CHAPTER 6
Production Inventory with Service Time and Interruptions

6.1 Introduction

In all the studies on inventory systems prior to Berman et al [8], it was assumed that the serving of inventory is instantaneous. However this is not the case in many practical situations. For example in a TV showroom, a customer usually spends some time with the salesperson before buying the TV or in a computer shop, after selecting the model, one might have to wait until all the required software are installed. In Berman et al [8] it has been assumed that the amount of time taken to serve an item is constant. This leads to the analysis of a queue of demands formed in an inventory system. This study was followed by numerous studies by several researchers on many kinds of inventory models with positive service time. Krishnamoorthy and Viswanath [47] introduced the idea of positive service time in to a production inventory model by considering MAP arrivals and a correlated production process. This model being a very general one as far as the modeling parameters are considered, only a numerical study of the model was carried out there. In a very recent paper by Krishnamoorthy and Viswanath [48], assuming all the underlying distributions as exponential, a
product form solution for the steady state has been obtained in a production inventory model with positive service time. The above paper had been motivated by the paper by Schwarz et al. [58], where a product form solution has been obtained in an (s, S) inventory model with positive service time.

The delay in the service caused by server interruptions being a common phenomenon in almost all practical situations, White and Christie [73] was the first study to introduce this in a queueing model. Following this, there had been extensive study on these type of queueing models. We refer to the survey paper by Krishnamoorthy and Pramod [38] for more details on such studies.

Though there had been numerous studies on inventory models, where interruption occurs due to an unreliable supplier [64,63,12] and the references therein], Krishnamoorthy et al [41] can be considered as the first paper to introduce the concept of service interruption, which occurs in the middle of a service, in an inventory system. They assume that there is no bound on the number of interruptions that can occur in the middle of a single service and also that an order is instantaneously processed (zero lead-time). The steady state distribution has been obtained explicitly in product form in the above paper. In another paper [40] by the same authors, the above model has been extended by considering positive lead-time.

In an (s, S) production inventory system, once the production process is switched on (as the inventory falls from S to s), it is switched off only after the inventory level goes back to S, the maximum inventory level. This makes it distinct from an (s, S) inventory system with positive lead-time, where once the order is placed (the moment at which the inventory level hits the re-order
level s), usually the ordering quantity is taken such that the inventory level goes above s as the order materializes.

In a queueing system, where the service process has certain number of phases, which are subject to interruptions, the concept of protecting certain phases of service (which may be so costly to afford an interruption) from interruption could be an important idea. Klimenok et al [25] study a multi-server queue with finite buffer and negative customers where the arrival is BMAP and service is PH-type. They assume that a negative customer can delete an ordinary customer in service if the PH-service process belongs to some given subset of the set of service phases; whereas if the service process belongs to some phase outside the above subset, the ordinary customer is protected from the effect of the negative customers. The above paper is extended by assuming an infinite buffer in Klimenok and Dudin [24]. Krishnamoorthy et al [28] introduces the idea of protection in a queueing system where the service process is subject to interruptions. They assume that the final m-n phases of the Erlang service process are protected from interruption. Whereas if the service process belongs to the first n phases, it is subject to interruption and an interrupted service is resumed/repeated after some random time. There is no reduction in the number of customers due to interruption and no bound was assumed on the number of interruptions that can possibly occur in the middle of a service. In this way, this study differs from the earlier one where at most one interruption was possible in the middle of a service and where the customer whose service got interrupted is removed from the system.
This chapter introduces the concept of service interruption to a production inventory model with positive service time. The service time and the time to produce one item are assumed to follow distinct Erlang distributions. The service process as well as the production process is subject to interruptions and certain number of phases in both these processes are protected from interruption.

This chapter introduces a production process into the sum of the inventory models studied earlier in this thesis. This adds an item to the inventory one at a time while the production is on. We use the (s, S) policy to control the production. This policy was used in the previous chapters for replenishment (with or with out lead type). Such a situation can be regarded as production in bulk in a production cycle. We incorporate interruption in production as well as in service. Another important entity that we introduce in this chapter is ‘protection’ against interruption, both for production and service processes. Unlike in previous chapters, were all distributions involved were exponential, in the present chapter we go for the Erlang distribution. We introduce protection from interruption to a few among the last stages of service and production. This involves economic consideration. We look for the optimal number of stages (phases) of service/production processes to be protected. Unlike in chapter 5 here the underlying Markov chain turns out to be level – independent quasi birth and death processes.

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inventory one at a time while the production is on. We use the \((s,S)\) policy to control the production. This policy was used in the previous chapters for replenishment (with or without lead time). Such a situation can be regarded as production in bulk in a production cycle. We incorporate interruption in production as well as in service. Another important entity that we introduce in this chapter is ‘protection’ against interruption, both for production and service processes. Unlike in previous chapters, where all distributions involved were exponential, in the present chapter we go for the Erlang distribution. We introduce protection from interruption to a few among the last stages of service and production. This involves economic consideration. We look for the optimal number of stages (phases) of service/production processes to be protected. Unlike in chapter 5 here the underlying Markov chain turns out to be level – independent quasi birth and death processes.

An example for the applicability of the model: In the production process, assume that less expensive components of a system are assembled first. (These could be done without any protection). Next the expensive parts are to be assembled. These need protection from negligent handling. Thus such stages of the assembly are protected which involves additional cost. Similar example could be given in the case of service in phases with last few service phases protected.

We apply a novel method, which works even if we assume general PH distributions for the production as well as the service processes, for finding an explicit expression for the stability of the system. Studies like [3, 4], have analyzed inventory system where customers are not allowed to join the system, when there is a shortage of inventory and had found that the stability of such
systems is not affected by the inventory parameters. However, in the above studies, the underlying distributions were all exponential. Our proof for the stability of the system shows that the above phenomenon holds even if the underlying distributions are general PH distributions and hence it gives a characterization of the stability of inventory systems where the customers are not allowed to join the system when there is shortage of inventory.

