CHAPTER III

AGEING PROPERTIES OF THE GENERALIZED PEARSON FAMILY

3.1 Introduction

In reliability theory the concept of ageing plays a central role as it helps to classify the lifetime models. Earlier works in reliability theory was centered around the problem of estimating the reliability function based on observed data. Recently a lot of interest have been evolved to modeling the lifetime data and to classify the life distributions based on certain ageing properties. Accordingly, large number of research papers have been published which examine the behavior of the life distributions based on certain criteria for ageing [See, Bryson and Siddique (1969), Rolski (1975), Klefsjo (1980), Basu and Ebrahami (1986), Singh and Deshpande (1985), Abouammoh (1988) and Jinhua, Cao and Wang (1991)].

One of the methods of describing the failure mechanism is to expose the manner in which its life length is affected by the advancement of age. Usually by ageing we mean that an older component has a shorter remaining lifetime than a newer or younger one. No ageing is equivalent to saying that, the age of a system has no effect on the distribution of the residual lifetime. Positive ageing implies that the age has an adverse effect on the residual lifetime. That is the residual lifetime tends to be smaller in some
probabilistic sense with increasing age. Negative ageing describes that the age has a beneficial effect on the residual lifetime. If the same type of ageing persists throughout the entire lifetime of a unit, the system is said to have monotonic ageing.

The phenomena of ageing had been first extensively studied by Bryson and Siddique (1969) and they had postulated a set of seven criteria for describing the ageing behaviour. Later, Basu and Ebrahami (1986) described, how ageing or wear out have been used to study lifetimes of systems and components. Abouammoh (1988) introduced a new criteria of ageing in terms of the conditional mean remaining life. The phenomenon of ageing can be described by using different reliability concepts such as failure rate, reliability function, MRL and VRL. In the present work, we discuss the ageing behaviour of the lifetime models belonging to the generalized Pearson system using failure rate and mean residual life function.

In reliability the ageing behaviour of the system is usually studied either by the failure rate function or by the mean residual life function. The increasing (decreasing) failure rate (IFR/DFR) property is a characteristic of the system that consistently deteriorate (improved) with age. This brings the relevance and need for classification of distributions based on failure rate function which provides information about the system reliability.

**Definition 3.1**

The distribution of $X$ possess the increasing (decreasing) failure rate property if $h(x)$ is an increasing (decreasing) function of $X$. 
In the following we discuss a method to identify an IFR(DFR) model in the generalized Pearson system (2.1) using

\[ \beta(x) = \frac{-f'(x)}{f(x)}. \]

The function \( \beta(x) \) was used earlier by Glaser (1980) for the analysis of bathtub models. Later Mukherjee and Roy (1989) used \( \beta(x) \) to characterize certain lifetime models. An important feature of the procedure is that the method can be applicable to the most of the models used in the lifetime data analysis.

**Lemma 3.1**

Suppose that the distribution of \( X \) belongs to the generalized Pearson system (2.1). Let \( \beta'(x) \) denote the derivative of \( \beta(x) \) with respect to \( x \). Then for \( b_2 \neq 0 \) in (2.1)

(A) \( \beta'(x) > 0 \) if (i) \( p_2 > 0 \) and either
   (a) \( \Delta = 0 \) and \( x \neq -d \) or
   (b) \( \Delta < 0 \) or
   (c) \( \Delta > 0 \) and \( x \notin (\alpha, \beta) \) or
   (ii) \( p_2 < 0, \Delta > 0 \) and \( x \in (\alpha, \beta) \) and

(B) \( \beta'(x) < 0 \) if (i) \( p_2 < 0 \) and either
   (a) \( \Delta = 0 \) and \( x \neq -d \) or
   (b) \( \Delta < 0 \) or
   (c) \( \Delta > 0 \) and \( x \notin (\alpha, \beta) \) or
   (ii) \( p_2 > 0, \Delta > 0 \) and \( x \in (\alpha, \beta) \)

where \( \alpha \) and \( \beta \) are the roots of the equation

\[ p_2 x^2 + 2p_1 dx + p_4 d - p_6 = 0 \]

with
The generalized Pearson system (2.1) can be written as

\[
p_i = \frac{b_i b_i}{a_i b_i - a_i b_i}, \quad i = 0, 1, 2, \quad b_2 a_1 - a_2 b_1
\]  

(3.2)

\[
d = \frac{a_i b_i - a_i b_i}{a_i b_i - a_i b_i}
\]  

(3.3)

and

\[
\Delta = 4p_i^2d^2 - 4p_i(p_1d - p_0).
\]

**Proof**

The generalized Pearson system (2.1) can be written as

\[
\frac{d \log f}{dx} = c + \frac{x + d}{p_0 + p_1 x + p_2 x^2}
\]  

(3.4)

where

\[
c = \frac{a_2}{b_2}
\]

and \( p_i \)'s and \( d \) are given in (3.2) and (3.3).

