CHAPTER 5

A-EXPANSIVENESS ON G-SPACES

Motivated by the concept of expansiveness of a homeomorphism on a metric space, in Chapter 2 we defined the notion of A-expansiveness of a homeomorphism on a topological space $X$ relative to a subset $A$ of $X \times X$; while in Chapter 3 we defined the notion of G-expansiveness of a homeomorphism on a metric G-space wherein $G$ is any topological group acting on the metric space. It is therefore natural to consider the general case of a G-space $X$, that is, a topological space $X$ on which a topological group $G$ acts; and to define and study the notion of expansiveness of a homeomorphism $h$ in this setting. We take up this task in the present chapter. In fact, we define the notion of expansiveness of a homeomorphism $h$ on a G-space $X$ relative to a subset $A$ of $X \times X$ and terming it GA-expansive homeomorphism we carry out their study obtaining some interesting results. Naturally, in case of a metric G-space, for a specific choice of $A$, the concept of GA-expansive homeomorphism coincides with that of G-expansive homeomorphism.

Let $H(X)$ throughout denote the collection of all homeomorphisms on the topological space $X$.

1. GA-expansiveness.

The considerations of the following examples help us to motivate the concept of GA-expansiveness.
Examples 5.1(a). Let $X = [0,1]$ with usual metric. Choose either $A = [b,1] \times [c,d]$ where $b \in (1/2,1)$ and $c, d \in X$ or $A = [0,a] \times [c,d]$ where $a \in (0,1/2)$ and $c, d \in X$. Define $h : X \times X$ by $h(x) = 1 - x$.

Then $h$ is $A$-expansive:

Let $x, y \in X$ be such that $x \neq y$. If $(x,y) \in A$, then for $n = 0$,

$(h^n(x),h^n(y)) \not\in A$

and if $(x,y) \in A$, then

$(h(x),h(y)) \not\in A$.

Next, let the topological group $G = \mathbb{Z}_2 = \{-1,1\}$ act on $X$ with the action $t \cdot x = x$ and $-t \cdot x = 1 - x$, where $t \in X$. Then it can be easily seen that there exist $x, y$ in $X - \{1/2\}$ with distinct $G$-orbits such that for no $n$ in $\mathbb{Z}$

$[ h^n(G(x)) \times h^n(G(y)) ] \cap A = \emptyset$.

5.1(b). Let $X$ be as in Example 5.1(a), $A = [1/5,1/2] \times [1/3,2/3]$ $\subset X \times X$ and $h : X \times X$ be defined by

$h(x) = 3x$, if $x \in [0,1/5]$;

$= (11x+5)/12$, if $x \in [1/5,1/2]$, and

$= (x+3)/4$, if $x \in [1/2,1]$.

It can be easily seen that $h$ is an $A$-expansive homeomorphism on $X$.

Let $G = \mathbb{Z}_2$ act on $X$ as defined in Example 5.1.(a). Then it can be observed that whenever $x, y \in X$ with distinct $G$-orbits, there exists an $n$ in $\mathbb{Z}$ satisfying

$[ h^n(G(x)) \times h^n(G(y)) ] \cap A = \emptyset$.

In Example 5.1(b), given any $A = [a,b] \times [c,d] \subset X \times X$, where

$a, b, c, d \not\in \{0,1\}$, one can construct a suitable $h$ satisfying the
same property, namely given any pair of distinct $G$-orbits $G(x)$ and $G(y)$, there exists an $n$ in $\mathbb{Z}$ such that 
$$[ h^n(G(x)) \times h^n(G(y)) ] \cap A = \emptyset.$$ 
In fact one may define $h$ in such a way that $h([a,b]) \cap [c,d] = \emptyset$. But here we observe that $A$ does not contain the diagonal in $X \times X$. However, we do have similar situation even if $A$ is a regular closed set containing the diagonal as can be seen in the following example.