In the section to follow, the mathematical formulation of the model is provided. Section 3 is concerned with the investigation of the stability of the system. The long run system state distribution is also given in that section. In section 4, numerous system performance measures are provided. Numerical investigation of performance measures is extensively discussed in section 5. Finally, section 6 concludes the discussion.

### 6.2. The Mathematical Model

The model under study is described as follows: Customers arrive to a single server counter according to a Poisson process of rate $\lambda$ where inventory is served. Service time duration follows Erlang distribution, with Phase-type representation $(T, \beta)$ of order $m$ where $\beta = (1,0,\ldots,0)$ and

$$
T = \begin{bmatrix}
-\mu_1 & \mu_1 & 0 & \cdots & 0 \\
0 & -\mu_1 & \mu_1 & \cdots & 0 \\
0 & 0 & -\mu_1 & \mu_1 & \cdots & 0 \\
0 & \cdots & 0 & -\mu_1 \\
0 & \cdots & 0 & 0 & -\mu_1
\end{bmatrix}
$$

Production is by one unit at a time. The production process starts whenever the inventory level falls to $s$ and continues until the inventory level reaches $S$. The
production process follows Erlang distribution with Phase-type representation \((U, \alpha)\) of order \(n\) with \(\alpha = (1,0,\ldots,0)\) and

\[
U = \begin{bmatrix}
-\mu & \mu & 0 & \ldots & 0 \\
0 & -\mu & \mu & 0 & \ldots \\
0 & 0 & -\mu & \mu & 0 \\
0 & 0 & 0 & -\mu & \mu \\
0 & 0 & 0 & 0 & -\mu \\
\end{bmatrix}
\]

Interruption to the service process occurs according to a Poisson process of rate \(\delta_1\); the server is subject to at most one interruption at a time and, further only when service is going on, the server is subject to interruption (that is an idle server is not affected by the interruption process). An exponentially distributed amount of time with parameter \(\delta_1\) is required to resume service from where it was stopped. That is the system recovers from the interruption after a repair having exponentially distributed duration with parameter \(\delta_1\). Similarly, the production process also encounters interruptions, with the interruption process following Poisson process with parameter \(\delta_2\) and recovers from it on being repaired, with repair time following an exponential distribution with parameter \(\delta_2\). In contrast to the case of service interruption, after repair an interrupted production process needs to be restarted from the beginning. In other words, we assume that an item being produced is discarded due to an interruption. For reducing the adverse effect of interruptions, we apply the concept of protection of certain phases of service as well as the production process from interruptions. Precisely, the last \(l_1\) phases of the service process are assumed to be protected in the sense that the service will not be interrupted while being in these phases; so interruptions to the service can occur only while in service in the first \(m-l_1\) phases. Similarly, the
final \( l \) phases of the production process are assumed to be protected in the sense that the item being produced will not be affected and the processor will not be subject to interruption while being in these phases; so interruptions to production can occur in the first \( n-l \) phases. It is important to note our assumption concerning the two interruption processes: while in interruption (service or production) another interruption cannot befall in the sense that the system behaves like a Type I counter (see Karlin and Taylor [14]). Further, a customer’s service may encounter any number of interruptions. However, it may be noted that since the item being produced is discarded consequent to an interruption of the production process, there can be at most one interruption to a unit being produced. All distributions involved are assumed to be mutually independent.

For the model under discussion, we make the following assumptions:

- No inventory is lost due to a service interruption.
- The customer being served when interruption occurs, waits there until his service is completed.
- The inventory being produced is lost due to a production interruption.
- A customer, who finds no inventory on his/her arrival, leaves the system forever.
- Only when production is on, that process could get interrupted and the service gets interrupted only while the server is busy providing service.

Let \( N_c(t) \) denote the number of customers in the system including the one getting service (if any) and \( N_i(t) \) denote the inventory level in the system at time \( t \). Further let

\[
V(t) = \begin{cases} 
0 & \text{if the server is idle} \\
1 & \text{if the server is busy} \\
2 & \text{if the server is on interruption}
\end{cases}
\]
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\[
P(t) = \begin{cases} 
0 & \text{if the production process is in off mode} \\
1 & \text{if the production process is in on mode}
\end{cases}
\]

\[
V_1(t) = \begin{cases} 
0 & \text{if the production process is not interrupted/off} \\
1 & \text{if the production process is interrupted}
\end{cases}
\]

Finally, let \( Z_1(t) \) and \( Z_2(t) \) denote the phases of the service and production processes, respectively. Then \( \Psi = \{(N_c(t), V(t), N_1(t), P(t), V_1(t), Z_1(t), Z_2(t)); \ t \geq 0\} \) forms a continuous time Markov chain on the state space \( U \hat{L}(i) \), where \( \hat{L}(i) \)'s, which are called the levels, are the collection of states defined as follows:

\[
\hat{L}(0) = \hat{L}(0,0) = U_{j=0}^{s} \bar{L}(0,0,j)
\]

For \(0 \leq j \leq s\), \( \bar{L}(0,0,j) = \bar{L}(0,0,j,1) = L(0,0,j,1,0) \cup L(0,0,j,1,1)\),

for \(s+1 \leq j \leq S-1\), \( \bar{L}(0,0,j) = L(0,0,j,0) \cup L(0,0,j,1,0) \cup L(0,0,j,1,1)\), and

\[
\bar{L}(0,0,S) = L(0,0,S,0).
\]