From (3.4), we have

\[
\beta(x) = -\frac{f'(x)}{f(x)} = -\left[ c + \frac{x + d}{p_0 + p_1 x + p_2 x^2} \right]
\]

and hence

\[
\beta'(x) = \frac{p_2 x^2 + 2p_1 dx + p_1 d - p_0}{(p_0 + p_1 x + p_2 x^2)^2}.
\]  

(3.5)

Thus from (3.5), it is obvious that the sign of \( \beta'(x) \) is determined by the sign of equation (3.1). Then from the elementary algebra we have the following results.

If \( \Delta = b^2 - 4ac \) is the discriminant of the expression \( ax^2 + bx + c = 0 \), we have
(a) if $\Delta = 0$, $ax^2 + bx + c$ has the same sign as that of ‘$a$’ for all $x \neq \frac{-b}{2a}$ and $ax^2 + bx + c = 0$ when $x = \frac{-b}{2a}$

(b) if $\Delta < 0$, $ax^2 + bx + c$ has the same sign as ‘$a$’ for all real $x$.

(c) if $\Delta > 0$ and the roots of $ax^2 + bx + c = 0$ are $\alpha$ and $\beta$ with $\alpha > \beta$, then

(i) $ax^2 + bx + c$ has the same sign as that of ‘$a$’ whenever $x > \alpha$ or $x < \beta$ and

(ii) $ax^2 + bx + c$ has the sign opposite that of ‘$a$’ whenever $\beta < x < \alpha$.

Thus $\beta'(x) > 0$, when (A) holds and $\beta'(x) < 0$, when (B) holds. This completes the proof.

**Theorem 3.1**

A distribution belonging to the generalized Pearson family (2.1) has IFR property in a region if condition (A) of Lemma 3.1 holds in that region and has DFR property in a region if condition (B) of Lemma (3.1) holds in that region.

**Proof**

The proof directly follows from Lemma (3.1) and theorem given in Glaser (1980 p.667).

**Remark 3.1**

When $b_2 = 0$ in (2.1), (3.5) becomes

$$\beta'(x) = \frac{a_\delta b_1 - a_\beta b_\delta - 2a_\delta b_\delta x - a_\beta b_1 x^2}{(b_\delta + b_1 x)^2}.$$  \hfill (3.6)
Thus \( \beta'(x) > (\leq) 0 \) according as \( a_b b_i - a_i b_o - 2a_x b_o x - a_x b_1 x^2 > (\leq) 0 \).

In this case, we have

\[
p_d = p_o - 2a_o b_o, \quad 2p_x d = -2a_x b_o \quad \text{and} \quad p_x = -a_x b_1.
\]

**Corollary 3.1**

When \( a_2 = 0 \), the above result reduces to the Pearson family given by Sankaran and Sindu (2001).

For the verification of the theorem, consider inverse Gaussian distribution with p.d.f (2.11). Then we have

\[
\beta(x) = \frac{-\lambda \mu^2 + \lambda x^2 + 3x \mu^2}{2x^2 \mu^2}
\]

and hence

\[
\beta'(x) = \frac{\lambda}{x^3} \frac{3}{2x^2}.
\]

Thus the distribution is IFR if

\[
\frac{\lambda}{x^3} \frac{3}{2x^2} > 0
\]

or

\[
0 < x < \frac{2\lambda}{3}
\]

and DFR if

\[
x > \frac{2\lambda}{3}.
\]

Table 3.1 gives the region where the distribution possesses the IFR (DFR) property based on \( \beta(x) \) for some popular models belonging to the family (2.1).
### Table 3.1
The region where the distribution possesses the IFR (DFR) property