5.1 (c). Let $X = [0,1]$ with the usual metric and consider the subset $A^\delta$ of $X \times X$ given by 
$$A^\delta = \{ (x/(x+1), y/(y+1)) | x, y \geq 0 \text{ with } |x - y| \leq \delta \} \cup \{(1,1)\},$$ 
where $\delta > 0$ is a fixed real number. Define $h : X \rightarrow X$ by 
$$h(x) = \frac{\beta \cdot x}{1 - (\beta - 1) \cdot x + 1}, \quad x \in X,$$ 
where $\beta$ is a fixed positive real number and $\beta \neq 1$. Then as observed in the Note following Example 2.3 of Chapter 2, $h$ is $A^\delta$-expansive on $X$. Let $G = \mathbb{Z}_2$ act on $X$ as in Example 5.1(a). Then it can be seen that whenever $x, y \in X$ with $G(x) \neq G(y)$, there exists an $r$ in $\mathbb{Z}$ satisfying $[ h^r(G(x)) \times h^r(G(y)) ] \cap A = \emptyset$. For example take $\beta = 2$. Then $\text{Fix} h = \{0,1\}$ and for any $x \in X - \{0,1\}$, $h^n(x) \rightarrow 1$ and $h^{-n}(x) \rightarrow 0$ as $n \rightarrow \infty$.

Thus, there exist integers $i, m, n, k$ such that 
$$(h^i(x), 0) \not\in A^\delta; \quad (0, h^m(y)) \not\in A^\delta;$$ 
$$(h^n(1 - x), 0) \not\in A^\delta \text{ and } (0, h^k(1 - y)) \not\in A^\delta.$$ 
If $r = \max \{i, m, n, k\}$, then it follows that 
$$[ h^r(G(x)) \times h^r(G(y)) ] \cap A^\delta = \emptyset.$$
5.1 (d). Consider the unit circle $S^1$ and the usual action of the multiplicative group $G = U(n)$ of $n$th roots of unity on $S^1$. Let $C_k$ denote the arc $(e^{ik\pi/n}, e^{i(k+1)\pi/n})$ of $S^1$, $k = 0, 1, \ldots, n-1$ and $f_k$ denote the homeomorphism from $I_k = [0,1]$ to $C_k$ given by

$$f_k(s) = e^{i\pi(s+k)/n}$$

where $s \in [0,1]$ and $k = 0, \ldots, n-1$. Since the homeomorphism $g_k$ on $I_k$ defined by $g_k(x) = \beta x/[(\beta-1)x+1]$, for a fixed $\beta$, $\beta > 0$ and $\beta \neq 1$ is $A^\delta$-expansive, where $A^\delta$ is as described in Example 5.1(c), it follows from Theorem 2.3 of Chapter 2 that $f_k g_k f_k^{-1} = h_k$ is $[(f_k \times f_k)(A^\delta)]$-expansive on $C_k$. Define $h : S^1 \to S^1$ by $h|_{C_k} = h_k$,

where $k = 0, 1, \ldots, n-1$. Obviously $h$ is in $H(S^1)$. In fact $h$ is an $\bigcap_{k=0}^{n-1} ((f_k \times f_k)(A^\delta))$-expansive homeomorphism on $S^1$ and the subset

$$B_n = \bigcap_{k=0}^{n-1} [(f_k \times f_k)(A^\delta)]$$

of $S^1 \times S^1$ is a regular closed set which contains the diagonal in $S^1 \times S^1$. In this example also one can verify that for distinct $G$-orbits $G(x)$ and $G(y)$, there exists an $n$ in $Z$ satisfying $[ h^n(G(x)) \times h^n(G(y)) ] \cap B_n = \emptyset$.

The observations made in the above examples lead us to the following definition of GA-expansiveness.

**Definition 5.1.** Let $X$ be a topological space on which a topological group $G$ acts, $A \subseteq X \times X$ and $h \in H(X)$. Then $h$ is called **GA-expansive** if whenever $x, y \in X$ with $G(x) \neq G(y)$, there exists an integer $n$ satisfying $[ h^n(G(x)) \times h^n(G(y)) ] \cap A = \emptyset$.