For \(i \geq 1\), \( \hat{L}(i) = \hat{L}(i,0) \cup \hat{L}(i,1) \cup \hat{L}(i,2)\), where

\[
\hat{L}(i,0) = \bar{L}(i,0,0) = \bar{L}(i,0,0,1) = L(i,0,0,1,0) \cup L(i,0,0,1,1)
\]

\[
\hat{L}(i,l') = U_{j=1}^{s} \bar{L}(i,l',j), \ l' = 1,2, \text{ where}
\]

\[
\bar{L}(i,l',j) = \bar{L}(i,l',j,1) = L(i,l',j,1,0) \cup L(i,l',j,1,1), \ 1 \leq j \leq s
\]

\[
\bar{L}(i,l',j) = L(i,l',j,0) \cup L(i,l',j,1,0) \cup L(i,l',j,1,1), \ s + 1 \leq j \leq S - 1
\]

\[
\bar{L}(i,l',S) = L(i,l',S,0)
\]

Finally, with \(\delta_j\) as Kronecker delta,

\[
L(i,0,0,1,j_1) = \{(i,0,0,1,j_1, j_2); 1 \leq j_2 \leq n - \delta_{1,j_1} I\}, \ i \geq 0, j_1 = 0,1
\]

\[
L(0,0,j,1,j_1) = \{(0,0,j,1,j_1, j_2); 1 \leq j_2 \leq n - \delta_{1,j_1} I\}, \ 1 \leq j \leq S - 1, j_1 = 0,1
\]

\[
L(0,0,j,0) = L(0,0,j,0,0) = \{(0,0,j,0,0)\}, \ s + 1 \leq j \leq S
\]
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\[ L(i, i', j, 1, j_1) = \{(i, i', j, 1, j_1, j_2, j_3); 1 \leq j_2 \leq m - \delta_{2,i}, 1 \leq j_3 \leq n - \delta_{1,i}, l\}, \]
\[ i \geq 1; i' = 1, 2; 1 \leq j \leq S - 1; j_1 = 0, 1 \]
\[ L(i, i', j, 0) = \{(i, i', j, 0, 0, j_2); 1 \leq j_2 \leq m - \delta_{2,i}, l\}, i \geq 1; i' = 1, 2; s + 1 \leq j \leq S. \]

It turns out that the continuous-time Markov chain \( \Psi \) is a Level Independent Quasi Birth-Death (LIQBD) process with infinitesimal generator given by

\[
W = \begin{bmatrix}
A_{00} & B_{00} & 0 & 0 & 0 & 0 & 0 \\
A_{10} & A_{1} & A_{0} & 0 & 0 & 0 \\
0 & A_{2} & A_{1} & A_{0} & 0 & 0 \\
0 & 0 & A_{2} & A_{1} & A_{0} & 0 & 0 \\
& & & & & & \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
& & & & \ddots & & & \\
& & \ddots & & & & & \\
& & & & & & & \\
\end{bmatrix},
\]

where the matrix \( A_{00} \) records the transition rates within the level \( \tilde{L}(0) \), \( B_{00} \) those from level \( \tilde{L}(0) \) to \( \tilde{L}(1) \); \( A_{10} \) governs transitions from \( \tilde{L}(1) \) to \( \tilde{L}(0) \) and the matrices \( A_{1} \) and \( A_{0} \) constitute respectively, the transition rates within a level \( \tilde{L}(i) \) and from level \( \tilde{L}(i) \) to \( \tilde{L}(i + 1) \) for \( i \geq 1 \). Finally, the matrix \( A_{2} \) governs transitions from level \( \tilde{L}(i) \) to \( \tilde{L}(i - 1) \) for \( i \geq 2 \). A detailed description of the transitions that govern the generator matrix \( W \) can be found in Appendix I.

In the sequel, \( Q = I - s \cdot I_n \) denotes the identity matrix of order \( n \), \( e \) denotes a column matrix of 1’s of appropriate order, \( e_n \) denotes a column matrix of 1’s of order \( n \times 1 \) and \( 0_n \) denotes a zero matrix of order \( n \times n \).
6.3. Analysis of the Model

6.3.1 Stability Condition

In section 2, we have assumed that the service time follows a PH distribution of order \( m \), with representation \((T, \beta)\). It follows that the service process which is subject to possible interruptions has a PH distribution with representation \((T^*, \beta^*)\).

where

\[
\beta^* = (\beta, 0) \quad \text{and} \quad T^* = \begin{bmatrix} T - \delta_1 J_1 & \delta_1 J_2 \\ \delta_2 J_3 & \delta_2 J_{(m-l_1)} \end{bmatrix}, \quad \text{with} \quad J_1 = \begin{bmatrix} I_{(m-l_1)} & 0 \\ 0 & 0 \end{bmatrix}_{(m \times m)},
\]

\[
J_2 = \begin{bmatrix} I_{(m-l_1)} \\ 0 \end{bmatrix}_{(m \times m-l_1)} \quad \text{and} \quad J_3 = \begin{bmatrix} I_{(m-l_1)} & 0 \end{bmatrix}_{(m-l_1 \times m)}.
\]

Let \( T^o \) be the column matrix such that \( T^* e + T^o = 0 \) and let \( 2m - l_1 = m' \).

Similarly the production process subject to possible interruptions also has a PH distribution with representation \((U^*, \alpha^*)\) where \( \alpha^* = (\alpha, 0) \) and

\[
U^* = \begin{bmatrix} U - \delta_3 J_4 & \delta_3 J_5 \\ \delta_4 J_6 & -\delta_4 J_{(n-l)} \end{bmatrix}, \quad J_4 = \begin{bmatrix} I_{n-l} & 0 \\ 0 & 0 \end{bmatrix}_{(n \times n)},
\]

\[
J_5 = \begin{bmatrix} I_{n-l} \\ 0 \end{bmatrix}_{(n \times n-l)} \quad \text{and} \quad J_6 = \begin{bmatrix} e_{n-l} & 0 \end{bmatrix}_{(n-l \times n)}.
\]

Let \( U^o \) be the column matrix such that \( U^* e + U^o = 0 \) and let \( 2n - l = n' \).

