduces to the simple form for Ord family of distributions as shown below.

Theorem 5.5

Let the distribution of $X$ belong to the Ord family (4.1). Then $X$ is said to have discrete IFR(DFR) property if and only if

$$r(x) \geq (\leq) p_1-p_2+2p_2x$$

(5.8)

for all $x=0,1,2...$ where $p_i = \frac{k_i}{1-2k_2}$, $i=1,2$.

Proof

When the distribution of $X$ belong to the Ord family (4.1), we have (Nair and Sankaran (1991))

$$r(x)+x = (p_0+p_1x+p_2x^2)h(x+1)+\mu$$

(5.9)

where $\mu = E(x)$.

Changing $x$ to $(x-1)$ in (5.9) and subtracting the resulting expression from (5.9), we get

$$r(x) - r(x-1)+1 = (p_0+p_1x+p_2x^2)[h(x+1)-h(x)]+h(x) (p_1-p_2+p_2x).$$

(5.10)

Substituting the relationship (5.6), (5.10) becomes,

$$h(x)[r(x)-(p_1-p_2+2p_2x)]=(p_0+p_1x+p_2x^2)[h(x+1)-h(x)].$$

(5.11)

It is easy to verify that $(p_0+p_1x+p_2x^2)\geq 0$. Thus from (5.11),

$$[h(x+1)-h(x)] \geq (\leq) 0$$

if and only if

$$r(x) \geq (\leq) (p_1-p_2+2p_2x).$$

This completes the proof.
Remark 5.3

The $p_i$'s in (5.8) are directly related to the moments of the distributions. To apply the result in a practical situation one need to take the sample moments and sample MRL function as estimators.

For the verification of Theorem 5.5, consider the Waring distribution with

$$p(x) = \frac{(a-b)(b)x}{(a)_x}, x=0,1,2... \quad a>b>0$$

where $(b)_x = b(b+1)...(b+x-1)$.

By direct computation we get,

$$r(x) = \frac{a+x}{a-b-1}, \quad p_1 = \frac{a+1}{a-b-1} \quad \text{and} \quad p_2 = \frac{1}{a-b-1}$$

Since $r(x) < p_1 - p_2 + 2p_2x$ for any $x=0,1,2,...$ Waring distribution has DFR property.

5.3. Length Biased Models

In this section we discuss the form-invariant length biased models from generalized Ord family.

Analogous to the continuous case, the length biased distribution of a discrete random variable $X$ with the set of non-negative integers as the support is defined as (Gupta 1979),

$$g(x) = \frac{xp(x)}{\mu}, \quad x=1,2...$$

(5.12)

where $\mu = E(X) < \infty$. Clearly the above random variable $Y$ will have no zero in its support. Applying a displacement of $Y$ to the left, by
taking $Z=Y-1$, $Z$ would be realized by length biased sampling on $X$ with the above displacement and the support becomes the set of non negative integers (See Patil and Ord 1976). The resulting probability mass function of $Z$ is

$$p(x)=g(x+1) \quad \text{for } x=0,1,2...$$

For the application of (5.12) to reliability we can refer to Patil and Rao (1977), and Gupta and Kirmani (1990).

5.3.1 Form Invariance

The distribution of $X$ with p.m.f. $p(x)$ is said to be form-invariant under length biased sampling if observed variable $Z$ has the same distribution as $X$, with a change in parameter. The major relationships between the survival function, failure rate and MRL of the original distribution and its corresponding length biased version is given as

$$G(x)=\frac{m(x)R(x+1)}{\mu} \quad (5.13)$$

$$k(x+1)=\frac{(x+1)h(x+1)}{m(x)} \quad (5.14)$$

$$e(x-1)=\frac{\sum_{t} R(t+1)m(t)}{R(x+1)m(x)} \quad (5.15)$$

where $G(x)$, $k(x)$ and $e(x)$ are respectively the survival function, failure rate and MRL of $Y$. The above identities connecting reliability characteristics of $X$ and $Y$ can be employed in the characterization of the distribution of $X$. Sankaran and Nair (1993) derived conditions under which models belonging to the Ord family retain the same form for their length biased distributions.
In reliability the ageing patterns of system can be studied by comparing the structural properties of their life lengths with those from the corresponding length biased distributions. In the following we derive the conditions under which the members of the generalized Ord family (4.2) are form invariant with respect to the formation of their length biased distributions.