Observe that a metric space can always be regarded as a metric $G$-space by considering the trivial action of any group $G$ on
it; and hence by choosing $A = A_\delta \equiv d^{-1}[0,\delta]$ for some $\delta > 0$ when $X$ is a metric space with metric $d$ in this definition, one sees that GA-expansiveness of $h$ in $H(X)$ is equivalent to expansiveness of $h$ with expansive constant $\delta$. However, if $X$ is any $G$-space with action of $G$ on $X$ trivial, then the GA-expansiveness of $h$ in $H(X)$ is equivalent to $A$-expansiveness of $h$. Also, in case $X$ is a metric $G$-space and $A = A_\delta$ for some $\delta > 0$, then GA-expansiveness of $h$ in $H(X)$ is equivalent to $G$-expansiveness of $h$ with $G$-expansive constant $\delta$.

Example 5.1 (a) shows that an $A$-expansive homeomorphism need not be GA-expansive and on the other hand Example 3.2 of Chapter 3 shows that a GA-expansive homeomorphism is not necessarily an $A$-expansive homeomorphism.

2. Properties of GA-expansive homeomorphisms.

We study some properties of GA-expansive homeomorphisms. To begin with, the following result regarding the restriction of a GA-expansive homeomorphism follows from the definition.

**Theorem 5.1.** Let $X$ be a $G$-space, $Y$ be a $G$-invariant subspace of $X$, $h \in H(X)$ be GA-expansive where $A \subseteq X \times X$ and $h(Y) = Y$. Then $h|_Y$ is GB-expansive, where $B$ is trace of $A$ in $Y \times Y$.

**Proof.** Suppose $x$ and $y$ are two points in $Y$ with distinct $G$-orbits. Then GA-expansiveness of $h$ on $X$ gives an integer $n$ satisfying $[h^n(G(x)) \times h^n(G(y))] \cap A = \emptyset$. Now the Theorem follows if we take $B = A \cap Y \times Y$. 

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Next, we have a result regarding product of two GA-expansive homeomorphisms.

Theorem 5.2. Let $X, Y$ be $G$-spaces, $A \subset X \times X$, $B \subset Y \times Y$, $h \in H(X)$ be GA-expansive and $f \in H(Y)$ be GB-expansive. Then $\psi = h \times f$ is $G(q^{-1}(A \times B))$-expansive on $W = X \times Y$, where $q : W \times W \rightarrow (X \times X) \times (Y \times Y)$ is defined by $q(x, y, x', y') = (x, x', y, y')$, $x, x' \in X$, $y, y' \in Y$ and $W$ is considered to be a $G$-space under the diagonal action of $G$.

Proof. Let $(x, y), (x', y') \in W$ be such that $G(x, y) \neq G(x', y')$. Since action of $G$ on $W$ is diagonal action, i.e., $g(x, y) = (gx, gy)$, $g \in G$, $(x, y) \in W$, the following cases arise: (i) $G(x) \neq G(x')$ or (ii) $G(y) \neq G(y')$. In case (i) since $G(x) \neq G(x')$, from GA-expansiveness of $h$ there exists an $n \in \mathbb{Z}$ satisfying $[h^n(G(x)) \times h^n(G(x'))] \cap A = \emptyset$ which implies

$$[h^n(G(x)) \times h^n(G(x')) \times f^n(G(y)) \times f^n(G(y'))] \cap (A \times B) = \emptyset.$$ 

Further, as $q$ is a homeomorphism we obtain

$$q^{-1}[h^n(G(x)) \times h^n(G(x')) \times f^n(G(y)) \times f^n(G(y'))] \cap q^{-1}(A \times B) = \emptyset$$

which implies

$$[h^n(G(x) \times G(y)) \times (h \times f)^n(G(x') \times G(y'))] \cap q^{-1}(A \times B) = \emptyset.$$ 

Since $G(x, y) \subseteq G(x) \times G(y)$ and $G(x', y') \subseteq G(x') \times G(y')$, we therefore obtain

$$[(h \times f)^n(G(x, y)) \times (h \times f)^n(G(x', y'))] \cap q^{-1}(A \times B) = \emptyset$$

and hence $h \times f$ is $G(q^{-1}(A \times B))$-expansive on $W$. Similarly Case(ii) follows from GB-expansiveness of $f$ on $Y$.