Then \( \Psi^* = (N(t), L(t), P(t), Z_1^*(t), Z_2^*(t)) \), where \( Z_1^*(t) \) and \( Z_2^*(t) \) denote the phases of the above service and production process, models the same Markov process as \( \Psi \). The infinitesimal generator matrix of the process \( \Psi^* \) is given by
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\[ W^* = \begin{bmatrix}
    A_{00}^* & B_{00}^* & 0 & 0 & 0 & 0 \\
    A_{10}^* & A_1^* & A_0^* & 0 & 0 & 0 \\
    0 & A_2^* & A_1^* & A_0^* & 0 & 0 \\
    0 & 0 & A_2^* & A_1^* & A_0^* & 0 \\
    & & & & & \ddots \\
\end{bmatrix} \]

Since the form of the matrices \( A_{00}^*, B_{00}^* \) and \( A_{10}^* \) will not affect the stability of the Markov chain \( \Psi^* \), we do not give their detailed description here; the other block matrices are given as follows:

\[ A_0^* = \begin{bmatrix}
    0 & 0 \\
    0 & \lambda I 
\end{bmatrix}; \]

\[ A_2^* = \begin{bmatrix}
    0 & A_2^{(0,0)} (1) & 0 & A_2^{(0,1)} (1) & 0 & \cdots & A_2^{(0,1)} (1) & 0 & A_2^{(0,1)} (1) & 0 \\
    & A_2^{(0,2)} (1) & 0 & A_2^{(0,2)} (1) & 0 & \cdots & A_2^{(0,2)} (1) & 0 & A_2^{(0,2)} (1) & 0 \\
    & & A_2^{(0,3)} (1) & 0 & A_2^{(0,3)} (1) & 0 & \cdots & A_2^{(0,3)} (1) & 0 & A_2^{(0,3)} (1) & 0 \\
    & & & A_2^{(0,4)} (1) & 0 & A_2^{(0,4)} (1) & 0 & \cdots & A_2^{(0,4)} (1) & 0 & A_2^{(0,4)} (1) & 0 \\
\end{bmatrix} \]

where \( A_2^{(0,0)} (1) = T^0 \otimes I_{n'\ell}, A_2^{(0,1)} (1) = T^0 \beta^* \otimes I_{n'\ell}, A_2^{(0,4)} (1) = [T^0 \beta^* 0], \)

\[ A_2^{(0,2)} (1) = \begin{bmatrix}
    T^0 \beta^* \otimes \alpha^* \\
    T^0 \beta^* \otimes I_{n'\ell} 
\end{bmatrix}, \ A_2^{(0,3)} (1) = \begin{bmatrix}
    T^0 \beta^* & 0 \\
    0 & T^0 \beta^* \otimes I_{n'\ell} 
\end{bmatrix} \]

and
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Let \( A = A_* + A_+ + A_0 \) and \( x = (x_{i,0}, x_{i,1}, \ldots, x_{s+1,0}, x_{s+1,1}, \ldots, x_{S-1,0}, x_{S-1,1}, \ldots) \) be the steady state vector of \( A \), where \( x_{i,0} \) and \( x_{i,1} \), \( s + 1 \leq i \leq S - 1 \), represent the probabilities corresponding to the production \( \text{off} \) and \( \text{on} \) states respectively. Then \( xA = 0 \) gives us the following system of equations.

\[
x_0 U^* + x_1 T^{0^*} \otimes I_n = 0 \tag{3.1.1}
\]

\[
x_0 \beta^* \otimes U^{0^*} \alpha^* + x_1 (T^* \otimes U^*) + x_2 T^{0^*} \beta^* \otimes I_n = 0 \tag{3.1.2}
\]

\[
x_{i-1} I_{m^*} \otimes U^{0^*} \alpha^* + x_i (T^* \otimes U^*) + x_{i+1} T^{0^*} \beta^* \otimes I_n = 0, \quad 2 \leq i \leq S - 1 \tag{3.1.3}
\]

\[
x_{i-1} I_{m^*} \otimes U^{0^*} \alpha^* + x_i (T^* \otimes U^*) + x_{s+1,0} T^{0^*} \beta^* \otimes \alpha^* + x_{s+1,1} T^{0^*} \beta^* \otimes I_n = 0 \tag{3.1.4}
\]
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\[ x_i I_m \otimes U^{0^*} \alpha^* + x_{i+1} (T^* \oplus U^*) + x_{s+2,i} T^{0^*} \beta^* \otimes I_n = 0 \]  
(3.1.5)

\[ x_{i-1,i} I_m \otimes U^{0^*} \alpha^* + x_{i,1} (T^* \oplus U^*) + x_{i+1,i} T^{0^*} \beta^* \otimes I_n = 0, \ s + 2 \leq i \leq S - 2 \]  
(3.1.6)

\[ x_{s-2,i} I_m \otimes U^{0^*} \alpha^* + x_{s-1,i} (T^* \oplus U^*) = 0 \]  
(3.1.7)

\[ x_{s-1,i} I_m \otimes U^{0^*} + x_s T^* = 0 \]  
(3.1.8)

\[ x_{i,0} T^* + x_{i+1,0} T^{0^*} \beta^* = 0, \ s + 1 \leq i \leq S - 2 \]  
(3.1.9)

\[ x_{s-1,0} T^* + x_s T^{0^*} \beta^* = 0. \]  
(3.1.10)

Noting that \( T^* e = -T^{0^*} \) and that \( \beta^* e = 1 \), right multiply each equation in the system of equations (3.1.9) and (3.1.10) by the column vector \( e \) to obtain the system of equations:

\[ x_{i,0} T^{0^*} = x_{i+1,0} T^{0^*}, \ s + 1 \leq i \leq S - 2 \]  
(3.1.11)

\[ x_{s-1,0} T^{0^*} = x_s T^{0^*} \]  
(3.1.12)

Now, (3.1.2) and (3.1.3) together result in:

\[ x_{i,0} (T^* + T^{0^*} \beta^*) = 0, \ s + 1 \leq i \leq S - 1. \]  
(3.1.13)

From (3.1.11) and (3.1.13), we get that \( x_{i+1,0} = x_{i+2,0} = \ldots = x_{s-1,0} \), which is the left eigen vector of the irreducible generator matrix \( (T^* + T^{0^*} \beta^*) \).

