Let $p_0(x), p_1(x), p_2(x), \ldots, p_n(x)$ be the system of normalized orthogonal polynomials in the interval $(-1, 1)$, corresponding to a positive bounded and summable weight function $w(x)$. The Fourier expansion corresponding to a function $f(x) \in L^2[-1, 1]$ in this system is given by

\begin{equation}
(7.1.1) \quad f(x) \sim \sum_{n=0}^{\infty} C_n p_n(x).
\end{equation}

Let

\begin{align*}
S_n(x) &= \sum_{v=0}^{n} C_v p_v(x) \\
t_n(x) &= \frac{1}{p_n} \sum_{v=0}^{n} p_{n-v} S_v(x) \\
\sigma_n^\alpha(x) &= \frac{1}{A_n^{\alpha}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} S_v(x)
\end{align*}

be sequence of partial sums, $(N, p_n)$ means and $(C, \alpha)$ means respectively, of the series (7.1.1).
Let \( \{p_n(x)\} \) be a system of orthonormal polynomial system belonging to the distribution \( d\mu(x) \).

Let \( \mathcal{H} \) denote a class of functions and

\[
S_n(x) = \sum_{k=0}^{n} a_{nk} p_k(x)
\]

be a linear combination of \( p_0(x), p_1(x), \ldots, p_n(x) \), with \( a_{nk} \). Let \( d_n^{(2)}(f) \) denote the lower bound of all the numbers,\footnote{\[ \int_a^b \left[ f(x) - S_n(x) \right]^2 d\mu(x) \]}

\[
\text{formed with arbitrary } S_n(x).
\]

Then,

\[
p_n(\mathcal{H}, 2) = \sup_{f \in \mathcal{H}} d_n^{(2)}(f)
\]

is called the best degree of approximation for the class \( \mathcal{H} \) in the sense of \( L^2 \) approximation.

If for some sequence \( \{p_n\} \) the condition

(i) \( 0 < p_n < p_{n+1} \) for \( n = 0, 1, 2, \ldots \)

or

\( 0 < p_{n+1} < p_n \) for \( n = 0, 1, 2, \ldots \)
(ii) \( p_0 + p_1 + \ldots + p_n = p_n \uparrow \infty \)

(iii) \[
\lim_{n \to \infty} \frac{n \Delta p_n}{p_n} = \alpha
\]

where \( \alpha > 0 \),

are satisfied, then we shall say that the sequence \( \{p_n\} \)

belongs to the class \( M^\alpha \).

Let \( w(f, \delta, a, b) \) denote the continuity modulus of the function \( f(x) \) in the interval \([a, b]\) i.e.

\[
w(f, \delta, a, b) = \sup_{|t - x| < \delta} |f(t) - f(x)|
\]

\[t, x \in [a, b].\]

\( w(f, \delta, a, b) \) can be represent as \( w(f, \delta) \). We denote by \( w(\delta) \) a majorant function of \( w(f, \delta, a, b) \) i.e. a function satisfying the condition

\( w(\delta) \geq w(f, \delta, a, b). \)
Cesàro, Kiesz, Euler and Nörlund summability of series of orthogonal functions (not necessarily polynomials) have been studied by Alexits G.\(^1\), Kaczmarz\(^2\), Menchoff\(^3\), Meder\(^4\), Tandori\(^5\), Lorentz\(^6\), Zygmund\(^7\), Patel C. M.\(^8\), Bhatnagar P. C.\(^9\) and Leindler L.\(^10\), Jackson D.\(^11\) applied the Cesàro summability to series of orthogonal polynomials for the first time. He proved the following theorem for Cesàro mean of order 1.

**Theorem A:**

If the weight function \(g(t)\) is a bounded function and if \(g(t) \varphi^2(t)\) is summable in the interval \((-1, 1)\), then the series \((7.1.1)\) is summable \((C, 1)\) to a function \(f(x)\) in \((-1, 1)\). Where,

\[
\phi(t) = \frac{f(t) - f(x)}{t - x}
\]

2) Kaczmarz [52] 8) Patel C.M. [92]
4) Meder J. ([79], [80]) 10) Leindler [64]
5) Lorentz [71] 11) Jackson [48]
6) Tandori ([127], [128])
Similar results were proved by Patel\textsuperscript{1}), Bhatnagar\textsuperscript{2}) for Riesz and Euler means of order 1.