The above result extends to any finite product of GA-expansive
homeomorphisms and can be proved in a similar way by using induction principle. Next we obtain a result regarding integral powers of a GA-expansive homeomorphism.

Theorem 5.3. Let \( X \) be a paracompact Hausdorff \( G \)-space, \( \mathcal{U} \) be the uniformity on it consisting of all the neighbourhoods of the diagonal in \( X \times X \) and \( h \in H(X) \) be such that \( h^m, m \neq 0 \) is uniformly continuous with respect to \( \mathcal{U} \). Then \( h \) is GA-expansive for some \( A \in \mathcal{U} \) iff \( h^m, m \neq 0 \), is GB-expansive for a suitable \( B \in \mathcal{U} \).

Proof. Consider any integer \( m \) different from 0. Suppose \( i \in \{ \pm 1, \ldots, \pm m \} \). Since for each \( i \), \( h^{-i} \) is uniformly continuous, there exists a \( B_i \) in \( \mathcal{U} \) for each \( i \) such that

\[
(h^{-i} \times h^{-i})(B_i) \subseteq A
\]

or equivalently

\[
(h^i \times h^i)(X \times X - A) \subseteq X \times X - B,
\]

where

\[
B = \cap \{ B_i \mid i \in \{ \pm 1, \ldots, \pm m \} \}.
\]

Let \( x, y \in X \) with \( G(x) \neq G(y) \). Then from the GA-expansiveness of \( h \) there exists an \( n \) in \( \mathbb{Z} \) satisfying \( \{ h^n(G(x)) \times h^n(G(y)) \} \cap A = \emptyset \).

But this gives

\[
\{ h^i(h^n(G(x))) \times h^i(h^n(G(y))) \} \cap B = \emptyset \quad (\ast)
\]

for each \( i \in \{ \pm 1, \ldots, \pm m \} \). Let \( r \) be in \( \mathbb{Z} \) such that \( 0 < |r - n/m| \leq 1 \), i.e., \( 0 < |rm - n| \leq |m| \). Then putting \( i = rm - n \) in \((\ast)\) we get

\[
\{ (h^m)^r(G(x)) \times (h^m)^r(G(y)) \} \cap B = \emptyset
\]

Thus \( h^m \) is GB-expansive, where \( B \in \mathcal{U} \).

Conversely, let \( h \) in \( H(X) \) be such that \( h^m \) is GA-expansive
for some $m$ in $\mathbb{Z} - \{0\}$. Then, for $x, y$ in $X$ with $G(x) \neq G(y)$, the
GA-expansiveness of $h^m$ implies that there exists an $n$ in $\mathbb{Z}$
satisfying
\[ (h^m)^n(G(x)) \times (h^m)^n(G(y)) \cap A = \varnothing. \]
Now put $r = m \cdot n$ to see that $h$ is also GA-expansive.

The following result shows that admitting a GA-expansive
homeomorphism is a topological property for $G$-spaces under some
condition.

Theorem 5.4. Let $X$ and $Y$ be $G$-spaces, $A \subseteq X \times X$ and $f : X \to Y$ be a
pseudoequivariant homeomorphism. Then an $h$ in $H(X)$ is GA-expansive
iff $fh^{-1}$ is a $G((fxf)(A))$-expansive homeomorphism of $Y$.

Proof. Let $y, y' \in Y$ with $G(y) \neq G(y')$. Since $f$ is a homeomorphism,
there exist $x, x'$ in $X$ such that $f(x) = y$, $f(x') = y'$; and
therefore
\[ G(f(x)) \neq G(f(x')). \]
Now, pseudoequivariance of $f$ implies
\[ f(G(x)) \cap f(G(x')) = \varnothing \]
and therefore, $f$ being bijective, we get
\[ G(x) \neq G(x'). \]
Now, GA-expansiveness of $h$ implies the existence of an integer $n$
satisfying
\[ h^n(G(x)) \times h^n(G(y)) \cap A = \varnothing. \]
Again using bijectivity of $f$, it follows that
\[ fh^n(G(f^{-1}(y))) \times fh^n(G(f^{-1}(y'))) \cap (fxf)(A) = \varnothing. \]
As $f$ is pseudoequivariant, from Lemma 3.1 of Chapter 3 it follows that $f^{-1}$ is also pseudoequivariant. Hence