**Theorem 5.6**

Among the members of family (4.3), $X$ and $Y$ have the same type of distribution if and only if $a_0 + b_0 = 0$, and the p.m.f of $Y$ satisfies

$$\frac{g(x+1) - g(x)}{g(x)} = \frac{p_0 + p_1 x + p_2 x^2}{q_0 + q_1 x + q_2 x^2}$$

(5.16)

where $p_0, p_1, p_2, q_0, q_1$ and $q_2$ are real constants.

**Proof**

Suppose that (4.3) holds and $X$ and $Y$ have the same distributional form. Then from (5.12), we have

$$\frac{g(x+1) - g(x)}{g(x)} = \frac{x+1}{x} \frac{f(x+1)}{f(x)} - 1$$

(5.17)

which gives,

$$\frac{g(x+1) - g(x)}{g(x)} = \frac{x+1}{x} \left[ 1 + \frac{c_0 + c_1 x + c_2 x^2}{d_0 + d_1 x + d_2 x^2} \right] - 1.$$  

(5.18)

Since $Y$ also must belong to the family (4.3), (5.18) must be of the form,

$$\frac{p_0 + p_1 x + p_2 x^2}{q_0 + q_1 x + q_2 x^2} = \frac{c_2 x^3 + (c_1 + c_2 + d_2)x^2 + (c_0 + c_1 + d_1)x + c_0 + d_0}{(d_0 x + d_1 x^2 + d_2 x^3)}$$

or

$$\frac{p_0 + p_1 x + p_2 x^2}{q_0 + q_1 x + q_2 x^2} = \frac{c_2 x^3 + (c_1 + c_2 + d_2)x^2 + (c_0 + c_1 + d_1)x + c_0 + d_0}{(d_0 x + d_1 x^2 + d_2 x^3)}$$

or
\[(d_0 x + d_1 x^2 + d_2 x^3) (p_0 + p_1 x + p_2 x^2) = [c_2 x^3 + (c_1 + c_2 + d_2) x^2 + (c_0 + c_2 + d_1) x + (c_0 + d_0)] (q_0 + q_1 x + q_2 x^2) \] (5.19)

Equating the coefficients of like powers of \(x\) in (5.18), we have six equations,

\[q_0 (c_0 + d_0) = 0 \] (5.20)

\[p_0 d_0 = q_0 (c_0 + c_1 + d_1) + q_1 (c_0 + d_0) \] (5.21)

\[p_0 d_1 + p_1 d_0 = q_0 (c_1 + c_2 + d_2) + q_1 (c_0 + c_1 + d_1) + q_2 (c_0 + d_0) \] (5.22)

\[p_2 d_0 + p_1 d_1 + p_0 d_2 = q_1 (c_1 + c_2 + d_2) + q_2 (c_0 + c_1 + d_1) + q_0 c_2 \] (5.23)

\[p_2 d_1 + p_1 d_2 = q_2 (c_1 + c_2 + d_2) + q_1 c_2 \] (5.24)

\[p_2 d_2 = q_2 c_2. \] (5.25)

From (5.20), we have the following cases,

(i) \(q_0 \neq 0\) and \(c_0 + d_0 = 0\), which leads to

\[g(x + 1) - g(x) = \frac{c_2 x^2 + (c_1 + c_2 + d_2) x + c_0 + c_1 + d_1}{-c_0 + d_1 x + d_2 x^2}. \] (5.26)

It is easy to see that, \(p(x)\) and \(g(x)\) have same distributional form though with possibly different parameters.

(ii) When \(q_0 = 0\) and \(c_0 + d_0 \neq 0\),

\[g(x + 1) - g(x) = \frac{p_2 + p_0 q_2 + (p_2 q_2 - p_2 q_1) x}{q_2 + (q_1 x + q_2 x^2)} \]

which has not the same form as \(p(x)\).

(iii) When \(q_0 = 0\) and \(c_0 + d_0 = 0\), we obtain an equation

\[p_0 d_0 = 0\]

which leads to three different cases,

(a) \(p_0 = 0\), \(d_0 \neq 0\)

(b) \(p_0 \neq 0\), \(d_0 = 0\)

(c) \(p_0 = 0\), \(d_0 = 0\).

The discussions based on above three cases lead to the situations parallel to those we have already mentioned with \(c_0 + d_0 = 0\).
Conversely when $c_0 + d_0 = 0$, (4.3) and (5.18) provide that $X$ and $Y$ have the same type of distributions. This completes the proof.