In this chapter we prove the following theorems for Nörlund and Cesaro summability.

**Theorem 1:**

If \( g(t) \varphi^2(t) \) is summable in \((-1, 1)\) then the series \( \{7, 1, 1\} \) is \((N, p_n)\) summable to a function \( f(x) \) in \((-1, 1)\) if \( p_n \in \mathcal{M}^\alpha \).

**Theorem 2:**

If \( g(t) \varphi^2(t) \) is summable in \((-1, 1)\) then the series \( \{7, 1, 1\} \) is \((C, g)\) summable in \((-1, 1)\).

Jackson D.\textsuperscript{3}) has proved several theorems while discussing the degree of approximation with the help of trigonometrical system. One of the theorems,

\begin{align*}
1) & \text{ Patel } [91] \\
2) & \text{ Bhatnagar } [15] \\
3) & \text{ Jackson } [46]
\end{align*}
for the degree of approximation in the sense of $L^2$ approximation, proved by Jackson\textsuperscript{1)} is as follows.

**Theorem B:**

Let \( \mathcal{G}_n(\mathcal{K}, 2) \) denote the best approximation in the $L^2$ space of the class $\mathcal{K}$ of all the $2\pi$ periodic functions $f(x)$ possessing an $L^2$ continuity modulus $w_2(f, \delta) < w_2(\delta)$,

where $w_2(\delta)$ is a majorant of $w_2(f, \delta)$,

Then we have

\[
\mathcal{G}_n(\mathcal{K}, 2) = O\left[ w_2 \left( \frac{1}{n} \right) \right].
\]

In this chapter we extend the above result for orthogonal polynomial system under a weaker condition.

**Theorem 3:**

Let \( \{p_n(x)\} \) be an orthonormal polynomial

1) Jackson [46]
system in the interval \([-1, 1]\) belonging to the
weight function \(g(x)\).

\[ o < \varphi = \mathcal{O}\left(\frac{1}{1 - x^2}\right) . \]

Let \(g_n(\overrightarrow{R}, 2)\) denote the best degree of
approximation, by linear forms of the system, in the \(L^2\)
-space of the class \(\overrightarrow{R}\) of all functions which possess
continuity modulus \(w(\phi, 6) < w(6)\), then we have

\[ \rho_n(\overrightarrow{R}, 2) = \mathcal{O}(w(\phi, \frac{1}{n})) . \]

Moreover, while proving the converging almost
everywhere of general orthogonal series in the intervals
of continuity, Alexits and Kralik\(^1\) has proved the
following theorem.

**Theorem C:**

Let \(\{\phi_n(x)\}\) denote a constant preserving polynomial like orthonormal system with respect to \((x)\)
whose functions \(\phi_n(x)\) are uniformly bounded. The condition,

\[ \sum_{k=0}^{n} \phi_k^2(x) = \mathcal{O}(n) \]

\(^1\) Alexits ([5], p 312).
is satisfied in the interval of orthogonality while the condition (5.1.7) satisfied uniformly in the subinterval

\[ [c, d] \text{ of } [a, b]. \]

If \( f(x) \) is an \( L^2 \)-integrable function, continuous in \([c, d]\) with the continuity modulus

\[ w(f, \delta, c, d) = o\left( \frac{1}{\varnothing(1/\delta)} \right) \]

where \( \varnothing(x) > 0 \), is an arbitrary function, monotone increasing for \( x > 1 \) and satisfying the condition

\[ \int_{2}^{\infty} \frac{dx}{x\varnothing(x-1)} < \infty \]

then the expansions

\[ (7.1.2) \quad f(x) \sim \sum_{n=0}^{\infty} c_n \varnothing_n(x) \]

converges to \( f(x) \) almost everywhere in \([c, d]\).

In this chapter we extend the above theorem for orthogonal polynomial system under weaker condition for certain class of functions.

