CERTAIN SUMMABILITY MEANS OF GENERAL ORTHOGONAL SERIES

Let \( \{ \varphi_n(x) \} \) \((n=0, 1, 2, \ldots)\) be an orthonormal system (ONS) of \( L^2 \) integrable functions defined in the closed interval \([a, b]\). We consider the orthogonal series

\[
(2.1.1) \quad \sum_{n=0}^{\infty} c_n \varphi_n(x)
\]

with real coefficients \(c_n\)'s.

Let us denote the partial sums, \((G, \alpha)\) means, \((R, 1)\) means, \((N, p_n)\) means, Riesz means and de-la Vallée Poussin's means, of the series \((2.1.1)\) by

\[
S_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x),
\]

\[
\sigma_n^\alpha(x) = \frac{1}{\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k(x)
\]

\[
L_n(x) = \frac{1}{\log n} \sum_{k=0}^{n} \frac{S_k(x)}{k}
\]

\[
\overline{T}_n(x) = \frac{1}{p_n} \sum_{k=0}^{n} p_k S_k(x),
\]
\[ \sigma_n(\lambda, x) = \sum_{k=0}^{n} \left( 1 - \frac{\lambda^k}{\lambda^{n+1}} \right) c_k \phi_k(x), \]

\[ V_n(x) = \sum_{k=n}^{2n-1} \frac{S_k(x)}{n}, \]

respectively.

We denote as usual the \((C,1)\) means, \((E, 1)\) means, \((R, \lambda_n, 1)\) means and \((N, p_n)\) means of the orthogonal series \(\xi_{\lambda, l}^{(1)}\) by

\[ \sigma_n(x), T_n(x), \sigma_n(\lambda, x) \text{ and } t_n(x). \]

An increasing sequence of natural numbers

\[ n_1 < n_2 < \ldots < n_k < \ldots \]

is said to satisfy the condition \((L)\), if the series \(\sum \frac{1}{n_k}\) satisfies condition \((L)\) i.e.,

\[ \sum_{k=m}^{\infty} \frac{1}{n_k} = O\left( \frac{1}{n_m}\right)^{\frac{1}{2}} \]

Sunouchi \(2)\) has proved the following theorem concerning the \((C, 1)\) means of \(\xi_{\lambda, l}^{(1)}\).

Theorem :- If

\[ |\phi_n(x)| \leq k \quad (n=0,1,2,\ldots) \]

then

1) Bary N.K. \([12]\)
2) Sunouchi G. \([118]\)
\[ \int_{a}^{b} \sum_{n=1}^{\infty} \frac{|S_n(x) - t_n(x)|}{n} \, dx \leq A \sum_{n=1}^{\infty} n^{q-2} |c_n|^q, \quad q > 1. \]

Similarly, Patel\(^1\) and Kantawala\(^2\) have found similar orders of approximation of,

(2.1.3) \[ \sum_{n=1}^{\infty} \frac{|S_n(x) - T_n(x)|^k}{n^k}, \quad k \geq 2 \]

(2.1.4) \[ \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(\lambda, x)|^k}{n^k}, \quad k \geq 2. \]

(2.1.5) \[ \sum_{n=1}^{\infty} \frac{|S_n(x) - t_n(x)|^k}{n^k}, \quad k \geq 2 \]

The convergence of the series (2.1.3), (2.1.4) and (2.1.5) for \( k = 2 \) have been studied by Meder\(^3\), Patel\(^4\) and Kantawala\(^5\).

In this chapter we first prove the analogous results for \((C, \alpha)\) summability, \((\bar{N}, p_n)\) summability, \((R, 1)\) summability, Riesz summability of order 1, de-la Vallee'poussin's summability for \( k = 2 \) and then we extend the above result for \((C, \alpha)\) summability, \((\bar{N}, p_n)\) summability by considering \( k \geq 2 \). We prove the following theorems:

1) Patel R. K. [94] 4) Patel [92]
2) Kantawala [1] 5) Kantawala [50]
3) Meder [76]
Theorem 1: If the coefficients of the orthogonal series (2,1,1) satisfy the condition

\[(2.1.6) \quad \sum_{n=0}^{\infty} c_n^2 < \infty, \]

then

\[\sum_{n=1}^{\infty} \frac{[S_n(x) - q^\alpha_n(x)]^2}{n} < \infty\]

almost everywhere.

