Let \( \{ \phi_n(x) \} \) \((n = 0, 1, 2, \ldots)\) be an orthonormal system (ONS) of \( L^2 \) - integrable functions defined in the closed interval \([a, b]\). We consider the orthogonal series

\[
\sum_{n=0}^{\infty} c_n \phi_n(x)
\]

with real coefficients \(c_n\)'s.

The \((N, p_n)\) means and \((\overline{N}, p_n)\) means of the orthogonal series (6.1.1) are given by,

\[
t_n(x) = \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} S_k(x)
\]

and

\[
\overline{t}_n(x) = \frac{1}{p_n} \sum_{k=0}^{n} p_k S_k(x),
\]

respectively.

The series (6.1.1) is said to be \((N, p_n)\) and \((\overline{N}, p_n)\) summable to \(S(x)\) if

\[
\lim_{n \to \infty} t_n(x) = S(x)
\]

and
\[
\lim_{n \to \infty} T_n(x) = S(x), \text{ respectively.}
\]

Let \( \lambda = \{\lambda_n\} \) be a monotonic non-decreasing sequence of natural numbers with \( \lambda_{n+1} - \lambda_n \leq 1 \), and \( \lambda_1 = 1 \). The transformation

\[
V_n(\lambda) = \frac{1}{\lambda_n} \sum_{v=n}^{n+1} S_v
\]
defines the sequence \( \{V_n(\lambda)\} \) of generalized de-la Valle'e Poussin means of (6.1.1) generated by \( \lambda \).

The series (6.1.1) is said to be absolute \( N \)-Lund summable, absolute \( (N, p_n) \) summable and \( |V, \lambda| \) summable if,

\[
\sum_{n=1}^{\infty} \left| t_n - t_{n-1} \right|
\]

\[
\sum_{n=1}^{\infty} \left| \overline{T}_n - \overline{T}_{n-1} \right|
\]

and

\[
\sum_{n=1}^{\infty} \left| V_{n+1}(\lambda) - V_n(\lambda) \right|
\]

are convergent respectively.

Billiard \(^2\) and Tandori \(^3\) have studied the \( |(\omega, 1)| \)

\(^1\) Sharma P.L. and Jain R.K. [111]
\(^2\) Billiard P. [20]
\(^3\) Tandori K. ([131], [132])
summability of (6.1.1), Tandori\textsuperscript{1}) has proved the following theorem.

**Theorem A**: The condition

\[
(6.1.2) \quad \sum_{m=0}^{\infty} \left\{ \sum_{k=2^m+1}^{2^{m+1}} c_k ^2 \right\}^{1/2} < \infty
\]

is the necessary and sufficient condition for the series (6.1.1) to be \((C,1)|\) - summable almost everywhere.

If for some sequence \(\{p_n\}\) the conditions

1. \(0 < p_n < p_{n+1}\) for \(n = 0, 1, 2, \ldots\)

or \(0 < p_{n+1} < p_n\) for \(n = 0, 1, 2, \ldots\)

2. \(p_0 + p_1 + \ldots + p_n = p_n \uparrow \infty\)

3. \(\lim_{n \to \infty} \frac{n \Delta p_{n-1}}{p_n} = 1 - \alpha, \text{ where } \alpha > 0,\)

\(\Delta p_{n-1} = p_{n-1} - p_n\) are satisfied, then we shall say that the sequence \(\{p_n\}\) belongs to the class \(N^\alpha\).

In the special case when \(p_n = \frac{1}{n+1}\), \(p_n \sim \log n\), and the Nörlund means \(t_n\) reduces to harmonic means \(3)\).

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1) Tandori K. ([131], [132])
2) Meder J. (80]
3) Bhatt S.N. (16]
In the same direction the same result was generalized by different authors for different summabilities. Tandori K.\(^1\) and P. Srivastava\(^2\) have proved the result for absolute Riesz summability. Meder J.\(^3\) has proved the following theorem for absolute Nörlund summability:

**Theorem B:**

Let \( \{ p_n \} (-M, \alpha > \frac{1}{2}) \), then

\[
\sum_{m=0}^{\infty} \sum_{k=2^{m+1}}^{2^m} a_k^2 < \infty,
\]

is the necessary and sufficient condition for the series \((6.1.1)\) to be \( |N, p_n| \) summable in interval \([0, 1]\).