**Theorem 4:**

If \( \{p_n(x)\} \) is any orthonormal polynomial
system belonging to the distribution \( du(x) \) and \( f(x) \) is a function with the continuity modulus

\[
(7.1.3) \quad w(f, \varepsilon) = O\left(\frac{1}{\varepsilon(1/\varepsilon)}\right)
\]

then

\[
(7.1.4) \quad \int_{1}^{\infty} \frac{dx}{x\phi(x)} < \infty
\]

implies the expansion (7.1.1) converges almost everywhere in \([a, b]\) for any order of its term.

In order to prove the above theorems we need the following Lemmas:

Lemma 1.1:

If \( \{p_n\} \subset M^\alpha, \quad \alpha > \frac{1}{2} \),

then

\[
\lim_{n \to \infty} \frac{n}{\sum_{k=0}^{\infty} \frac{p_k^2}{(k+1)^2}} = \frac{1}{2\alpha - 1}.
\]

1) Meder [78]
Lemma 2:

If \( \varphi(x) \) satisfies the condition (7.1.4) and if \( R_\varphi \) be the class of all functions \( f(x) \) for which

\[
E_n = O\left(\frac{1}{\varphi(n)}\right)
\]

then the expansion (7.1.2) converges almost everywhere in \([a, b]\) for any order of its term. Where \( E_n \) denote the best degree of approximation of \( f(x) \) by linear forms in the space \( L^2 \).

Proof of Theorem 1:

\[
t_n(x) - f(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_n k S_k(x) - f(x)
\]

\[
= \frac{1}{P_n} \sum_{k=0}^{n} p_n k S_k(x) - \frac{1}{P_n} \sum_{k=0}^{n} p_n k f(x)
\]

\[
= \frac{1}{P_n} \sum_{k=0}^{n} p_n k (S_k(x) - f(x))
\]

(7.1.7) \( P_n \left\{ t_n(x) - f(x) \right\} = \sum_{k=0}^{n} p_n k (S_k(x) - f(x)) \)

We have,

\[
S_k(x) = C_0 p_0(x) + C_1 p_1(x) + \ldots + C_k p_k(x)
\]

1) Leindler L. [63]
\[ \begin{align*} &\int_{-1}^{1} f(t) g(t)p_0(t) p_0(x) \, dt + \int_{-1}^{1} f(t) g(t)p_1(t)p_1(x) \, dt \\
&\quad + \int_{-1}^{1} f(t) g(t)p_2(t)p_2(x) \, dt = \int_{-1}^{1} f(t) g(t)p_k(t)p_k(x) \, dt \\
&= \int_{-1}^{1} f(t) g(t) \left( p_0(x) p_0(t) + p_1(t)p_1(x) + \cdots \right) \\
&\quad + p_k(t)p_k(x) \int_{-1}^{1} g(t) \, dt \\
\end{align*} \]

\[ (7.1.8) \quad S_k(x) = \int_{-1}^{1} f(t) K_k(x, t) g(t) \, dt \]

where

\[ K_k(x, t) = p_0(x) p_0(t) + p_1(t)p_1(x) + \cdots + p_k(t)p_k(x) \]

\[ = \sum_{v=0}^{k} p_v(x) p_v(t) \]

Let \( \alpha_k \) denotes the positive coefficients of \( x^k \) in \( p_k(x) \).

Since 1 is a polynomial of degree 0,
\[
\int_{-1}^{1} g(t) p_k(t) \, dt = \int_{-1}^{1} g(t) p_k(t) \cdot 1 \, dt
\]
\[
= 0 \quad \text{for } k > 1.
\]
and
\[
p_0(x) = \varphi_0(t) = \alpha_0
\]
Here
\[
\int_{-1}^{1} g(t) p_0(x) p_0(t) \, dt = \int_{-1}^{1} g(t) \varphi_0(t) \, dt = 1
\]
by the orthonormality property.