Theorem 2: If \( p_0 > 0, \ p_n \geq 0, \ np_n = O(p_n) \) and the condition (2.1.6) is satisfied, then the series

\[\sum_{n=1}^{\infty} \frac{(S_n(x) - T_n(x))^2}{n} < \infty\]

almost everywhere.

Theorem 3: If the coefficients of the orthogonal series (2,1,1) satisfy the condition

\[\sum_{k=1}^{\infty} c_k^2 \log k < \infty, \]

then,

\[\sum_{n=1}^{\infty} \frac{(S_n(x) - L_n(x))^2}{n} < \infty\]

almost everywhere.
Theorem 4 :- If the coefficients of the orthogonal series \((2,1,1)\) satisfy the condition \((2.1.6)\), and

\[1 < q \leq \frac{\lambda_{n+1}}{\lambda_n}\]

then

\[
\lim_{n \to \infty} \frac{\sum_{n=1}^{\infty} (S_n(x) - \sigma_n(\lambda, x))^2}{\sum_{n=1}^{\infty} n^p} < \infty, \quad p > 1
\]

almost everywhere.

Theorem 5 :- If the coefficients of the orthogonal series \((2,1,1)\) satisfy the condition \((2.1.6)\) then

\[
\lim_{n \to \infty} \frac{\sum_{n=1}^{\infty} (S_n(x) - \sigma_n(\lambda, x))^2}{\sum_{n=1}^{\infty} n^p} < \infty, \quad p > 1
\]

almost everywhere.

Theorem 6 :- If \(|\phi_n(x)| \leq k, \ n = 0, 1, 2, \ldots\) then

\[
\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n^\alpha(x)|^q}{n^p} \ dx < k \int_0^\infty \sum_{n=1}^{\infty} ^\infty n^{q-2} |c_n|^q, \quad q > 2.
\]

Theorem 7 :- If \(p_n > 0, p_n > 0, \ np_n = \bigO(p_n)\) and the condition \((2.1.2)\) are satisfied then

\[
\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n^\alpha(x)|^q}{n^p} \ dx = \bigO(1) \sum_{n=1}^{\infty} |c_n|^q n^{q-2}, \quad q > 2.
\]

Dealing with the \((c, 1)\) summability Kolmogoroff\(^1\) has proved the following theorem.

\(^1\) Kolmogoroff [58]
Theorem A: Under the condition
\[ \sum_{n=0}^{\infty} c_n^2 < \infty \]
the relation \( S_{V_n}(x) - \sigma_{V_n}(x) = O_x(1) \) is valid almost everywhere for every index sequence \( \{V_n\} \) with
\[ \frac{V_{n+1}}{V_n} \geq q \geq 1. \]

Similar result was proved by Sharma\(^1\) for \((N, p_n)\) summability using lacunarity. The same result was proved by Sapre\(^2\) with \((L)\) condition, weaker than lacunarity. In this chapter we extend the above results for Nörlund summability and \((N, p_n)\) summability.

Theorem 8: Let \( \{p_n\} \) be a nonnegative monotone sequence of real numbers such that \( p_n \to \infty \) as \( n \to \infty \) and \( np_n = O(P_n) \). If an increasing sequence of natural numbers \( \{V_n\} \) satisfies the condition \((L)\), then we have under the condition \((2.1.6)\), the relation:
\[ S_{V_n}(x) - t_{V_n}(x) = O_x(1) \]
almost everywhere.

Theorem 9: Under the same condition as of theorem 8, we have,
\[ S_{V_n}(x) - \bar{T}_{V_n}(x) = O_x(1) \]
almost everywhere.

1) Sharma [110] 2) Sapre A. R. [106]
We need the following lemmas for proving the above theorems:

**Lemma 1** (Paley's theorem): Let \( \{\phi_n(x)\} \) be an ONS over the interval \((a, b)\) and \( |\phi_n(x)| \leq M \) for \( a < x < b \).

(i) If \( f \in L^p, 1 < p \leq 2 \) and \( c_1, c_2, \ldots, c_n, \ldots \) are the Fourier coefficients of \( f \) with respect to \( \phi_1, \phi_2, \ldots, \phi_n, \ldots \), then

\[
\left\{ \sum_{n=1}^{\infty} |c_n|^p \right\}^{\frac{1}{p}} \leq A_p \left\{ \frac{1}{a} \int_{b}^{d} |f|^p \, dx \right\}^{\frac{1}{p}}
\]

where \( A_p \) depends only on \( p \) and \( M \).