<table>
<thead>
<tr>
<th>Sl. No</th>
<th>Distributions with pdf</th>
<th>$\beta(x)$</th>
<th>Region</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Gamma $\frac{m^p e^{-mx}}{\Gamma(p)} x^{p-1}$, $x&gt;0, p,m&gt;0$</td>
<td>$\frac{mx-(p-1)}{x}$</td>
<td>IFR if $p&gt;1$ DFR if $0&lt;p&lt;1$</td>
</tr>
<tr>
<td>2</td>
<td>Pareto $ak^a x^{(a+1)}$, $a&gt;0$, $x \geq k &gt; 0$</td>
<td>$\frac{a+1}{x}$</td>
<td>DFR</td>
</tr>
<tr>
<td>3</td>
<td>Normal $\frac{1}{\sqrt{2\pi}\sigma} \exp \left{ -\frac{(x-\mu)^2}{2\sigma^2} \right}$, $-\infty&lt;x&lt;\infty$, $\sigma&gt;0$, $-\infty&lt;\mu&lt;\infty$</td>
<td>$\frac{x-\mu}{\sigma^2}$</td>
<td>IFR</td>
</tr>
<tr>
<td>4</td>
<td>Finite Range $\frac{d}{R} \left(1 - \frac{x}{R}\right)^{d-1}$, $0&lt;x&lt;R$, $d&gt;1$.</td>
<td>$\frac{d-1}{R-x}$</td>
<td>IFR</td>
</tr>
<tr>
<td>5</td>
<td>Exponential $\lambda e^{-\lambda x}$, $x&gt;0$, $\lambda&gt;0$</td>
<td>$\lambda$</td>
<td>Both IFR and DFR</td>
</tr>
<tr>
<td>6</td>
<td>Inverse Gaussian $\frac{\lambda}{\sqrt{2\pi}\lambda^2} \exp \left{ -\lambda(x-\mu)^2 \right}$, $x&gt;0$, $\lambda, \mu&gt;0$</td>
<td>$\frac{-\lambda\mu^2 + \lambda x^2 + 3\mu^2 x}{2\mu^2 x^2}$</td>
<td>IFR if $0&lt; x &lt; \frac{2\lambda}{3}$ DFR if $x &gt; \frac{2\lambda}{3}$</td>
</tr>
<tr>
<td>7</td>
<td>Maxwell $4 \left(\frac{\lambda}{\pi}\right)^{\frac{1}{2}} x^2 e^{-\lambda x^2}$, $x&gt;0$, $\lambda&gt;0$</td>
<td>$\frac{2\lambda x^2 - 2}{x}$</td>
<td>IFR</td>
</tr>
<tr>
<td>8</td>
<td>Rayleigh $2\lambda x e^{-\lambda x^2}$, $x&gt;0$, $\lambda&gt;0$</td>
<td>$\frac{2\lambda x^2 - 1}{x}$</td>
<td>IFR</td>
</tr>
</tbody>
</table>

### 3.2 Characterization using Mean Residual Life

As mentioned in Chapter I, mean residual life (MRL) function is extensively used in the analysis of lifetime data. It is shown that an increasing (decreasing) failure rate class of distributions is a
subclass of decreasing (increasing) MRL class of distributions. In the following we prove a characterization result for IFR (DFR) class of distributions in the generalized Pearson family using MRL function.

**Definition 3.2**

Let \( X \) be a non-negative continuous random variable with survival function \( R(x) \). Then the distribution of \( X \) is said to have increasing mean residual life (IMRL) property if

\[
r(x) = \frac{1}{R(x)} \int_x^\infty R(t) dt
\]

is increasing in \( x > 0 \) and have decreasing mean residual life (DMRL) property if

\[
r(x) = \frac{1}{R(x)} \int_x^\infty R(t) dt
\]

is decreasing in \( x > 0 \).

**Theorem 3.2**

Let the distribution of \( X \) belong to the generalized Pearson family (2.1) with \((b_0+b_1x+b_2x^2)\geq 0\). Then \( X \) has IFR (DFR) property if and only if

\[
a_2 V(x) + r(x) [a_2x+a_1] + m_1(x)[a_2 r(x)+2b_2] +b_1 \leq (\geq) 0 \tag{3.7}
\]

where

\[
V(x)=\text{E}[(X-x)^2 \mid X>x],
\]

\[
r(x)=\text{E}[(X-x) \mid X>x]
\]

and

\[
m_1(x)=\text{E}[X \mid X>x].
\]
**Proof**

When the distribution of \( X \) belongs to the family (2.1), we have (2.29),

\[
a_2 \ m_2(x) + (a_1+2b_2) \ m_1(x) + a_0 + b_1 + (b_0+b_1x+b_2x^2)h(x) = 0 \quad (3.8)
\]

where

\[
m_2(x) = E[X^2 \mid X > x].
\]

Using (1.12) and

\[
m_2(x) = V(x) + 2x \ m_1(x) - x^2 + r^2(x)
\]

(3.8) becomes

\[
a_2 \ [V(x) + 2x(x+r(x)) - x^2 + r^2(x)] + (a_1+2b_2)(x+r(x)) + a_0 + b_1 + (b_0+b_1x+b_2x^2)h(x) = 0. \quad (3.9)
\]

Differentiating (3.9), we obtain

\[
a_2 \ [V'(x) + 2(xr'(x)+ r(x)) + 2x + 2r(x) r'(x)] + (a_1+2b_2)[1+ r'(x)]
\]

\[
+ (b_0+b_1x+b_2x^2) h'(x) + (b_1+2b_2x) h(x) = 0. \quad (3.10)
\]