**Corollary 5.4**

When $c_2=0$, Theorem 5.6 reduces to the result of Sankaran and Nair (1993) for the Ord family of distributions.

To verify Theorem 5.6, consider the confluent hypergeometric distribution with p.m.f (4.6), then the LBD can be obtained as

$$g(x) = K \frac{\Gamma x + \gamma^\gamma + x}{\Gamma x + \gamma^\gamma + x} \theta^{-1} , x = 1, 2, \ldots$$

where $K$ is the normalizing constant. This has the same form as parent distribution with different parameters.

Next we prove the condition under which the length biased distribution of $X$ belongs to the generalized Ord family when the original belongs to Ord family.

Suppose that the distribution of $X$ belongs to the Ord family (4.1) and that of $Y$ belong to (4.3). Then we have,

$$\frac{g(x+1) - g(x)}{g(x)} = \frac{x+1}{x} \frac{p(x+1)}{p(x)} - 1$$

or

$$\frac{c_0 + c_1 x + c_2 x^2}{d_0 + d_1 x + d_2 x^2} = \frac{k_0 - u + (k_1 - d_1 - 1)x + (k_2 - 1)x^2}{k_0 x + k_1 x^2 + k_2 x^3}$$

which provides,

$$(k_0 x + k_1 x^2 + k_2 x^3)(c_0 + c_1 x + c_2 x^2) = (d_0 + d_1 x + d_2 x^2)$$

$$[k_0 - u + (k_1 - d_1 - 1)x + (k_2 - 1)x^2] .$$
Equating the coefficients of like powers of \( x \) in (5.28), we obtain the following equations

\[
\begin{align*}
d_0[k_0-u] &= 0 \\
c_0k_0 &= d_0(k_1-u-1)+d_1(k_0-u) \\
c_1k_0+c_0k_1 &= d_0(k_2-1)+d_1(k_1-u-1)+d_2(k_0-d_k) \\
c_2k_0+c_0k_2+c_1k_1 &= d_1(k_2-1)+d_2(k_1-u-1) \\
c_1k_2+c_2k_1 &= d_2(k_2-1) \\
c_2k_2 &= 0.
\end{align*}
\]

Now from (5.29) and (5.34), we have the following cases.

(i) \( d_0=0, k_2=0, k_0-u\neq0, c_2\neq0 \), then

\[
\frac{g(x+1)-g(x)}{g(x)} = \frac{k_0-u+(k_1-d_1-1)x+x_2}{k_0+k_1x+k_2x^2}
\]

and

\[
\frac{p(x+1)-p(x)}{p(x)} = \frac{-(x+u)}{k_0+k_1x+k_2x^2}
\]

clearly \( p(x) \) and \( g(x) \) have different forms.

(ii) \( d_0=0, c_2=0, k_2\neq0, k_0-u\neq0 \), then

\[
\frac{p(x+1)-p(x)}{p(x)} = \frac{-(x+u)}{k_0+k_1x+k_2x^2}
\]

and

\[
\frac{g(x+1)-g(x)}{g(x)} = \frac{k_0-u+(k_1-d_1-1)x+k_2x^2}{(k_0+k_1x+k_2x^2)x}
\]

here also, \( p(x) \) and \( g(x) \) have different forms.

(iii) \( d_0\neq0, c_2=0, k_2\neq0, k_0-u=0 \), then

\[
\frac{g(x+1)-g(x)}{g(x)} = \frac{(k_1-d_1-1)+(k_2-1)x}{k_0+k_1x+k_2x^2}
\]

and

\[
\frac{p(x+1)-p(x)}{p(x)} = \frac{-(x+u)}{k_0+k_1x+k_2x^2}
\]
Thus \( p(x) \) and \( g(x) \) have same distributional forms but with different parameters.

(iv) \( d_0 \neq 0, k_2 = 0, c_2 \neq 0, k_0 - u = 0 \), then
\[
\frac{p(x+1) - p(x)}{p(x)} = \frac{-(x+u)}{k_1 x + u}
\]
and
\[
\frac{g(x+1) - g(x)}{g(x)} = \frac{k_1 - d_1 - 1 - x}{k_1 x + u}.
\]
Thus \( p(x) \) and \( g(x) \) have same distributional forms.