$$[ (fh^{-1}f^{-1})(G(y)) \times (fh^{-1}f^{-1})(G(y')) ] \cap (xf)(A) = \emptyset$$

or equivalently

$$[ (fh^{-1})^n(G(y)) \times (fh^{-1})^n(G(y')) ] \cap (xf)(A) = \emptyset.$$  

This proves that $fh^{-1}$ is $G((xf)(A))$-expansive on $Y$.

Conversely, suppose $x, y \in X$ with distinct $G$-orbits, i.e., $G(x) \neq G(y)$. Then bijectivity of $f$ gives $f(G(x)) \cap f(G(y)) = \emptyset$. Since $f$ is pseudoequivariant, we have $G(f(x)) \cap G(f(y)) = \emptyset$, i.e., $f(x)$ and $f(y)$ also has distinct $G$-orbits. Further, since $fh^{-1}$ is $G((xf)(A))$-expansive on $Y$ it follows that there exists an integer $n$ satisfying

$$[ (fh^{-1})^n(G(f(x))) \times (fh^{-1})^n(G(f(y))) ] \cap (xf)(A) = \emptyset$$

that is

$$[ (fh^{-1})^n(G(f(x))) \times (fh^{-1})^n(G(f(y))) ] \cap (xf)(A) = \emptyset.$$  

Another application of pseudoequivariance of $f$ then gives

$$[ fh^n(G(x)) \times fh^n(G(y)) ] \cap (xf)(A) = \emptyset.$$  

Finally, apply bijectivity of $f$ to obtain

$$[ h^n(G(x)) \times h^n(G(y)) ] \cap A = \emptyset.$$  

Hence $h$ is $G\times A$-expansive on $X$.

3. Extension and characterization of $G\times A$-expansive homeomorphisms.

Next result is regarding extension of $G\times A$-expansive homeomorphisms. If $X$ is a $G$-space and $S$ is a $G$-invariant subspace of $X$, then by $G\times A$-expansiveness of an $h$ in $H(X)$ on $S$ we mean there exists a subset $A$ of $X\times X$ such that whenever $x, y \in S$ with
G(x) ≠ G(y), an integer n will exist satisfying
\[ [h^n(G(x)) \times h^n(G(y))] \cap A = \varnothing. \]

**Theorem 5.5.** Let X be a paracompact Hausdorff G-space, S ⊆ X be such that S is G-invariant and X - S is finite. If h in H(X) is GU-expansive on S, where U is a neighbourhood of the diagonal in X×X, then h is GV-expansive on X for a suitable neighbourhood V of the diagonal in X×X.

**Proof.** Let X - S = \{x_0, x_1, ..., x_n\}. We first show h is GV-expansive on S ∪ \{x_0\}. Since X is a paracompact Hausdorff space and U is a neighbourhood of the diagonal in X×X, there exists a symmetric neighbourhood V' of the diagonal in X×X such that V'∩V' ⊆ U, where V'∩V' = \{ (x,y) ∈ X×X | there exists z in X satisfying (x,z) ∈ V' and (z,y) ∈ V' \}

Since V' contains the diagonal, V' ⊆ V'∩V' ⊆ U.

First note that h being GU-expansive on S, there does not exist two points p_1, p_2 in S such that G(p_1) ≠ G(p_2) and for some g_1, k_1, g_2 in G
\[ (h^n(g_1p_1),h^n(k_1x_0)) \in V' \text{ and } (h^n(g_2p_2),h^n(k_1x_0)) \in V' \]
for each integer n, i.e., there exists at most one point p in S such that for some g, k_1 in G,
\[ (h^n(gp),h^n(k_1x_0)) \in V' \]
for each integer n. In case no such p exists in S then h is GV-expansive on S ∪ \{x_0\}, where V = V'. On the other hand if such a point p exists, then by taking
\[ V = V' - \{ [ (G(p)×G(x_0)) \cup (G(x_0)×G(p)) ] \cap V' \}, \]
one can easily verify that $h$ is GV-expansive on $S \cup \{x_0\}$.