Now,
\[
g(t) p_0(x) p_0(t) + g(t) p_1(t) p_1(x) + \cdots + g(t) \cdot p_k(t) p_k(x) = g(t) K_k(x,t),
\]
Therefore,
\[
(7.1.9) \quad \int_{-1}^{1} g(t) K_k(x,t) \, dt = 1
\]
as a special case of (7.1.8) with \( f(t) = 1 \).
As \( f(x) \) is constant with respect to the variable of integration, multiplying (7.1.9) by \( f(x) \) we get
\[
(7.1.10) \quad f(x) = \int_{-1}^{1} g(t) f(x) K_k(x,t) \, dt
\]
From (7.1.8) and (7.1.10) we have

\[ S_k(x) - f(x) = \int_{-1}^{1} g(t) \left\{ f(t) - f(x) \right\} K_k(x, t) \, dt \]

Hence, (7.1.7) gives

\[ P_n \left\{ t_n(x) - f(x) \right\} = \sum_{n-k}^{1} \alpha_k \frac{p_{n-k} g(t) \left\{ f(t) - f(x) \right\}}{(t-x)} (t-x)K_k(x, t) \, dt \]

By applying Christoffel-Darboux formula 1) we get,

\[ (7.1.11) = \sum_{n-k}^{1} \alpha_k \frac{p_{n-k} g(t) \left\{ f(t) - f(x) \right\}}{(t-x)} (t-x)K_k(x, t) \, dt \]

Putting \( H_n(\lambda, x, t) = \sum_{k=0}^{n-1} \frac{p_{n-k} \alpha_k p_{k+1}(t) p_k(x)}{\alpha_{k+1}} \)

R.H.S. of (7.1.11) is

\[ (7.1.12) \int_{-1}^{1} g(t) \phi(t) H_n(\lambda, x, t) \, dt - \int_{-1}^{1} g(t) \phi(t) H_n(\lambda, x, t) \, dt \]

1) Alexits G. ([5], p. 26)
Using Schwarz inequality we obtain,

\[(7.1.13) \left\{ \int_{-1}^{1} \varphi(t) H_{n}(\lambda, x, t) \, dt \right\}^2 \]

\[\leq \int_{-1}^{1} \varphi(t) \varphi^2(t) \, dt \int_{-1}^{1} \varphi(t) \left[ H_{n}(\lambda, x, t) \right]^2 \, dt \]

\[= O(1) \int_{-1}^{1} \left\{ H_{n}(\lambda, x, t) \right\}^2 \, dt \varphi(t) \]

\[= O(1) \sum_{k=0}^{n} p_{n-k}^2 (x) \]

As \( \frac{\alpha_k}{\alpha_{k+1}} \) is bounded for large \( k \).

\[= O(n) \sum_{k=0}^{n} \frac{p_{n-k}}{(n-k)^2} \frac{(n-k)^2}{p_{2}^2} \]

\[= O(n) \sum_{k=0}^{n} \frac{p_{n-k}}{(n-k)^2} \]

\[= O(1) \frac{n}{p_{n}^2} \sum_{v=0}^{n} \frac{p_{v}^2}{(v+1)^2} \]

\[= O(1) \frac{p_{n}^2}{p_{n}^2} \]

1) Alexits G. [5], p. 28
this implies that
\[
\lim_{n \to \infty} t_n(x) = f(x)
\]
Hence the proof.

**Proof of Theorem 2:**

\[
\sigma_n^{\infty}(x) - f(x) = \frac{1}{\alpha} \sum_{k=0}^{n} \alpha^{-1} A_{n-k} S_k(x) - \frac{1}{\alpha} \sum_{k=0}^{n} A_{n-k} f(x)
\]

\[
= \frac{1}{\alpha} \sum_{k=0}^{n} A_{n-k} (S_k(x) - f(x)).
\]

(7.1.14) \( A_n^\alpha \left\{ \sigma_n^{\infty}(x) - f(x) \right\} = \sum_{k=0}^{n} A_{n-k} (S_k(x) - f(x)) \)

From theorem 1

\[
S_k(x) - f(x) = \int_{-1}^{1} g(t) \left\{ f(t) - f(x) \right\} k_k(x, t) dt
\]

Hence (7.1.14) gives,

\[
A_n^\alpha (\sigma_n^{\infty}(x) - f(x) = \sum_{k=0}^{n} A_{n-k} g(t) \left\{ f(t) - f(x) \right\} k_k(x, t) dt.
\]
By applying Christoffel – Darbou formula, \\

\[ (7.1.15) = \sum_{k=0}^{n} A_{n-k} \varphi(t) \frac{\alpha_k}{\alpha_{k+1}} \left\{ p_{k+1}(t) p_k(x) - p_{k+1}(x) p_k(t) \right\} \int \varphi(t) \, dt . \]

Putting \( H_n (\lambda, x, t) = \sum_{k=0}^{n} A_{n-k} \frac{\alpha_k}{\alpha_{k+1}} p_{k+1}(t) p_k(x) \).