(ii) If \( q > 2 \) and \( c_1, c_2, \ldots, c_n, \ldots \) is a sequence of numbers for which

\[
\sum_{n=1}^{\infty} |c_n|^q < \infty
\]

then a function \( f(x) \in L^q (a, b) \) exists, for which the numbers \( c_n \) are Fourier coefficients with respect to the system \( \{\phi_n(x)\} \), and

\[
\left\{ \frac{1}{a} \int_{b}^{d} |f|^q \, dx \right\}^{\frac{1}{q}} \leq B_q \left\{ \sum_{n=1}^{\infty} |c_n|^q \right\}^{\frac{1}{q}}
\]

1) Bary N.K. [12]
where \( B_q \) depends only on \( q \) and \( M \).

**Lemma 2** - Suppose that \( p_n \) is non-increasing and that \( p_n \geq \sigma > 0, \ n = 0, 1, 2, \ldots \).

Then \((N, p_n)\) summability reduces to \((N, p^n)\) summability.

**Lemma 3** - If the coefficients of the orthogonal series (2.1.1) satisfies the condition (2.1.6) and

\[
\left\{ p_n \right\} (- M^\alpha, \ \alpha > 0),
\]

then the series,

\[
\sum_{n=1}^{\infty} \frac{(S_n(x) - T_n(x))^2}{n} < \infty,
\]

almost everywhere.

**Lemma 4** - If \( p > 0, p_n \geq 0 \) and the condition, \( \eta_k = O(p_n) \)

\[
|\phi_n(x)| \leq k
\]

is satisfied then

\[
\int_{a}^{b} \left[ \sum_{n=1}^{\infty} \frac{|S_n(x) - T_n(x)|^q}{n} \right] dx = O(1) \sum_{n=1}^{\infty} |c_n|^q \]

where \( q > 2 \).

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1) Ishiguro Kazuo [45]
2) Kantawala P.S. [50]
Proof of Theorem 1.

We have,

\[ S_n(x) - \sigma_n^\alpha(x) = \sum_{k=0}^{n} c_k \phi_k(x) - \frac{1}{A_n^\alpha} \sum_{V=0}^{n} A_{n-V} S_V(x) \]

\[ = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{V=0}^{n} A_{n-V} - \frac{1}{A_n^\alpha} \sum_{V=0}^{n} A_{n-V} S_V(x) \]

consequently,

\[ (2.1.7) \quad \sum_{n=1}^{\infty} \frac{1}{n} \int_{a}^{b} \left[ S_n(x) - \sigma_n^\alpha(x) \right]^2 \, dx \]

\[ = \sum_{n=1}^{\infty} \frac{1}{n(A_n^\alpha)^2} \sum_{k=0}^{n} c_k^2 \sum_{V=0}^{n} A_{n-V}^\alpha \]
Zygmund\(^1\) has proved that \(A_n^\alpha\) is increasing (as a function of \(n\)) for \(\alpha > 0\) and \(A_n^\alpha\) is decreasing for \(-1 < \alpha < 0\).

For \(\alpha > 1\), the condition (2.1.7) gives
\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_{a}^{b} \left( S_n(x) - \sigma_n^\alpha(x) \right)^2 \ dx
\]
\[
\leq \sum_{n=1}^{\infty} \frac{1}{n(n^\alpha)^2} \sum_{k=0}^{n} k^2 c_k^2 \left( A_n^{\alpha-1} \right)^2
\]
\[
= O(1) \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n} \frac{n^{2\alpha-2}}{n^{2\alpha}} k^2 c_k^2
\]
\[
= O(1) \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 c_k^2
\]
\[
= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \frac{1}{k^2}
\]
\[
= O(1) \sum_{k=1}^{\infty} c_k^2 < \infty.
\]

Therefore, by B. Levy's theorem\(^2\),
\[
\sum_{n=1}^{\infty} \left( \frac{[S_n(x) - \sigma_n^\alpha(x)]^2}{n} \right)
\]
converges almost everywhere in \((a, b)\).