Right multiplying equation (3.1.1) by \( e' \), we get the equation

\[ -x_0 U^{0^*} + x_i T^{0^*} \otimes e'_{n'} = 0 \]

Noticing that \( \beta^* \otimes x_0 U^{0^*} = x_0 \beta^* \otimes U^{0^*} \) and that \( \beta^* \otimes T^{0^*} = T^{0^*} \beta^* \), we take the left Kronecker product in the above equation with \( \beta^* \) to obtain the equation:

\[ -x_0 \beta^* \otimes U^{0^*} + x_i T^{0^*} \beta^* \otimes e'_{n'} = 0 \]  
(3.1.14)
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Right multiplying each equation (3.1.2) to (3.1.7) by \( I_{m'} \otimes e_n \), we get the following equations

\[ x_0 \beta^* \otimes U_{0}^{0'} + x_1 (T^* \otimes e_n^*) - x_1 (I_{m'} \otimes U_{0}^{0'}) + x_2 T_{0}^{0'} \beta^* \otimes e_n = 0 \]

(3.1.15)

\[ x_{i-1} I_{m'} \otimes U_{0}^{0'} + x_i (T^* \otimes e_n^*) - x_i (I_{m'} \otimes U_{0}^{0'}) + x_{i+1} T_{0}^{0'} \beta^* \otimes e_n = 0 \quad 2 \leq i \leq s - 1 \]

(3.1.16)

\[ x_{s-1} I_{m'} \otimes U_{0}^{0'} + x_s (T^* \otimes e_n^*) - x_s (I_{m'} \otimes U_{0}^{0'}) + x_{s+1} T_{0}^{0'} \beta^* \otimes e_n = 0 \]

(3.1.17 A)

\[ x_s I_{m'} \otimes U_{0}^{0'} + x_{s+1} (T^* \otimes e_n^*) - x_{s+1} (I_{m'} \otimes U_{0}^{0'}) + x_{s+2} T_{0}^{0'} \beta^* \otimes e_n = 0 \]

(3.1.18)

\[ x_{i-1} I_{m'} \otimes U_{0}^{0'} + x_i (T^* \otimes e_n^*) - x_i (I_{m'} \otimes U_{0}^{0'}) + x_{i+1} T_{0}^{0'} \beta^* \otimes e_n = 0 \quad s + 2 \leq i \leq S - 2 \]

(3.1.19)

\[ x_{S-1} I_{m'} \otimes U_{0}^{0'} + x_S (T^* \otimes e_n^*) - x_S (I_{m'} \otimes U_{0}^{0'}) = 0 \]

(3.1.20)

From equations (3.1.11) and (3.1.12), we note that \( x_{s+1} T_{0}^{0'} = x_{S-1} T_{0}^{0'} = x_3 T_{0}^{0'} \); using which, we re-write the equation (3.1.17 A) as

\[ x_{i-1} I_{m'} \otimes U_{0}^{0'} + x_i (T^* \otimes e_n^*) - x_i (I_{m'} \otimes U_{0}^{0'}) + x_{i+1} T_{0}^{0'} \beta^* \otimes e_n = 0 \]

(3.1.17)

Now, adding equations (3.1.8) and (3.1.13) to (3.1.20), we get the equation

\[ \left( x_i + x_{i+1} + \ldots + x_{s+1} + \ldots + x_{S-1}\right)(T^* + T_{0}^{0'} \beta^*) \otimes e_n \]

\[ + \left( x_{s+1} + \ldots + x_{S-1} + x_3\right)(T^* + T_{0}^{0'} \beta^*) = 0 \]

Noticing that, for any row vector \( \xi \) of dimension \( 1 \times m' n' \), we have
6.3 Analysis of the Model

\[ \tilde{\xi}((T^* + T^0 \beta^*) \otimes e_n) = (\tilde{\xi}(I_m \otimes e_n))(T^* + T^0 \beta^*) \]
we write the above equation as,

\[ ((x_1 + x_2 + \ldots + x_s + x_{s+1,1} + \ldots + x_{S-1,1})(I_m \otimes e_n))(T^* + T^0 \beta^*) + \\
(x_{s+1,0} + \ldots + x_{S-1,0} + x_S)(T^* + T^0 \beta^*) = 0. \]

(3.1.21)

Since the generator matrix \( T^* + T^0 \beta^* \) is irreducible, equation 3.1.21 implies that

\[ (x_1 + x_2 + \ldots + x_s + x_{s+1,1} + \ldots + x_{S-1,1})(I_m \otimes e_n) + (x_{s+1,0} + \ldots + x_{S-1,0} + x_S) = a \rho, \]

where \( \rho \) is the steady state vector of the generator matrix \( T^* + T^0 \beta^* \) and \( a \) is a scalar. Now, since \( xe = 1 \), it follows that \( a = 1 - x_0 e_n \).

The Markov chain \( \Psi^* \) and hence the Markov chain \( \Psi \) are stable if and only if \( xA^* e < xA^* e \) (see Neuts [15]). From the structure of the matrices \( A^*_0 \) and \( A^*_2 \), it follows that,

\[ xA^*_0 e = (1 - x_0 e_n) \hat{\lambda} \quad \text{and} \]

\[ xA^*_2 e = ((x_1 + x_2 + \ldots + x_s + x_{s+1,1} + \ldots + x_{S-1,1})(I_m \otimes e_n) + (x_{s+1,0} + \ldots + x_{S-1,0} + x_S))T^0 \]

\[ = (1 - x_0 e_n) \rho T^0 \]

Hence the inequality \( xA^*_0 e < xA^*_2 e \) reduces to

\[ (1 - x_0 e_n) \hat{\lambda} < (1 - x_0 e_n) \rho T^0. \]

That is, the Markov chain \( \Psi \) is stable if and only if \( \hat{\lambda} < \rho T^0 \). We summarize the above as:

**Theorem 6.3.1**

A necessary and sufficient condition for the stability of the Markov chain \( \Psi \) is

\[ \hat{\lambda} < \rho T^0. \]
Note:
We notice that the above stability condition is the same as that for an $M/PH/1$ queueing system, where the arrival process is Poisson with parameter $\lambda$ and the service time follows a Phase-type distribution with representation $(T^*, \beta^*)$ of order $m'$. Also, notice that the production process as well as the reorder and maximum inventory levels have no influence on the stability of the model studied. The proof of Theorem 3.1 reveals that barring customers from joining the system when there is shortage of inventory is the reason behind this phenomenon and hence characterizes the stability of such inventory systems.