R.H.S. of (7.1.15) is

\[
\int_{-1}^{1} \varphi(t) \varphi(t) H_n (\lambda, x, t) \, dt - \int_{-1}^{1} \varphi(t) \varphi(t) H_n (\lambda, x, t) \, dt
\]

Using Schwarz inequality we have,

\[
\left\{ \int_{-1}^{1} \varphi(t) \varphi(t) H_n (\lambda, x, t) \, dt \right\}^2 \leq \int_{-1}^{1} \varphi(t) \varphi(t) \, dt \int_{-1}^{1} \varphi(t) \varphi(t) \, dt
\]

\[ = O(1) \int_{-1}^{1} \left[ H_n (\lambda, x, t) \right]^2 \varphi(t) \, dt \]

\[ = O(1) \sum_{k=0}^{n} \frac{\alpha_k}{\alpha_{k+1}}^2 p_k(x). \]

as \( \frac{\alpha_k}{\alpha_{k+1}} \) is bounded for large \( k \).

\[ = O(1) n^{2a-1} \]

1) Alexits G. ([5], p. 26)
This implies that
\[
\lim_{n \to \infty} \sigma_n^\alpha(x) = f(x)
\]
Hence the proof.

Proof of Theorem 3 :-

Let \( S_n(x) \) be the \( n^{th} \) partial sums of the orthonormal expansion of \( f(x) \).

Then,
\[
\int_{-1}^{1} \left[ f(x) - S_n(x) \right]^2 \sigma(x) \, dx
\]
\[
= \bigO(1) \int_{-1}^{1} \left[ f(x) - p_n(x) \right]^2 \sigma(x) \, dx
\]
\[
= \bigO(1) \int_{-1}^{1} \left[ f(x) - p_n(x) \right]^2 \frac{1}{\sqrt{1-x^2}} \, dx
\]
\[
= \bigO(1) \int_{0}^{\pi} \left[ f(\cos \Theta) - p_n(\cos \Theta) \right]^2 \, d\Theta
\]
\[
= \bigO(1) \int_{0}^{\pi} \left[ f(\cos \Theta) - p_n(\cos \Theta) \right]^2 \, d\Theta.
\]

1) Alexits G. ([5], p. 6)
The function \( G(\varphi) = f(\cos \varphi) \) is defined in \([0, \pi]\) and possesses a continuity modulus \( w(G, \delta) \). Also we have,

\[
\sup_{|\varphi_1 - \varphi_2| \leq \delta} |G(\varphi_1) - G(\varphi_2)| < \sup_{|x_1 - x_2| \leq \delta} |f(x_1) - f(x_2)|
\]

\[
= w(f, \delta).
\]

Therefore, by Theorem B and using condition \( w_2(f, \delta) \leq w(f, \delta) \).

\[
\frac{1}{2} \int_{-1}^{1} [f(x) - s_n(x)]^2 \varrho(x) \, dx \leq O(1) \left[ w_2^2 \left( f, \frac{1}{n} \right) \right]
\]

Hence,

\[
\frac{1}{2} \int_{-1}^{1} [f(x) - s_n(x)]^2 \varrho(x) \, dx = O(1) \left[ w(f, \frac{1}{n}) \right]
\]

Hence,

\[
\varrho_n(\mathbb{R}, 2) = O\left[ w \left( f, \frac{1}{n} \right) \right].
\]

**Proof of Theorem 4:**

From theorem 3 we evidently have the relation

\[
E_n^2 \leq O\left[ w^2 \left( f, \frac{1}{n} \right) \right]
\]
From this condition and by (7.1.9) we have,

\[ f \left( \mathcal{R}_\emptyset^{(\mu)} \{ p_n(x) \} \right). \]

Hence the proof directly follows by applying Lemma 2.