1) Zygmund \([149]\)
2) Alexits G. \([5]\)
Again for \( \alpha < 1 \), the condition (2.1.7) gives

\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_{a}^{b} \left[ S_n(x) - \sigma_n^{\alpha}(x) \right]^2 \, dx
\]

\[
\leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n} c_k^2 (A_n - k + 1)^{\alpha-1}
\]

\[
= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{(n-k+1)^{2\alpha-2}}{n^{2\alpha+1}}
\]

\[
= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^{2\alpha+1}}
\]

\[
= O(1) \sum_{k=1}^{\infty} c_k^2 < \infty
\]

Therefore, by B. Levy's theorem,

\[
\sum_{n=1}^{\infty} \frac{(S_n(x) - \sigma_n^{\alpha}(x))^2}{n} < \infty
\]

almost everywhere in \((a, b)\).

Thereby the theorem is completely proved.

Proof of Theorem 2 :- We have

\[
S_n(x) - \bar{T}_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x) - \frac{1}{p_n} \sum_{r=0}^{n} p_r S_r(x)
\]

\[
= \frac{1}{p_n} \sum_{k=0}^{n} c_k \varphi_k(x) \sum_{r=0}^{n} p_r - \frac{1}{p_n} \sum_{r=0}^{n} p_r \sum_{k=0}^{n} c_k \varphi_k(x)
\]
\[
\frac{1}{p} \sum_{k=0}^{n} c_k \phi_k(x) - \sum_{i=0}^{r} p_i \leq \frac{1}{p} \sum_{k=0}^{n} c_k \phi_k(x) - \sum_{i=k}^{r} p_i.
\]

Consequently,
\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_{a}^{b} (S_n(x) - T_n(x))^2 \, dx
\]
\[
= \sum_{n=1}^{\infty} \frac{1}{n p_n^2} \sum_{k=0}^{n} c_k^2 \left( \sum_{i=0}^{k} p_i^2 \right).
\]

If \( \{ p_n \} \) is increasing, then the condition (2.1.6) and \( n p_n = \mathcal{O}(P_n) \) gives,
\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_{a}^{b} (S_n(x) - T_n(x))^2 \, dx \leq \sum_{n=1}^{\infty} \frac{1}{n n p_n^2} \sum_{k=0}^{2} c_k^2 \sum_{n=k}^{\infty} \frac{p_n^2}{n p_n^2}
\]
\[
= \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^3}
\]
\[
= \mathcal{O}(1) \sum_{k=1}^{\infty} c_k^2 < \infty.
\]

Hence by B. Levy's theorem,
\[
\sum_{n=1}^{\infty} \frac{(S_n(x) - T_n(x))^2}{n} < \infty.
\]
almost everywhere in \([a, b]\).

If \([p_n]\) is non-increasing then by Lemma 2 \((N, p_n)\) summability reduces to \((N, p_n)\) summability and for \((N, p_n)\) summability the same result was discussed by Agrawal and Kantawala\(^1\) (see Lemma 3).

**Proof of Theorem 3:**

We have,

\[
S_n(x) - L_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x) - \frac{1}{\log n} \sum_{v=1}^{n} \frac{S_v}{V} \]

\[
= \sum_{k=0}^{n} c_k \varphi_k(x) - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{V} \sum_{v=1}^{k} c_v \varphi_v(x) \]

\[
= \sum_{k=0}^{n} c_k \varphi_k(x) - \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{V} \sum_{v=1}^{k} c_v \varphi_v(x) \sum_{v=k}^{n} \frac{1}{V} \]

Since

\[
\lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n \right) \text{ is finite,}
\]

and

\[
o < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n, \text{ for } n = 1, 2, \ldots
\]

\[
\ldots, \text{ there exist such a constant } M > 1 \text{ such that}
\]

\(^1\) Agrawal S. R. and Kantawala P. S. [1]
\[(2.1.8)\quad 0 < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \quad \log n < M\quad \text{for} \quad n = 1, 2, \ldots \quad \text{so by inequality,}\]
\[
(a + b)^2 \leq 2(a^2 + b^2)
\]

We have,
\[
(2.1.9)\quad \left( \sum_{\nu=1}^{n} \frac{1}{\nu} \right)^2 \leq 2(M^2 + \log^2 n)\quad \text{for} \quad n = 1, 2, \ldots
\]