6.3.2 Computation of the Steady State Vector

Next we compute the steady state vector of $\Psi$ numerically. Let $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$, be the steady state vector of $\Psi$, where $\pi_0 = \pi_{00} = (\pi_{00}(1), \pi_{00}(2), \pi_{00}(3))$;
\[\pi_{00}(1) = (\pi_{00}(0,1,0,*), \pi_{00}(0,1,1,\bullet), \pi_{00}(1,1,0,*), \pi_{00}(1,1,1,\bullet), \pi_{00}(s,1,0,*), \pi_{00}(s,1,1,\bullet)) ;\]
\[\pi_{00}(2) = (\pi_{00}(s+1,0,0), \pi_{00}(s+1,1,0,*), \pi_{00}(s+1,1,1,\bullet), \pi_{00}(s+2,0,0), \pi_{00}(s+2,1,0,*), \pi_{00}(s+2,1,1,\bullet), \ldots, \pi_{00}(S-1,0,0), \pi_{00}(S-1,1,0,*), \pi_{00}(S-1,1,1,\bullet)) ;\]
\[\pi_{00}(3) = \pi_{00}(S,0,0) . \text{ Here } \pi_{00}(i,1,0,*) \text{ is a row vector of dimension } n \text{ and that of } \pi_{00}(i,1,1,\bullet) \text{ is } n-l . \text{ For } r \geq 1 ,\]
\[\pi_{r0}(0) = (\pi_{r0}(0,1,0,*), \pi_{r0}(0,1,1,\bullet)) ;\]
\[\pi_{r0}(1) = (\pi_{r0}(1,1,0,\circ,*) \pi_{r0}(1,1,1,\circ,\bullet) \pi_{r0}(2,1,0,\circ,*) \pi_{r0}(2,1,1,\circ,\bullet) \ldots, \pi_{r0}(s,1,0,\circ,*), \pi_{r0}(s,1,1,\circ,\bullet)) ;\]
\[\pi_{r0}(2) = (\pi_{r0}(s+1,0,0,\circ), \pi_{r0}(s+1,1,0,\circ,*), \pi_{r0}(s+1,1,1,\circ,\bullet), \pi_{r0}(s+2,0,0,\circ), \pi_{r0}(s+2,1,0,\circ,*), \pi_{r0}(s+2,1,1,\circ,\bullet), \ldots, \pi_{r0}(S-1,0,0,\circ), \pi_{r0}(S-1,1,0,\circ,*), \pi_{r0}(S-1,1,1,\circ,\bullet)) ; \pi_{r0}(3) = (\pi_{r0}(S,0,0,\circ)) .\]

Here $\pi_{r0}(0,1,0,*)$ is a row vector of length $n$, $\pi_{r0}(0,1,1,\bullet)$ is a row vector of length $n-l$, $\pi_{r0}(i,1,0,\circ,*)$ is a row vector of length $mn$, $\pi_{r0}(i,1,1,\circ,\bullet)$ is a row vector of length $n$.
vector of length \( m(n-l) \), \( \pi_{r_0}(i,0,0,0) \) is a row vector of length \( m \). The description of \( \pi_{r_1} \) is identical to that of \( \pi_{r_0} \) except for \( \pi_{r_1} = (\pi_{r_1}(1),\pi_{r_1}(2),\pi_{r_1}(3)) \) and \( m-l \) coming in place of \( m \).

The sub vectors \( \pi_i \) of the vector \( \pi \) are given by \( \pi_i = -\pi_0B_0(A_i + RA_i)^{-1}R_i, \ i \geq 1 \), where \( R \) is the minimal non-negative solution of the matrix-quadratic equation \( R^2A_i + RA_i + A_0 = 0 \) (see Neuts [15]). The sub vector \( \pi_0 \) is first found as the steady state probability vector of the generator matrix \( A_{00} - B_{00}(A_i + RA_i)^{-1}A_{10} \) and then normalized using the condition \( \pi_0e + \sum_{i \geq 1} \pi_i e = \pi_0 e + \pi_1(I-R)^{-1} e = 1 \). For computing the \( R \) matrix, we applied the logarithmic reduction algorithm by Latouche and Ramaswami [17].

6.4. System Performance Measures

1. The probability that server is busy

\[
PSB = \sum_{r \geq 0} \pi_{r_0} e
\]

2. The probability that server is on interruption

\[
PSI = \sum_{r \geq 1} \pi_{r_1} e
\]

3. The expected inventory level when there are no customers in the system

\[
EIL(0) = \sum_{i=1}^{S-1} \sum_{k=1}^{n} i\pi_{00}(i,1,0,k) + \sum_{i=1}^{S-1} \sum_{k=1}^{n-1} i\pi_{00}(i,1,1,k) + \sum_{i=1}^{S-1} i\pi_{00}(i,0,0) + S\pi_{00}(S,0,0)
\]