In this chapter we extend the above result for \( |N, p_n| \) and \( |N, \Lambda| \) summability. We also generalize the Theorem B of Meder. Our theorems are as follows.

**Theorem 1:**

If \( np_n = O(p_n) \), then

\[
\sum_{m=0}^{\infty} \left\{ \sum_{v=0}^{m} C_v^2 \right\}^{\frac{1}{2}} < \infty
\]

implies the \( |N, p_n| \) summability of \((6.1.1)\).

---

1) Tandori K. ([131], [132])
2) P. Srivastava [113]
3) Meder J. [77]
Theorem 2: 

If \( n p_n = \bigcirc(P_n) \) then

\[
\sum_{n=0}^{\infty} \left\{ \sum_{v=0}^{n} c_v^2 \right\}^{\frac{1}{2}} < \infty
\]

implies the \(|\bar{N}, p_n|\) summability of (6.1.1).

Theorem 3: 

Let \( \lambda=\{\lambda_n\} \) be a monotonic nondecreasing sequence of natural numbers with

\[
\lambda_{n+1} - \lambda_n < 1 \quad \text{and} \quad \lambda_1 = 1,
\]

then

\[
\sum_{n=1}^{\infty} \left\{ \sum_{k=n-1}^{n+1} \lambda_n + 2c_k^2 \right\}^{\frac{1}{2}} < \infty
\]

implies the \(|v, \lambda|\) summability of orthogonal series (6.1.1).

In order to prove above theorems, we need the following Lemma:

1) Lemma: 

If \( \{p_n\} \leftarrow M^\alpha, \quad \alpha > \frac{1}{2} \), then

\[
\lim_{n \to \infty} \frac{n}{p_n^2} \sum_{k=0}^{n} \frac{p_k^2}{(k+1)^2} = \frac{1}{2\alpha - 1}.
\]

1) Meder J. [78]
Proof of Theorem 1:

We have,

\[ t_n(x) - t_{n-1}(x) = \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} S_k(x) - \frac{1}{p_{n-1}} \sum_{k=0}^{n} p_{n-k-1} S_k(x) \]

\[ = \frac{1}{p_n} \sum_{r=0}^{n} \sum_{k=r}^{n} p_{n-k} C_{r,k}(x) S_k(x) - \frac{1}{p_{n-1}} \sum_{r=0}^{n} \sum_{k=r}^{n} p_{n-k-1} C_{r,k}(x) S_k(x) \]

\[ = \frac{1}{p_n} \sum_{r=0}^{n} \sum_{k=r}^{n} p_{n-k} C_{r,k}(x) - \frac{1}{p_{n-1}} \sum_{r=0}^{n} \sum_{k=r}^{n} p_{n-k-1} C_{r,k}(x) \]

\[ = \frac{1}{p_n} \sum_{k=0}^{n} p_n C_{k,0}(x) - \frac{1}{p_{n-1}} \sum_{k=0}^{n} p_{n-1} C_{k,0}(x) \]

\[ = \frac{1}{p_n} \sum_{k=0}^{n} p_n C_{k,0}(x) - \frac{1}{p_{n-1}} \sum_{k=0}^{n} p_{n-1} C_{k,0}(x) \]

\[ = O(1) \left\{ \frac{1}{p_{n-1}} \sum_{k=0}^{n} p_{n-k} C_{k,0}(x) - \frac{1}{n p_{n-1}} \sum_{k=0}^{n} p_{n-k} C_{k,0}(x) \right\} \]
\[
= \mathcal{O}(1) \left\{ \frac{1}{p_{n-1}} \sum_{v=1}^{n} p_v c_{n-v} \varnothing_{n-v}(x) - \frac{1}{np_{n-1}} \right\}
\]

\[
= \mathcal{O}(1) \left\{ \frac{1}{p_{n-1}} \sum_{v=1}^{n} \frac{p_v}{v} c_{n-v} \varnothing_{n-v}(x) - \frac{1}{np_{n-1}} \right\}
\]

\[
= \mathcal{O}(1) \left\{ \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} c_{n-v} \varnothing_{n-v}(x) - \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} c_{n-v} \varnothing_{n-v}(x) \right\}
\]