Hence by (2.1.8) and (2.1.9) we have

\[
\int_{a}^{b} (S_n(x) - L_n(x))^2 \, dx < \frac{4M^2}{\log^2 n} \sum_{k=1}^{n} c_k^2 + \sum_{k=1}^{n} c_k^2 \log^2 k.
\]

Let
\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_{a}^{b} (S_n(x) - L_n(x))^2 \, dx = I_1.
\]

Therefore,

\[
I_1 < 4M^2 \left[ \sum_{n=4}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} c_k^2 + \sum_{n=4}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} c_k^2 \log^2 k \right]
\]

\[
= 4M^2 \left[ \Sigma_1 + \Sigma_2 \right].
\]

Now
\[
\Sigma_1 = \sum_{n=4}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} c_k^2
\]

\[
< \sum_{k=1}^{\infty} c_k^2 \sum_{n=k}^{\infty} \frac{1}{n \log^2 n}
\]

\[
< \infty.
\]
Therefore,

\[ \sum_{n=1}^{\infty} \frac{\left( \frac{\sqrt{2\pi}}{\Gamma(n)} \right)^n}{n!} \frac{1}{\log(n)} \leq 2. \]

Hence by B. Levy's theorem,

\[ \sum_{n=1}^{\infty} \frac{(S_n(x) - L_n(x))^2}{n} \]

is convergent almost everywhere.

Proof of Theorem 4 :-

\[ S_n(x) - \sigma_n(\lambda, x) = \sum_{k=0}^{n} c_k \phi_k(x) - \sum_{k=0}^{n} \frac{\lambda}{\lambda_{n+1}} \phi_k(x) \]

\[ = \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} c_k \phi_k(x) \lambda_k \]
i.e.

\[
\int_a^b \left[ S_n(x) - \sigma_n(\lambda, x) \right]^2 \, dx = \frac{1}{2} \sum_{n=1}^{\infty} \lambda_{n+1}^2 c_k^2
\]

i.e.

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \int_a^b \left[ S_n(x) - \sigma_n(\lambda, x) \right]^2 \, dx = \sum_{n=1}^{\infty} \frac{1}{n^p} \lambda_{n+1}^2 c_k^2
\]

\[
= \sum_{k=0}^{\infty} c_k^2 \sum_{n=1}^{\infty} \frac{\lambda_k^2}{\lambda_{n+1}^2} \frac{1}{n^p}
\]

\[
= \sum_{k=0}^{\infty} c_k^2 \left[ \frac{\lambda_k^2}{\lambda_{k+1}^2} \frac{1}{k^p} + \frac{\lambda_k^2}{\lambda_{k+2}^2} \frac{1}{(k+1)^p} + \ldots \right]
\]

\[
= \sum_{k=0}^{\infty} c_k^2 \frac{1}{q^2} \sum_{l=k}^{\infty} \frac{1}{q^2(l+1)^p}
\]

\[
= O(1) \sum_{k=0}^{\infty} c_k^2
\]

\[
< \infty
\]

Therefore, by B.Levy's theorem,

\[
\sum_{n=1}^{\infty} \frac{\left[ S_n(x) - \sigma_n(\lambda, x) \right]^2}{n^p} < \infty, \quad p > 1.
\]
Proof of Theorem 5: 

\[
[S_n(x) - V_n(x)]^2 = [\sigma_n(\lambda, x) - V_n(x) + S_n(x) - \sigma_n(\lambda, x)]^2
\]

\[
\leq 2 \left\{ [\sigma_n(\lambda, x) - V_n(x)]^2 + [S_n(x) - \sigma_n(\lambda, x)]^2 \right\}
\]

\[
\sum_{n=1}^{\infty} \frac{[S_n(x) - V_n(x)]^2}{n^p} \leq 2 \left\{ \sum_{n=1}^{\infty} \frac{(\sigma_n(\lambda, x) - V_n(x))^2}{n^p} + \sum_{n=1}^{\infty} \frac{(S_n(x) - V_n(x))^2}{n^p} \right\}
\]

\[
= 2 \left[ M_1 + M_2 \right].
\]

But \( M_1 \) is convergent by Patel\(^1\)

\( M_2 \) is convergent by Theorem 2.

Hence Theorem is proved.