4. The expected inventory level when there is at least one customer in the system

and server is busy
6.4 System Performance Measures

\[ EIL(1) = \sum_{r=1}^{S-1} \sum_{m} \sum_{k=1}^{n} i\pi_{r0}(i,1,0,k_1,k_2) + \sum_{r=1}^{S-1} \sum_{m} \sum_{k=1}^{n-l} i\pi_{r0}(i,1,k_1,k_2) + \sum_{r=1}^{S-1} \sum_{m} \sum_{k=1}^{l} i\pi_{r0}(i,0,0,k_1) + \sum_{r=1}^{S-1} \sum_{m} i\pi_{r0}(S,0,0,k_1) \]

5. The expected inventory level when there is at least one customer in the system and server is on interruption

\[ EIL(2) = \sum_{r=1}^{S-1} \sum_{m-l} \sum_{k=1}^{a} i\pi_{r1}(i,1,0,k_1,k_2) + \sum_{r=1}^{S-1} \sum_{m-l} \sum_{k=1}^{n-l} i\pi_{r1}(i,1,k_1,k_2) + \sum_{r=1}^{S-1} \sum_{m-l} \sum_{k=1}^{l} i\pi_{r1}(i,0,0,k_1) + \sum_{r=1}^{S-1} \sum_{m-l} S\pi_{r1}(S,0,0,k_1) \]

6. The expected inventory level in the system

\[ EIL = EIL(0) + EIL(1) + EIL(2) \]

7. The expected number of customers in the system

\[ ENCS = \sum_{r \geq 0} r\pi_{r0} + \sum_{r \geq 0} r\pi_{r1} \]

8. The expected production switch off rate

\[ PSWOF = \pi_{00}(S-1,1,0,\bullet)U^0 + \sum_{r=1}^{\infty} \pi_{r0}(S-1,1,0,\bullet,\bullet)e_m \otimes U^0 + \sum_{r=1}^{\infty} \pi_{r1}(S-1,1,0,\bullet,\bullet)e_{m-l} \otimes U^0. \]

9. The expected production commencement rate

\[ PCOM = \sum_{r=1}^{\infty} \pi_{r0}(s+1,0,0,k)T^0 \]

10. The expected interruption rate of production
6.4 System Performance Measures

\[ EIRP = \delta_3 \{ \sum_{i=0}^{S-1} \sum_{k=1}^{n-l} \pi_{00}(i,1,1,k) + \sum_{r=1}^{\infty} \sum_{m=0}^{S-1} \sum_{k_1=1}^{n-l} \pi_{r0}(i,1,1,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{m=1}^{S-1} \sum_{n=0}^{m-l} \pi_{1}(i,1,1,k_1,k_2) \} \]

11. The expected rate of interruption of service

\[ EIRS = \delta_t \sum_{r \geq 1} \pi_{r1} e \]

12. Fraction of time service is in protected stage

\[ FSP = \sum_{r=1}^{\infty} \sum_{m=0}^{S-1} \sum_{n=0}^{m} \pi_{r0}(i,1,0,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{m=1}^{S-1} \sum_{n=1}^{m} \pi_{r0}(i,0,0,k_1) + \sum_{r=1}^{\infty} \sum_{m=1}^{S-1} \sum_{n=1}^{m} \pi_{r0}(i,1,1,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{m=1}^{S-1} \sum_{n=1}^{m} \pi_{r0}(S,0,0,k_1) \]

13. Fraction of time production is in protected stage

\[ FPP = \sum_{i=0}^{S-1} \sum_{k=1}^{n-l} \pi_{00}(i,1,0,k) + \sum_{r=1}^{\infty} \sum_{m=1}^{S-1} \sum_{k_1=1}^{n-l} \pi_{r0}(i,1,0,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{m=1}^{S-1} \sum_{k_1=1}^{n-l} \pi_{r0}(i,1,1,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{m=1}^{S-1} \sum_{n=1}^{m} \pi_{1}(i,1,0,k_1,k_2) \]

14. Loss rate of customers due to zero inventory

\[ LZI = \lambda \{ \sum_{k=1}^{n} \pi_{00}(0,1,0,k) + \sum_{k_1=1}^{n-l} \pi_{00}(0,1,1,k) + \sum_{k_1=1}^{n-l} \pi_{r0}(0,1,0,k_1,k_2) + \sum_{k_1=1}^{n-l} \pi_{r0}(0,1,1,k_1,k_2) + \sum_{k_1=1}^{n-l} \pi_{1}(0,1,0,k_1,k_2) + \sum_{k_1=1}^{n-l} \pi_{1}(0,1,1,k_1,k_2) \} \]

15. Probability that inventory level is greater than s and production is in off mode
6.4 System Performance Measures

\[ P_{\text{prod off}} (I>s) = \sum_{i=1}^{S-1} \pi_{00}(i,0) + \pi_{00}(S,0) + \sum_{r=1}^{\infty} \sum_{i=1}^{m} \sum_{k_1=1}^{l} \pi_{r0}(i,0,0,k_1) + \]
\[ \sum_{r=1}^{\infty} \sum_{i=1}^{m-l} \pi_{r1}(i,0,0,k_1) + \sum_{r=1}^{\infty} \sum_{k_1=1}^{l} \pi_{r0}(S,0,0,k_1) \]

16. Probability that inventory level is greater than \( s \) and production is in \( \text{on} \) mode

\[ P_{\text{prod on}} (I>s) = \sum_{i=1}^{S-1} \sum_{i=1}^{n} \pi_{00}(i,1,0,k) + \sum_{r=1}^{\infty} \sum_{i=1}^{m} \sum_{k_1=1}^{l} \pi_{r0}(i,1,0,k_1,k_2) + \]
\[ \sum_{r=1}^{\infty} \sum_{i=1}^{m-l} \sum_{k_1=1}^{l} \pi_{r1}(i,1,0,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{r0}(i,1,1,k_1,k_2) + \]
\[ \sum_{r=1}^{\infty} \sum_{i=1}^{m-l} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{r1}(i,1,1,k_1,k_2) \]

17. Probability that inventory level is greater than \( s \),

\[ P (I>s) = P_{\text{prod off}} (I>s) + P_{\text{prod on}} (I>s) \]