Therefore,

\[
| t_n - t_{n-1} | = \mathcal{O}(1) \left\{ \left| \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} c_{n-v} \varnothing_{n-v}(x) - \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} c_{n-v} \varnothing_{n-v}(x) \right| \right\}
\]

\[
= \mathcal{O}(1) \left\{ \left| \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} c_{n-v} \varnothing_{n-v}(x) \right| + \left| \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} c_{n-v} \varnothing_{n-v}(x) \right| \right\}
\]
Therefore, by Schwarz's inequality,

\[ \sum_{n=1}^{\infty} \int_{a}^{b} |t_n(x) - t_{n-1}(x)| \, dx = O(1) \sum_{n=1}^{\infty} \left\{ \int_{a}^{b} (t_n(x) - t_{n-1}(x))^2 \, dx \right\}^{\frac{1}{2}} \]

\[ = O(1) \sum_{n=1}^{\infty} \left\{ \frac{p}{p_{n-1}} \sum_{v=0}^{n} C_v^2 \right\}^{\frac{1}{2}} \]

\[ + \sum_{n=1}^{\infty} \left\{ C_n^2 \right\}^{\frac{1}{2}} \]

\[ = O(1) \sum_{n=1}^{\infty} \left\{ \sum_{v=0}^{n} C_v^2 \right\}^{\frac{1}{2}} \]

\[ < \infty. \]

Hence, by B. Levy's theorem, we have

\[ \sum_{n=1}^{\infty} |t_n(x) - t_{n-1}(x)| < \infty. \]

Hence the proof.

Remark :- Under the same condition as of Theorem 1, the given series is absolutely harmonic summable.

Proof of Theorem 2 :-

\[ |\overline{T}_n(x) - \overline{T}_{n-1}(x)| = \frac{1}{p_n} \sum_{k=0}^{n} s_k(x) - \frac{1}{p_{n-1}} \sum_{k=0}^{n-1} s_k(x) \]
\[
\begin{align*}
&= -\frac{1}{p_1} \sum_{n=0}^{n-1} p_k s_k(x) - \frac{1}{p_1} \sum_{n=0}^{n-1} p_k s_k(x) \\
&
+ \frac{p_n}{p_1} s_n(x) \\
&= \left( -\frac{1}{p_1} - \frac{1}{p_1} \right) \sum_{k=0}^{n-1} p_k s_k(x) + \frac{p_n}{p_1} s_n(x) \\
&= \frac{-p_n}{p_1} \sum_{n=0}^{n-1} p_k \sum_{v=0}^{k} c_{\varnothing_v} (x) + \frac{p_n}{p_1} s_n(x) \\
&= \frac{-p_n}{p_1} \sum_{v=0}^{n-1} c_{\varnothing_v} (x) \sum_{k=0}^{n-1} p_k + \frac{p_n}{p_1} s_n(x) \\
&= \frac{-p_n}{p_1} \sum_{v=0}^{n-1} c_{\varnothing_v} (x) \sum_{k=0}^{n-1} p_k + \frac{p_n}{p_1} s_n(x) \\
&= \frac{-p_n}{p_1} \sum_{v=0}^{n-1} c_{\varnothing_v} (x) \sum_{k=0}^{n-1} p_k + \frac{p_n}{p_1} s_n(x) \\
&= \frac{-p_n}{p_1} \sum_{v=0}^{n-1} c_{\varnothing_v} (x) \sum_{k=0}^{n-1} p_k + \frac{p_n}{p_1} s_n(x) \\
&= \frac{p_n}{p_1} (s_n(x) - s_{n-1}(x)) + \frac{p_n}{p_1} p_{n-1} \\
&= \frac{p_n}{p_1} \sum_{v=0}^{n-1} c_{\varnothing_v} (x) p_{v-1} \\
&= \frac{p_n}{p_1} \sum_{v=0}^{n-1} c_{\varnothing_v} (x) p_{v-1}.
\end{align*}
\]
By Schwartz inequality,