Finally, the required result is proved using induction on the number of elements in $X - S$.

Recall that at the end of Section 1 of the present Chapter, we have observed that the notion of $A$-expansiveness and the notion of GA-expansiveness are independent of each other. In view of this the following characterization of GA-expansive homeomorphism is interesting. We first give a definition.

**Definition 5.2.** Let $X$ be a $G$-space, $A \subset X \times X$ and $h \in H(X)$. Then $h$ is said to **GA-separate** $h$-orbits if given any basis $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ of $X$ with respect to $h$, whenever $G(x_\alpha) \neq G(x_\beta)$, there exists an integer $n$ satisfying $[h^n(G(x_\alpha)) \times h^n(G(x_\beta))] \cap A = \emptyset$.

**Theorem 5.6.** Let $X$ be a $G$-space and $A \subset X \times X$. Suppose $h \in H(X)$ is pseudoequivariant. Then $h$ is GA-expansive iff

(a) $h$ GA-separates $h$-orbits

(b) given $p \in X$ and $n \in \mathbb{Z}$ such that $h^n(p) \notin G(p)$, there exists an integer $r$ satisfying $[h^r(G(p)) \times h^{-n}(G(p))] \cap A = \emptyset$.

**Proof.** Suppose $h$ is a GA-expansive homeomorphism. Then we show that (a) and (b) are true. For (a), let $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ be any basis of $X$ with respect to $h$. Consider $x_\alpha$ and $x_\beta \in \mathcal{B}$ with distinct $G$-orbits. Then by GA-expansiveness of $h$, there exists an $n \in \mathbb{Z}$ satisfying $[h^n(G(x_\alpha)) \times h^n(G(x_\beta))] \cap A = \emptyset$. This proves (a). For
we recall that (Lemma 3.1) as $h$ is pseudoequivariant, so we have

$$h^n(G(x)) = G(h^n(x)) \quad (*)$$

for each $x$ in $X$ and $n$ in $\mathbb{Z}$. Now, suppose there is a $p \in X$ and an $n$ in $\mathbb{Z}$ such that $h^n(p) \not\in G(p)$. Then we obtain an $r$ in $\mathbb{Z}$ for which $(b)$ holds. As $h$ is GA-expansive, so we find an integer $m$ satisfying

$$[h^mG(h^n(p)) \times h^mG(p)] \cap A = \emptyset.$$ 

Using $(*)$ we get

$$[h^{m+n}(G(p)) \times h^m(G(p))] \cap A = \emptyset.$$ 

On substituting $m + n = r$, we finally obtain

$$[h^r(G(p)) \times h^{r-n}(G(p))] \cap A = \emptyset.$$ 

Conversely, suppose $(a)$ and $(b)$ hold. Then we prove that $h$ is GA-expansive. Let $x, y \in X$ with $G(x) \not\subset G(y)$. Then two cases arise:

Either $x$ and $y$ have disjoint $h$-orbits or they intersect. In case $O(x) \cap O(y) = \emptyset$, we choose that basis of $X$ with respect to $h$ which has $x$ and $y$ as its members and then apply $(a)$ to obtain an integer $r$ satisfying

$$[h^r(G(x)) \times h^r(G(y))] \cap A = \emptyset.$$ 

This proves that $h$ is GA-expansive in this case. In the other case when the $h$-orbits of $x$ and $y$ intersect, there exists an integer $n$ for which $x = h^n(y)$. Since $x$ and $y$ are having distinct $G$-orbits, we get $G(y) \not\subset G(h^n(y))$ which implies $h^n(y) \not\subset G(y)$. Now we apply $(b)$ and obtain an integer $r$ satisfying

$$[h^r(G(y)) \times h^{r-n}(G(y))] \cap A = \emptyset$$

which implies

$$[h^r(G(h^{-n}(x))) \times h^{r-n}(G(y))] \cap A = \emptyset.$$
Once again we make use of (*) and obtain 
\[ [ \mathbb{h}^r(G(x)) \times \mathbb{h}^r(G(y)) ] \cap A = \emptyset. \]
This establishes the GA-expansiveness of \( h \) in this case.