Proof of Theorem 6: 

\[
S_n(x) - \sigma^\alpha_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x) - \frac{1}{\alpha} \sum_{V=0}^{n-\alpha} A_{n-V} S_V(x)
\]

\[
= \frac{1}{\alpha} \sum_{k=0}^{n} c_k \varphi_k(x) \sum_{V=0}^{n-\alpha} A_{n-V} - \frac{1}{\alpha} \sum_{V=0}^{n} S_V(x)
\]

\[
\alpha-1 \sum_{n=V} A_{n-V} S_V(x)
\]

\[\text{1) Patel C. M. [92]}\]
\[
\frac{1}{\alpha} \sum_{k=0}^{n} c_k \varnothing_k(x) \sum_{V=0}^{n \alpha-1} A_{n-V} = \frac{1}{\alpha} \sum_{V=0}^{n \alpha-1} A_{n-V} \sum_{k=0}^{n} c_k \varnothing_k(x) \]

Using Lemma 1 we have,

\[
\int_{a}^{b} |S_n(x) - \sigma_n(x)|^q dx = \int_{a}^{b} \sum_{k=0}^{n} c_k \varnothing_k(x) R_k |^q dx
\]

Therefore,

\[
\int_{a}^{b} \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(x)|^q}{n} dx \lesssim A_1 \sum_{k=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |c_k| |R_k|^k q q^{-2}
\]
Zygmund has proved that $A_n^\alpha$ is an increasing (as a function of $n$) for $\alpha > 0$ and $A_n^\alpha$ is decreasing for $-1 < \alpha < 0$.

For $\alpha > 1$, we have

$$\sum_{n=1}^{\infty} \frac{k^q}{n^q} \frac{1}{n^q} \cdot \mathcal{O}(1) \frac{1}{n}$$

$$= \mathcal{O}(1) \sum_{n=k}^{\infty} \frac{1}{n^q}$$

Therefore condition (2.1.1b) gives,

$$\int_a^b \frac{\left| S_n(x) - \sigma^\alpha_n(x) \right|^q}{n} \, dx = \mathcal{O}(1) \sum_{k=1}^{\infty} \frac{1}{k^q} \frac{1}{n^q}$$

For $0 < \alpha < 1$, we have

$$\sum_{n=k}^{\infty} \frac{k^q}{n^q} \frac{1}{n^q} \cdot \mathcal{O}(1) \frac{1}{n^q}$$

$$= \sum_{n=k}^{\infty} \frac{1}{n^q} \cdot \mathcal{O}(1) \left\{ \frac{1}{n} \right\}$$
Therefore, condition (2.1.10) gives
\[
\int_a^b \left( \frac{S_n(x) - \sigma^q_n(x)}{n} \right)^q \, dx = \mathcal{O}(1) \sum_{k=1}^q c_k^{k-2} k^{q-2}
\]

This completes the proof of our theorem.

Proof of Theorem 7 :-

We have,
\[
S_n(x) - \bar{T}_n(x) = \frac{1}{P_n} \sum_{k=0}^{n-1} c_k \varphi_k(x) \sum_{r=0}^{k-1} p_r
\]

where \( R_k = \frac{1}{P_n} \sum_{r=0}^{k-1} p_r \).

Therefore by using Lemma 1, we have,
\[
\int_a^b |S_n(x) - \bar{T}_n(x)|^q \, dx = \int_a^b \left( \sum_{k=0}^{n-1} c_k \varphi_k(x) \right)^q \left( \sum_{r=0}^{k-1} |R_k|^{q-2} \right) \, dx
\]

Hence,
(2.1.11) \[
\int_a^b \sum_{n=1}^\infty \frac{|S_n(x) - T_n(x)|^q}{n} \, dx < A_1 \sum_{n=1}^\infty \frac{1}{n} \sum_{k=1}^n |c_k|^q R_k^q \quad k^{q-2}
\]

\[
= A_1 \sum_{n=1}^\infty \sum_{k=1}^n |c_k|^q k^{q-2}
\]

If \( p_n \) is increasing, then

\[
\sum_{r=0}^{k-1} p_r \leq k p_n = k \bigvee (p_n)
\]

\[
\sum_{r=0}^{k-1} p_r = k \bigvee (p_n)
\]