Since \( V'(x) = h(x)[ V(x) - r^2(x)] \) and \( h(x) = \frac{1+r'(x)}{r(x)} \), (3.10) becomes

\[
a_2 \ [h(x)[ V(x) - r^2(x)] + h(x) r(x)[2x+2r(x)] + (a_1+2b_2) h(x) r(x)
\]

\[
+ (b_0+b_1x+b_2x^2) h'(x) + (b_1+2b_2x) h(x) = 0
\]

which provides,

\[
h(x)[a_2 \ V(x) - a_2 \ r^2(x) + r(x)[2a_2x+2a_2r(x)+a_1+2b_2] + (b_1+2b_2x)]
\]

\[
+ (b_0+b_1x+b_2x^2) h'(x) = 0. \quad (3.11)
\]

If \( (b_0+b_1x+b_2x^2) \geq 0 \), then \( h'(x) \geq 0 \), if and only if

\[
a_2 \ V(x) + r(x) [a_2x+a_1] + m_1(x)[a_2 \ r(x)+2b_2] + b_1 \leq (\geq) 0.
\]

This completes the proof.

For the Pearson family, Theorem 3.2 reduces to a much simpler form, as shown below.
Theorem 3.3

Let the distribution of \( X \) belongs to the Pearson family (1.18). Then \( X \) has IFR/DFR property if and only if for every \( x \) in \((a, b)\)
\[
r(x) \geq (\leq) c_1 + 2c_2x
\]
where
\[
c_i = \frac{b_i}{1-2b_2}, \quad i = 1, 2.
\]

Proof

Since \( X \) belongs to the Pearson family (1.18), we have (Nair and Sankaran, 1991)
\[
r(x) = (c_0 + c_1x + c_2x^2) h(x) + \mu \quad \text{(3.12)}
\]
where
\[
\mu = \frac{b_1-d}{2b_2} \quad \text{and} \quad c_i = \frac{b_i}{1-2b_2}, \quad i = 0, 1, 2.
\]
Differentiating (3.12) with respect to \( x \) and substituting the relationship between \( h(x) \) and \( r(x) \),
\[
h(x) r(x) = 1 + r'(x)
\]
we get,
\[
h(x) [r(x) - c_1 - 2c_2x] = (c_0 + c_1x + c_2x^2) h'(x). \quad \text{(3.13)}
\]
Now, to prove \((c_0 + c_1x + c_2x^2) \geq 0\), for all \( x \), we consider
\[
m_1(x) - m_1(a) = (c_0 + c_1x + c_2x^2) h(x) \quad \text{(3.14)}
\]
Since \( m_1(x) \) is non decreasing and \( h(x) \geq 0 \), (3.14) gives
\((c_0 + c_1x + c_2x^2) \geq 0\), for all \( x \).
Thus from (3.13) \( h'(x) \geq (\leq) 0 \) if and only if \( r(x) \geq (\leq) c_1 + 2c_2x \).
This completes the proof.

For example, consider
(i) the Lomax distribution with p.d.f
\[ f(x) = c \alpha^c (x+\alpha)^{c-1}, \ x > 0, \ \alpha > 0, \ c > 1 \]

then, \( c_1 = \frac{\alpha}{c-1} \), \( c_2 = \frac{1}{c-1} \) and \( r(x) = \frac{x+\alpha}{c-1} \).

Since \( r(x) < c_1 + 2c_2x \), the Lomax distribution has DFR property.

(ii) Consider the finite range distribution with p.d.f

\[ f(x) = \frac{d}{R} \left( 1 - \frac{x}{R} \right)^{d-1}, \ 0 < x < R, \ d > 0 \]

then \( c_1 = \frac{R}{d+1} \), \( c_2 = \frac{-1}{d+1} \) and \( r(x) = \frac{R-x}{d+1} \).

Since \( r(x) > c_1 + 2c_2x \), the finite range distribution has IFR property.