(v) \( k_0 - u = 0, d_0 = 0, k_2 = 0 \) and \( c_2 = 0 \), we obtain \( d_2 = 0 \), then
\[
\frac{p(x+1) - p(x)}{p(x)} = \frac{-(x+u)}{k_1 x + u}
\]
and
\[
\frac{g(x+1) - g(x)}{g(x)} = \frac{k_1 - d_1 - 1 - x}{k_1 x + u}
\]
thus \( p(x) \) and \( g(x) \) have same distributional forms.

When \( d_0 = -c_0 \) in (4.3), we obtain a sub class of the generalized Ord family. This sub-class, to be denoted by \( D \), contains many distributions of interest in reliability analysis.

Now we prove a characterization of the class \( D \) based on conditional moments.

**Theorem 5.7**

The distribution of \( X \) belongs to the class \( D \) if and only if
\[
c_2 m_3(x) + [(c_1 + 3d_2) - c_2 x] m_2(x) = [(c_1 + 2d_2) x + 3d_2 - c_0 - 2d_1] m_1(x) + (d_1 + c_0 - d_2)(x + 1)
\]
where \( m_i(x) = E[X^i|X>x], i=1,2,3. \)
Proof

Since \( d_0 = -c_0 \), (4.25) leads to
\[
(c_0 - d_1 x - d_2 x^2) h(x+1) = c_2 m_2(x) + (c_1 + 2d_2)m_1(x) + c_0 + d_1 - d_2.
\]
(5.36)

On similar lines, for the random variable \( Y \), (5.26) gives,
\[
(c_0 - d_1 x - d_2 x^2)k(x+1) = c_2 V_2(x) + (c_1 + c_2 + 3d_2)V_1(x) + c_0 + c_1 + 2d_1 - d_2
\]
(5.37)

where \( V_i(x) = \mathbb{E}[Y^i|Y>x], i = 1, 2 \), and \( k(x) \) is the failure rate of \( Y \).

From (5.14), we have
\[
k(x+1) = \frac{(x+1)h(x+1)}{m_1(x)}
\]
(5.38)

and
\[
V_i(x) = \frac{m_{i+1}(x)}{m_1(x)}, i = 1, 2.
\]
(5.39)

Dividing (5.37) by (5.36) and substituting the relationships (5.38) and (5.39) in the resulting equations, we obtain (5.35).

Conversely suppose that (5.35) holds. Then we have
\[
c_2 \sum_{x=1}^{\infty} t^1 p(t) + [c_1 + 3d_2 - c_2 x] \sum_{x=1}^{\infty} t^2 p(t) = [(c_1 + 2c_2)x + 3d_2 - c_0 - 2d_1] \sum_{x=1}^{\infty} t^1 p(t)
\]
\[+ (c_0 + d_1 - d_2)(x+1)R(x+1).
\]
(5.40)

Changing \( x \) to \( (x-1) \) in (5.40) and subtracting (5.40) from the resulting equation, we get
\[
d_2 x^2 p(x) + c_2 \sum_{x=1}^{\infty} t^1 p(t) = (2d_2 - d_1)x p(x) - (c_1 + 2d_2) \sum_{x=1}^{\infty} t^1 p(t)
\]
\[+ (c_0 + d_1 - d_2)R(x+1).
\]
(5.41)

Now changing the variable \( x \) to \( (x+1) \) in (5.41) and subtracting (5.41) from the resulting expression, we obtain
\[ p(x+1) [d_0 + d_1 x + d_2 x^2] = p(x) [(c_1 + d_1) x + (c_2 + d_2) x^2] \]

which is of the form (4.3) with \( d_0 = -c_0 \).

This completes the proof.

**Corollary 5.5**

When \( c_2 = 0 \), Theorem 5.7 reduces to the result of Sankaran and Nair (1993).

**Corollary 5.6**

The distribution of \( X \) is confluent hyper geometric with p.m.f (4.6) holds if and only if

\[ -m_3(x) + (x+\theta-b+1) m_2(x) = [(\theta-b)x-(\theta+3b+2)]m_1(x) + v\theta (x+1). \]

**Corollary 5.7**

The relationship

\[ -m_3(x) +[x+(n-3)\alpha+n-1)m_2(x) = \{[n(1-\alpha)-2\alpha-1]x+4-5n\alpha -3\alpha-n\} m_1(x) \]

\[ +(3n\alpha+n) (x+1) \]

holds if and only if \( X \) has Haight distribution with p.m.f (4.9).