Hence,

\[
\left( \sum_{r=0}^{k-1} p_r \right)^q
\]

\[
\frac{\left( \sum_{r=0}^{k-1} p_r \right)^q}{n p_n^q} = \bigvee (1) \quad \frac{k^q}{n^{q+1}}
\]

Hence R.H.S. of (2.1.11) is

\[
= A_1 \sum_{k=1}^\infty |c_k|^q \sum_{n=k}^\infty \frac{1}{n^{q+1}}
\]

Consequently, from (2.1.11)

\[
\int_a^b \sum_{n=1}^\infty \frac{|S_n(x) - T_n(x)|^q}{n} \, dx = \bigvee (1) \sum_{k=1}^\infty |c_k|^q k^{q-2}
\]
If \( p_n \) is non increasing then by Lemma 2 \((N, p_n)\) summability reduces to \((N, p_n)\) summability and for \((N, p_n)\) summability the same result was discussed by Agrawal and Kantawala \(^1\) (see Lemma 4). This completes the proof of our theorem.

**Proof of Theorem 8:**

We have,

\[
S_{V_n}(x) - t_{V_n}(x) = \sum_{k=0}^{V_n} c_k \hat{\phi}_k(x) - \frac{1}{p_{V_n}} \sum_{k=0}^{V_n} p_{V_n-k} S_k(x)
\]

\[
= \frac{1}{p_{V_n}} \sum_{k=0}^{V_n} c_k \hat{\phi}_k(x) \left( \sum_{i=0}^{k-1} p_{V_{n-i}} \right)
\]

Therefore,

\[
\int_a^b (S_{V_n}(x) - t_{V_n}(x))^2 \, dx = \frac{1}{p_{V_n}} \sum_{k=0}^{V_n} c_k^2 \left( \sum_{i=0}^{k-1} p_{V_{n-i}} \right)^2
\]

Now if \( \{p_n\} \) is increasing,

\[
\sum_{i=0}^{k-1} p_{V_{n-i}} = p_{V_n-k+1} + \cdots + p_{V_n}
\]

\[
< k \cdot p_{V_n}
\]

1) Agrawal and Kantawala \([1]\)
Hence,

\[
\int_{a}^{b} \left( S_{n}(x) - t_{n}(x) \right)^2 \, dx < \frac{1}{p_{V_{n}}} \sum_{k=0}^{V_{n}} c_k^2 \frac{2^{V_{n}}}{p_{V_{n}}} \sum_{k=0}^{V_{n}} k^2 c^2_k
\]

\[
= \mathcal{O}(1) \frac{1}{V_{n}} \sum_{k=0}^{V_{n}} k^2 c^2_k
\]

\[
(2.1.12) \sum_{n=1}^{\infty} \left( S_{n}(x) - t_{n}(x) \right)^2 \, dx = \mathcal{O}(1) \sum_{n=1}^{\infty} \frac{1}{V_{n}} \sum_{k=0}^{V_{n}} k^2 c^2_k
\]

Similarly of \( \{ p_{n} \} \) is decreasing,

\[
\sum_{i=0}^{k-1} p_{V_{n}-i} = p_{V_{n}-k+1} + \cdots + p_{V_{n}} < k \, p_{V_{n}-k+1}
\]

\[
\int_{a}^{b} \left( S_{n}(x) - t_{n}(x) \right)^2 \, dx < \frac{1}{p_{V_{n}}} \sum_{k=0}^{V_{n}} c_k^2 \frac{2^{V_{n}}}{p_{V_{n}}} \sum_{k=0}^{V_{n}-k+1} \frac{1}{V_{n}} \sum_{k=0}^{V_{n}} k^2 c^2_k
\]

\[
= \mathcal{O}(1) \frac{1}{V_{n}} \sum_{k=0}^{V_{n}} k^2 c^2_k
\]

Therefore,

\[
(2.1.13) \sum_{n=1}^{\infty} \int_{a}^{b} \left( S_{n}(x) - t_{n}(x) \right)^2 \, dx = \mathcal{O}(1) \sum_{n=1}^{\infty} \frac{1}{V_{n}} \sum_{k=0}^{V_{n}} k^2 c^2_k
\]
From (2.1.12) and (2.1.13), for \{p_n\} monotone,

\[
\sum_{n=1}^{\infty} \int_a^b \left( S_n(x) - \tau_n(x) \right)^2 \, dx = \mathcal{O}(1) \sum_{n=1}^{\infty} \frac{V_n}{V_n^2} \sum_{k=0}^{2} \frac{k^2 c_k}{V_n^k}.
\]