3.3 Form Invariant Length Biased Models from the Generalized Pearson Family

We have discussed basic properties and various applications of length biased models in chapter I. Gupta and Keating (1986) observed that it is worthwhile to investigate the structural relationships between the distributions of \( X \) and \( Y \) in the context of reliability. Later Jain, Singh and Bagai (1989) extended the Gupta-Keating results for an arbitrary weight function \( w(x) > 0 \). The major relationship established by Gupta and Keating (1986) are

(i) \[ G(x) = \frac{m(x)}{\mu} R(x) \]

(ii) \[ k(x) = \frac{x}{m(x)} h(x) \]

(iii) \[ s(x) = \frac{r(x)}{m(x)} \int \frac{t + r(t)}{r(t)} \exp \left\{ \int \frac{du}{r(u)} \right\} dt. \]

where \( G(x), k(x) \) and \( s(x) \) are respectively the survival function, failure rate and MRL of \( Y \).
The above identities along with some characterization theorems cited in Gupta and Kirmani (1990) show how length biased sampling affects the original distribution and how the corresponding reliability characteristics change under such a scheme of sampling. While comparing the distribution under length biased sampling with the parent model, it will be of some definite advantage if the original distribution keeps the same form under length biased sampling, except possibly for a change in the parameters. This will lead to the form invariance property of length biased models and is described as follows.

According to Patil and Ord (1976), the distribution of \( X \) with p.d.f \( f(x; \theta) \) is said to be form-invariant under length biased sampling of order \( \alpha \) if the observed variable \( Y \) has the same distribution as \( X \), with a change in parameter. In other words, \( f(x; \theta) = f(x; \eta) \). Also they proved that a necessary and sufficient condition for \( X \) to be form invariant under size-bias of order \( \alpha \) is that its p.d.f belongs to the log-exponential family. Some important members of this family are the log normal, Pareto, gamma, beta first and second kinds and Pearson type V. Motivated by the relevance of form-invariance in characterizing families of distributions, Sankaran and Nair (1993) derived the condition under which the Pearson family is form-invariant with respect to the length biased sampling. They proved that, the members of the Pearson family satisfying the differential equation (1.18) with \( b_2 \neq 1 \), have the same type of distribution for \( Y \) if and only if \( b_0 = 0 \) and the p.d.f of \( Y \) satisfies

\[
\frac{d \log g(x)}{dx} = \frac{- (x + d_i)}{c_1 x + c_2 x^2}
\]
where
\[ c_i = \frac{b_i}{1 - 2b_2} \quad i = 1, 2 \]
and
\[ d_1 = \frac{d - b_1}{1 - b_2}. \]

Now we prove a general theorem in this direction by identifying those distributions of \( X \) belonging to the generalized Pearson family (2.1) that retain the same form of the distribution of \( Y \). Note that we restrict our study to distribution of non-negative random variables belonging to the family (2.1).

**Theorem 3.4**

Among the members of (2.1), \( X \) and \( Y \) have the same type of distributions if and only if \( b_0 = 0 \) and the probability density function of \( Y \) satisfies

\[
\frac{d\log g(x)}{dx} = \frac{p_0 + p_1 x + p_2 x^2}{q_1 x + q_2 x^2} \tag{3.15}
\]

where \( p_0, p_1, p_2, q_1 \) and \( q_2 \) are real constants.

**Proof**

Suppose that (2.1) holds and that \( X \) and \( Y \) have the same distributional form. Then from (1.19), we have

\[
\frac{d\log g(x)}{dx} = \frac{f'(x)}{f(x)} + \frac{1}{x} \tag{3.16}
\]

or

\[
= \frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2} + \frac{1}{x}
\]
\[
\frac{d \log g(x)}{dx} = \frac{a_1 x + a_2 x^2 + a_3 x^3 + b_0 + b_1 x + b_2 x^2}{x(b_0 + b_1 x + b_2 x^2)}. \tag{3.17}
\]

Since \( Y \) also must belong to the same family, the equation (3.17) must be of the form,
\[
\frac{p_0 + p_1 x + p_2 x^2}{q_0 + q_1 x + q_2 x^2} = \frac{a_1 x + a_2 x^2 + a_3 x^3 + b_0 + b_1 x + b_2 x^2}{x(b_0 + b_1 x + b_2 x^2)}. \tag{3.18}
\]

Equating the coefficients of like powers of \( x \) in (3.18), we have six equations
\[
q_0 b_0 = 0 \tag{3.19}
\]
\[
q_0(a_0 + b_1) + q_1 b_0 = p_0 b_0 \tag{3.20}
\]
\[
q_1(a_0 + b_1) + q_1 b_0 + q_0(a_1 + b_1) = p_1 b_0 + p_0 b_1 \tag{3.21}
\]
\[
q_0 a_0 + q_1(a_1 + b_1) + q_2(a_0 + b_1) = p_1 b_1 + p_2 b_0 + p_0 b_2 \tag{3.22}
\]
\[
(a_1 + b_2) q_2 + a_2 q_1 = p_1 b_2 + p_2 b_1 \tag{3.23}
\]
\[
a_2 q_2 = p_2 b_2 \tag{3.24}
\]

Now consider the equation (3.19), we have three cases,

(i) when \( b_0 \neq 0, q_0 = 0 \) in (3.19), we get
\[
\frac{d \log f(x)}{dx} = \frac{a_2}{b_2} + \frac{(a_1 b_2 - a_2 b_1) x + a_0 b_2 - a_2 b_0}{b_2(b_0 + b_1 x + b_2 x^2)}
\]
and
\[
\frac{d \log g(x)}{dx} = \frac{p_2}{q_2} + \frac{(p_1 q_2 - p_2 q_1) x + p_0 q_2}{q_2(q_0 x + q_2 x^2)}
\]
in which \( f(x) \) and \( g(x) \) have different forms.