\[ \sum_{n=1}^{\infty} \int_{a}^{b} \left| \overline{T}_n(x) - \overline{T}_{n-1}(x) \right| \, dx = O(1) \sum_{n=1}^{\infty} \left\{ \int_{a}^{b} \left( \overline{T}_n(x) - \overline{T}_{n-1}(x) \right) \, dx \right\}^{\frac{1}{2}} \]

\[ = O(1) \sum_{n=1}^{\infty} \left\{ \frac{p_n^2}{\sqrt{p_n^2 \Sigma_{v=0}^{n} C_v^2}} \right\}^{\frac{1}{2}} \]

\[ = O(1) \sum_{n=1}^{\infty} \left\{ \frac{\Sigma_{i=0}^{n} p_i^2 \frac{p_v^2}{p_v^2 \Sigma_{i=0}^{n} p_i^2} \frac{p_v^2}{p_v^2 \Sigma_{i=0}^{n} p_i^2}}{\Sigma_{v=0}^{n} C_v^2} \right\}^{\frac{1}{2}} \]

\[ = O(1) \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{n}} \Sigma_{v=0}^{n} C_v^2 \right\}^{\frac{1}{2}} \]

\[ < \infty . \]

Hence the proof.
Proof of Theorem 3 :-

We have,

\[ |v_{n+1}(\lambda) - v_n(\lambda)| = \frac{1}{\lambda_n} \lambda_{n+1} \left\{ \sum_{k=n}^{n+1} \left( \lambda_{n+1} - \lambda_n \right) \right\} \]

\[ (k-n-1) + \lambda_n \{ C_k \phi_k(x) \} \]

\[ = \left| \frac{1}{\lambda_n} \frac{1}{\lambda_{n+1}} \sum_{k=n}^{n+1} (k-n-1) \right| C_k \phi_k(x) \]

Using Schwarz inequality and by B. Levy's theorem,

We have,

\[ \sum_{n=1}^{\infty} \int_{a}^{b} |v_{n+1}(\lambda) - v_n(\lambda)| \, dx = O(1) \sum_{n=1}^{\infty} \left\{ \frac{1}{\lambda_n} \frac{1}{\lambda_{n+1}} \right\} \]

\[ \sum_{k=n}^{n+1} (k-n-1) \frac{1}{\lambda_n} \frac{1}{\lambda_{n+1}} C_k^2 \]

\[ = O(1) \sum_{n=1}^{\infty} \left\{ \frac{1}{\lambda_n} \frac{1}{\lambda_{n+1}} \right\} \]

\[ \sum_{k=n}^{n+1} (k-n-1) \frac{1}{\lambda_n} \frac{1}{\lambda_{n+1}} C_k^2 \]
\[
\begin{align*}
\sum_{n=1}^{\infty} & \left\{ \frac{1}{2} \sum_{k=n-\lambda_n^+}^{n+1} (k-n-1)^2 c_k \right. \\
& + \frac{1}{2} \sum_{k=n-\lambda_n^+}^{n+1} (k-n-1)^2 c_k \\
& \left. + \frac{1}{2} \sum_{k=n-\lambda_n^+}^{n+1} (k-n-1)^2 c_k \right\}^{\frac{1}{2}} \\
= \mathcal{O}(1) & \sum_{n=1}^{\infty} \left\{ \frac{1}{2} \sum_{k=n-\lambda_n^+}^{n+1} (k-n-1)^2 c_k \right\}^{\frac{1}{2}} \\
= \mathcal{O}(1) & \sum_{n=1}^{\infty} \left\{ \frac{1}{2} \sum_{k=n-\lambda_n^+}^{n+1} (k-n-1)^2 c_k \right\}^{\frac{1}{2}} \\
< \infty. & \\
\end{align*}
\]

Hence the proof.