The above characterization of GA-expansive homeomorphisms gives the following sufficient condition for the homeomorphic extension of a GB-expansive homeomorphism on a G-invariant subspace \( Y \) of a G-space \( X \) to be GB-expansive on the whole space.

**Theorem 5.7.** Let \( Y \) be a G-invariant subspace of a G-space \( X \) and let \( h \) in \( H(Y) \) be pseudoequivariant GB-expansive, where \( B \subseteq X \times X \). Then a pseudoequivariant homeomorphic extension \( f \) of \( h \) to \( X \) is GB-expansive on \( X \) if

(i) \( f \) is GB-expansive on \( X - Y \) and

(ii) there exists a basis \( S \) of \( Y \) with respect to \( h \) such that

\[ [ G(y) \times (X - Y) ] \cap B = \emptyset, \]

for each \( y \) in \( S \).

**Proof.** For proving the GB-expansiveness of \( f \) in \( H(X) \), we show that conditions (a) and (b) of Theorem 5.6 are satisfied by \( f \).

For (a), choose any basis \( S' = \{ x_\alpha \mid \alpha \in \mathcal{A} \} \) of \( X \) with respect to \( f \) and consider any two members, say \( x_\alpha \) and \( x_\beta \), in \( S' \) with distinct G-orbits. We have following cases:

(i) \( x_\alpha, x_\beta \in Y \);

(ii) \( x_\alpha, x_\beta \in X - Y \) and

(iii) \( x_\alpha \in Y \) and \( x_\beta \in X - Y \) or \( x_\alpha \in X - Y \) and \( x_\beta \in Y \).

In cases (i) and (ii), we apply the fact that \( f\big|_X = h \) and...
are GB-expansive homeomorphisms and get the desired result (here we use the fact that a point lies in a G-invariant set iff the entire G-orbit of that point lies in that set).

Next we consider case (iii). Let us assume $x_\alpha \in Y$ and $x_\beta \in X - Y$. Then $x_\alpha \in G(y)$ for some $y \in \mathbb{R}$, i.e., $x_\alpha = h^n(y)$ for some integer $n$ and therefore by condition (ii) of the hypothesis

$$[ G(h^{-n}(x_\alpha)) \times (X - Y) ] \cap B = \varnothing.$$

Now $X - Y$ being G-invariant subspace of $X$, $G(x_\beta) \subseteq X - Y$. Also $f^{-n}(G(x_\beta)) \subseteq X - Y$. Therefore using pseudoequivariancy of $f|_X = h$, we obtain

$$[ f^{-n}(G(x_\alpha)) \times f^{-n}(G(x_\beta)) ] \cap B = \varnothing.$$

Hence $f$-orbits are GB-separated by $f$, i.e., $f$ satisfies condition (a) of Theorem 5.6. For condition (b), let $p$ in $X$ and integer $n$ be such that $f^n(p) \neq G(p)$. Again, either $p \in Y$ or $p \in X - Y$. If $p \in Y$, then $Y$ being G-invariant one gets $G(p) \subseteq Y$. Also, as $f|_X = h$ is GB-expansive Theorem 5.6 is applicable to the map $f$ on $Y$ and hence there will exists an integer $r$ which satisfies

$$[ f^r(G(p)) \times f^{r-n}(G(p)) ] \cap B = \varnothing.$$

For the case when $p \in X - Y$ we use the fact that $X - Y$ is G-invariant and $f|_{X - Y}$ is GB-expansive and argue exactly as we did when $p \in Y$ to obtain an $n \in \mathbb{Z}$ such that

$$[ f^r(G(p)) \times f^{r-n}(G(p)) ] \cap B = \varnothing.$$

Thus we obtain that $f$ is GB-expansive on whole of $X$.

It may be noted here that Theorems 5.6 and 5.7 reduce to respectively Theorems 1.8 and 1.7 stated in Chapter 1 due to Wine [42].