Similarly for case (ii) \( b_0 = 0, q_0 \neq 0 \), \( f(x) \) and \( g(x) \) have different forms.

On the other hand, for case (iii) when \( b_0 = 0, q_0 = 0 \), we get
\[
\frac{d \log f(x)}{dx} = \frac{a_2}{b_2} + \frac{(a_1 b_2 - a_2 b_1) x + a_0 b_2}{b_2(b_1 x + b_2 x^2)} \tag{3.25}
\]
and
\[
\frac{d \log g(x)}{dx} = \frac{a_z}{b_z} + \frac{(a_z b_z - a_z b_z + b_z^2)x + (a_z + b_z) b_z}{b_z (b_z x + b_z x^2)}.
\] (3.26)

Since the roots of the quadratic equation in the denominators of equations (3.25) and (3.26) are the same, \( f(x) \) and \( g(x) \) have the same distributional form though with possibly different parameters. Conversely suppose that \( b_0 = 0 \) in (2.1), then from (3.16), we have
\[
\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)} - \frac{1}{x}
\]
\[
= \frac{p_0 + p_1 x + p_2 x^2}{(q_1 x + q_2 x^2)} - \frac{1}{x}
\]
\[
= \frac{(p_0 - q_0) + (p_1 - q_1) x + p_2 x^2}{(q_1 x + q_2 x^2)}
\]
which is of the form (2.1) with \( a_0 = p_0 - q_1, a_1 = p_1 - q_2, a_2 = p_2, b_1 = q_1 \) and \( b_2 = q_2 \). This completes the proof.

**Corollary 3.2**

When \( a_2 = 0 \), Theorem 3.4 reduces to the result of Sankaran and Nair (1993) concerning the original Pearson family.

To verify Theorem 3.4, consider the generalized inverse Gaussian distribution with p.d.f (Johnson, Kotz and Balakrishnan, 1994)
\[
f(x) = K x^r \exp \left\{ -\frac{\lambda (x-\mu)^2}{2x\mu^2} \right\}, \quad x > 0, \ r, \ \lambda, \ \mu > 0
\] (3.27)
where \( K \) is a normalizing constant.

The length biased distribution (LBD) for (3.27) is obtained as
\[
g(x) = K' x^{r+1} \exp \left\{ -\frac{\lambda (x-\mu)^2}{2x\mu^2} \right\}, \quad x > 0, \ r, \ \lambda, \ \mu > 0
\]
which has the same form as (3.27), but different parameters.
Sankaran and Nair (1993) derived the conditions under which models belonging to the Pearson family retain the same form for their length biased distribution. But there are situations where both the original and LBD do not have the same form when they belong to Pearson family.

For example
1. Consider the exponential distribution with p.d.f
   \[ f(x) = \lambda e^{-\lambda x}, \ x > 0, \ \lambda > 0 \]
   then LBD is obtained as
   \[ g(x) = \lambda^2 x e^{-\lambda x}, \ x > 0, \ \lambda > 0 \]
   which has not the same form as \( X \). In fact \( Y \) is gamma.

2. When \( X \) is Pareto type II with p.d.f
   \[ f(x) = c \alpha^c (x+\alpha)^{-(c+1)}, \ x > 0, \ \alpha > 0, \]
   then LBD of \( X \) is
   \[ g(x) = c(c-1)\alpha^c x (x+\alpha)^{-(c+1)}, \ x > 0, c > 1. \]
   Here also \( g(x) \) has different form but both \( X \) and \( Y \) belong to the Pearson family.

3. When \( X \) is half normal with p.d.f
   \[ f(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \ x \geq 0, \ \sigma > 0 \]
   Then LBD of \( X \) is obtained as,
   \[ g(x) = \frac{x}{\sigma^2} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \ x \geq 0 \]
   and hence
   \[ \frac{g'(x)}{g(x)} = \frac{\sigma^2 - x^2}{\sigma^2 x} \]
which is of the form (2.1). Therefore from example 3, we can infer that, it is not necessary that \( X \) is in the Pearson family \( Y \) also belongs to that family. In this direction, next we investigate the condition under which the length biased distribution of \( X \) belong to the generalized Pearson family when the original belongs to the Pearson family.