Now we will prove the convergence of the series on the right of (2.1.14). Taking the first \(p\) terms of the series on the right side of (2.1.14),

\[
\frac{V_1}{V_1} \sum_{k=1}^{1} \frac{k^2 c_k}{V_1^k} + \frac{V_2}{V_2} \sum_{k=1}^{2} \frac{k^2 c_k}{V_2^k} + \cdots + \frac{V_p}{V_p} \sum_{k=V_p-1+1}^{4} \frac{k^2 c_k}{V_p^k}.
\]

But as the sequence \(\{V_n\}\) satisfy the condition (L) so also the sequence \(\{V^2_n\}\) (see Bary [2], page 8).

Therefore,

\[
\sum_{m=k}^{\infty} \frac{1}{V_m^2} < c \frac{1}{V_k^2}, \quad \text{where} \quad c \text{ is constant}.
\]

Hence,

\[
\sum_{n=1}^{p} \sum_{k=0}^{2} \frac{k^2 c_k}{V_n^k} < \sum_{k=1}^{V_1} \frac{k^2 c_k}{V_1^k} - \frac{1}{V_2^2} + \sum_{k=V_1+1}^{V_2} \frac{k^2 c_k}{V_2^k} - \frac{1}{V_3^2} + \cdots + \sum_{k=V_p-1+1}^{V_p} \frac{k^2 c_k}{V_p^k} - \frac{1}{V_{p+1}^2}.
\]
From this, the convergence of series on the right in (2.1.16) follows. So by B. Levy's theorem result directly follows. Hence the proof.

**Proof of Theorem 9:**

We have,

\[
S_n(x) - \overline{T}_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x) - \frac{1}{p_n} \sum_{k=0}^{n} p_k S_k(x)
\]

\[
= \frac{1}{p_n} \sum_{V=0}^{n} p_V \sum_{k=0}^{n} c_k \varphi_k(x) - \frac{1}{p_n} \sum_{k=0}^{n} p_k S_k(x)
\]

\[
= \frac{1}{p_n} \sum_{k=0}^{n} c_k \varphi_k(x) \sum_{V=0}^{k-1} p_V
\]

Now,

\[
\int_{a}^{b} (S_n(x) - \overline{T}_n(x))^2 \, dx = \frac{1}{p_n^2} \sum_{k=0}^{n} c_k^2 \sum_{V=0}^{k-1} p_V^2
\]

\[
\leq \frac{1}{p_n^2} \sum_{k=0}^{n} c_k^2 p_k^2
\]
Therefore,
\[
\sum_{n=1}^{\infty} \frac{b}{n^{1/a}} \int_{a}^{b} \left( S_n(x) - \Pi_n(x) \right)^2 \, dx < \sum_{n=1}^{\infty} \frac{1}{p_n^2} \sum_{k=0}^{n} c_k^2 p_k^2
\]

\[
= O(1) \sum_{n=1}^{\infty} \frac{1}{n^{2/p_n^2}} \sum_{k=0}^{n} c_k^2 p_k^2
\]

since \( \{ p_n \} \) is increasing we have,
\[
p_k < p_n \quad \text{for} \quad k < n.
\]

So,
\[
\sum_{n=1}^{\infty} \left( S_n(x) - \Pi_n(x) \right)^2 \, dx = O(1) \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n} c_k^2 k^2.
\]

Now replacing \( n \) by \( v_n \) in the above inequality we have,
\[
\sum_{n=1}^{\infty} \frac{b}{v_n^{1/a}} \int_{a}^{b} \left( S_{v_n}(x) - \Pi_{v_n}(x) \right)^2 \, dx = O(1) \sum_{n=1}^{\infty} \frac{1}{v_n^{2/p_n^2}} \sum_{k=0}^{v_n} c_k^2 k^2 v_n^2.
\]

The convergence of the right side of the above inequality follows from theorem \( \varphi \). Hence by B. Levy's theorem the result directly follows. Hence the proof.

If \( \{ p_n \} \) is nonincreasing then by Lemma 2, \( (\bar{N}, p_n) \) summability reduces to \( (N, p_n) \) summability and result follows from theorem \( \varphi \).

Hence the proof.