Suppose that the distribution of \( X \) belongs to the family (1.18) and that of \( Y \) belongs to the family (2.1), we have

\[
\frac{d \log g(x)}{dx} = \frac{d \log f(x)}{dx} + \frac{1}{x}.
\]

Thus

\[
\frac{p_0 + p_1 x + p_2 x^2}{q_0 + q_1 x + q_2 x^2} = \frac{-(x+d)}{b_0 + b_1 x + b_2 x^2} + \frac{1}{x}
\]

which leads to

\[
(p_0 + p_1 x + p_2 x^2)(b_0 + b_1 x + b_2 x^2) = (q_0 + q_1 x + q_2 x^2)(b_0 + (b_1 -d)x + (b_2 -1) x^2).
\]

Equating the coefficients of like powers of \( x \) in (3.28), we have the following six equations

\[
q_0 b_0 = 0 \quad \text{(3.29)}
\]
\[
q_0 (b_1 -d) + q_1 b_0 = p_0 b_0 \quad \text{(3.30)}
\]
\[
q_1 b_0 + q_1 (b_1 -d) + q_0 (b_2 -1) = p_1 b_0 + p_0 b_1 \quad \text{(3.31)}
\]
\[
q_1 (b_2 -1) + q_2 (b_1 -d) = p_1 b_1 + p_2 b_0 + p_0 b_2 \quad \text{(3.32)}
\]
\[
(b_2 -1) q_2 = p_1 b_2 + p_2 b_1 \quad \text{(3.33)}
\]
\[
p_2 b_2 = 0. \quad \text{(3.34)}
\]

Now we have the following cases arising from (3.29) and (3.34).

**Case I**

When \( b_0 = 0, \ b_2 = 0, \ q_0 \neq 0 \) and \( p_2 \neq 0 \), we obtain
This leads to exponential distribution with parameter $\frac{-1}{d}$ for $Y$.

**Case II**

When $b_0 = 0, p_2 = 0, q_0 \neq 0$ and $b_2 \neq 0$, we get

\[ b_1 = d = \frac{-q_0}{p_0}, \quad b_2 = \frac{-q_2}{p_2}. \]

This provides

\[ \frac{d\log g(x)}{dx} = \frac{dp_0}{q_0 + (dp_0 + q_0)x} \]

or

\[ g(x) = Y_0 \left[ \frac{q_0 + (dp_0 + q_0)x}{dp_0 + q_0} \right]^{\frac{p_0}{dp_0 + q_0}}, \quad 0 < x < \infty \]

leading to Lomax law, where $Y_0$ is the normalizing constant.

**Case III**

When $b_2 = 0, q_0 = 0$, we obtain

\[ p_0 = q_1, \quad b_1 = \frac{-q_2}{p_2}, \quad \text{and} \quad b_0 = \frac{dq_1}{q_2 - p_1}, \quad (p_1 \neq q_2). \]

This provides

\[ \frac{d\log g(x)}{dx} = \frac{q_1 + p_1x + p_2x^2}{q_1x + q_2x^2}. \]

Therefore the distribution of $Y$ belongs to generalized Pearson family (2.1).

**Case IV**

When $p_2 = 0, q_0 = 0$, we have,
\[ p_0 = q_0, \quad b_0 = \frac{dp_0}{q_2 - p_1}, \quad b_1 = \frac{q_1 + dq_1}{q_2 - p_1} \quad \text{and} \quad b_2 = \frac{q_2}{q_2 - p_1} \]

and hence

\[ \frac{d \log g(x)}{dx} = \frac{q_1 + p_1 x}{q_1 x + q_2 x^2} \]

or

\[ g(x) = Y_0 \left( q_1 + q_2 x \right)^{q_2} \left( q_1 x + q_2 x^2 \right)^{q_1}, \quad 0 < x < \infty \]

that leads to beta distribution with \( Y_0 \) as normalizing constant.

**Case V**

When \( p_2 = 0, \quad q_0 = 0, \quad b_0 = 0, \quad b_2 = 0 \), we have,

\[ q_2 = 0, \quad b_1 = \frac{dq_1}{q_1 - p_0}. \]

This provides

\[ \frac{d \log g(x)}{dx} = \frac{(p_0 - q_1)x + d(q_1 + p_0)}{dq_1 x} \]

or

\[ g(x) = Y_0 \exp \left\{ \frac{p_0 - q_1}{dq_1} x \right\} \left( q_1 x + q_2 x^2 \right)^{q_1}, \quad 0 < x < \infty \]

which is Pearson type III (Gamma) distribution, where \( Y_0 \) as normalizing constant.

The cases other than the above turns out to be special cases of the case V, will not be discussed further. The above discussion leads to the following characterization theorems whose proofs are direct.
Theorem 3.5

Suppose that the distribution of $X$ belongs to the Pearson family (1.18). Then the length biased distribution (LBD) of $X$ is exponential with parameter $\frac{-1}{d}$, if and only if $b_0 = 0$ and $b_2 = 0$.

Theorem 3.6

Suppose that the distribution of $X$ belongs to the Pearson family (1.18) and that of $Y$ belongs to the family (2.1). Then the distribution of $Y$ is

(a) Lomax (Pareto II) if and only if $b_0 = 0$ and $p_2 = 0$
(b) a member of generalized Pearson family if and only if $q_0 = 0$ and $b_2 = 0$
(c) beta if and only if $q_0 = 0$ and $p_2 = 0$
(d) Pearson Type III (Gamma) if and only if $q_0 = 0$, $p_2 = 0$, $b_0 = 0$ and $b_2 = 0$.

3.4 Characterization by Conditional Expectations

When $b_0 = 0$ in (2.1), we obtain a subclass of the generalized Pearson family. This subclass to be denoted by $C$, contains many distribution of interest in reliability analysis such as gamma, beta, inverted gamma and inverse Gaussian. In the following, we prove a characterization of the class $C$ based on conditional moments.

Theorem 3.7

Let $\lim_{x \to b} (b_1 x + b_2 x^2) f(x) = 0$. Then $f(x)$ belongs to $C$ if and only if

$$a_2 m_3(x) + [(a_1 + 3b_2 - a_2 x)m_2(x) = m_1(x) [(a_1 + 2b_2)x - (a_0 + 2b_1)] + x(a_0 + b_1) \quad (3.35)$$
where
\[
m_i(x) = \mathbb{E}(X^i | X > x), \quad i = 1, 2, 3.
\]

**Proof**

Let \( h(x) \) be the failure rate of \( X \). From theorem 2.3, for the class \( C \), we obtain,
\[
-(b_1x + b_2x^2) h(x) = a_2m_2(x) + (a_1 + 2b_2)m_1(x) + a_0 + b_1
\]
(3.36)

and
\[
-m_1(x)(b_1x + b_2x^2)k(x) = a_2m_3(x) + (a_1 + 3b_2)m_2(x) + (a_0 + 2b_1)m_1(x)
\]
(3.37)

where \( k(x) \) is the failure rate of \( Y \) (LBD).

Using the identity given by Gupta and Kirmani (1990), we have,
\[
k(x) = \frac{x}{h(x)} m_1(x)
\]
which gives,
\[
a_2m_3(x) + (a_1 + 3b_2)m_2(x) + (a_0 + 2b_1)m_1(x)
\]
\[= x[a_2m_2(x) + (a_1 + 2b_2)m_1(x) + a_0 + b_1]. \quad (3.38)
\]

Rearranging the terms in (3.38), we obtain (3.35).

Conversely suppose that (3.35) holds for all \( x \), then we have,
\[
a_2 \int_{x}^{b} t^i f(t) \, dt + (a_1 + 3b_1) \int_{x}^{b} t^2 f(t) \, dt
\]
\[= xR(x)(a_0 + b_1) + [(a_1 + 2b_2)x -(a_0 + 2b_1)] \int_{x}^{b} tf(t) \, dt. \quad (3.39)
\]

Differentiating (3.39) with respect to \( x \), we get
\[
\frac{d \log f(x)}{dx} = \frac{a_0 + a_1 x + a_2 x^2}{b_1 x + b_2 x^2}
\]
which has the same form as (2.1) with \( b_0 = 0 \). This completes the proof.
Corollary 3.3

When $a_2 = 0$, Theorem 3.7 reduces to the result of Sankaran and Nair (1993).

Corollary 3.4

The distribution of $X$ is inverse Gaussian with p.d.f (2.11) holds if and only if

$$2\mu^2 m_3(x) + (\lambda x + 3\mu^2)m_2(x) = m_1(x) (\mu^2 x - \lambda \mu^2) + \lambda \mu^2 x.$$

Corollary 3.5

The relationship

$$-2\lambda m_3(x) + 2\lambda m_2(x) = 3m_1(x) + 2x.$$

holds if and only if $X$ has Rayleigh distribution with p.d.f (2.13).

Corollary 3.6

The relationship

$$-m m_2(x) = m_1(x) [1-m-p]+px.$$

holds if and only if $X$ has gamma distribution with p.d.f

$$f(x) = \frac{m^p}{\Gamma(p)} e^{-mx} x^{p-1}, \ x>0, \ m, \ p>0.$$