18. Fraction of time production is in \( \text{on} \) mode

\[ F_{\text{P on}} = \sum_{r=1}^{\infty} \sum_{i=1}^{S-1} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{r0}(i,1,0,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{i=1}^{m} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{r0}(i,1,1,k_1,k_2) + \]
\[ \sum_{r=1}^{\infty} \sum_{i=1}^{m-l} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{r1}(i,1,0,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{r1}(i,1,1,k_1,k_2) + \]
\[ \sum_{i=0}^{S-1} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{00}(i,1,0,k) + \sum_{i=0}^{S-1} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{00}(i,1,1,k) \]

19. Fraction of time server is interrupted and production is in \( \text{on} \) mode

\[ F_{\text{P on/}SI} = \sum_{r=1}^{\infty} \sum_{i=1}^{S-1} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{r1}(i,1,0,k_1,k_2) + \sum_{r=1}^{\infty} \sum_{i=1}^{m-l} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{r1}(i,1,1,k_1,k_2) \]

20. Fraction of time production is in \( \text{on} \) mode with no customers in the system

\[ F_{\text{P on/}OC} = \sum_{i=0}^{S-1} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{00}(i,1,0,k) + \sum_{i=0}^{S-1} \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \pi_{00}(i,1,1,k) \]
21. Fraction of time production is in on mode with server active

\[ FP_{on}/SA = FP_{on} - (FP_{on}/OC + FP_{on}/SI) \]

### 6.5. Numerical Illustration

In this section, we provide the results of the numerical experiments that has been carried out for studying the impact of different parameters on various system performance measures.

#### 6.5.1 Effect of the Service Interruption Rate \( \delta \)

Intuitively, as the interruption rate \( \delta \) increases, the length of a service, which is subject to interruptions, increases, which in turn leads to lesser number of service completions and to increased queue length and loss rate. The increase in the measures \( PSI, ENCS \) and \( LZI \) in Table 1 supports the above intuition. From Table 1, one can observe that the sum of the two fractions \( PSB \) and \( PSI \) is increasing, whereas the fraction \( PSB \), which is the fraction of time, the server remains active, is decreasing. This shows that the decrease in the server idle probability is not in favor of the system. We think the possible decrease in the number of service completions with increase in \( \delta \) could be the reason behind the decrease in the server active probability \( PSB \). The same reasoning can be attributed to the decrease in the fraction of time the service process remains in protected phases \( FSP \). The decrease in the service completion rate as \( \delta \) increases, leading to a slow depletion rate of inventory in the system, can be pointed out as the reason for the increase in the expected inventory level \( EIL \). In the Table, one can see that \( EIL \) is above the production switch on level \( s \) and is increasing; this can be pointed out as the reason behind the increase in the production switch off rate \( PSWOF \) as well as the production commencement rate \( PCOM \).
### 6.5 Numerical Illustration

#### Table 1: Effect of the Service Interruption Rate $\delta_1$ on Various Performance Measures

$l = 3$, $l_1 = 3$, $n = 7$, $m = 7$, $\mu = 25$, $\mu_1 = 35$, $s = 4$, $S = 10$, $\lambda = 2$, $\delta_2 = 2$, $\delta_3 = 2$, $\delta_4 = 3$

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>PSB</th>
<th>PSI</th>
<th>EIL</th>
<th>ENCS</th>
<th>FSP</th>
<th>PSWOF</th>
<th>PCOM</th>
<th>LZI</th>
<th>FPon</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.378467</td>
<td>0.15138</td>
<td>5.5648</td>
<td>1.33858</td>
<td>0.1622</td>
<td>0.06733</td>
<td>0.06733</td>
<td>0.10766</td>
<td>0.795554</td>
</tr>
<tr>
<td>1.6</td>
<td>0.378193</td>
<td>0.172888</td>
<td>5.58341</td>
<td>1.49551</td>
<td>0.162083</td>
<td>0.06745</td>
<td>0.067451</td>
<td>0.10903</td>
<td>0.794978</td>
</tr>
<tr>
<td>1.8</td>
<td>0.377923</td>
<td>0.19436</td>
<td>5.6026</td>
<td>1.669662</td>
<td>0.161967</td>
<td>0.06755</td>
<td>0.067545</td>
<td>0.11038</td>
<td>0.794411</td>
</tr>
<tr>
<td>2</td>
<td>0.377658</td>
<td>0.215804</td>
<td>5.622427</td>
<td>1.86421</td>
<td>0.191854</td>
<td>0.06762</td>
<td>0.067616</td>
<td>0.111701</td>
<td>0.793854</td>
</tr>
<tr>
<td>2.2</td>
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<td>0.237222</td>
<td>5.6429</td>
<td>2.08313</td>
<td>0.161743</td>
<td>0.06766</td>
<td>0.067648</td>
<td>0.112988</td>
<td>0.79331</td>
</tr>
<tr>
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<td>0.258615</td>
<td>5.66404</td>
<td>2.331542</td>
<td>0.161635</td>
<td>0.06769</td>
<td>0.067692</td>
<td>0.114237</td>
<td>0.792782</td>
</tr>
</tbody>
</table>

#### 6.5.2 Effect of the Service Repair Rate $\delta_2$

When the repair rate increases, one would expect faster service completions, which leads to decreased queue length and loss rate. The decrease in the measures PSI, ENCS and LZI in Table 2 supports the above intuition. In contrast to the case of increase in the interruption rate $\delta_1$, here the server active probability PSB and the server idle probability are increasing. For a moment one may suspect that the increase in the server idle probability is not in favor of the system; however, a closer look at Table 2 reveals that the increase in the server idle probability = $1-PSB-PSI$ is due to the high decrease in the fraction PSI as compared to the increase in the fraction PSB. The increase in the server active probability suggests increasing the repair rate as a remedy to interruption; however to what extent one can achieve this may depend on the particular situation at hand. Faster recovery from interruption can be thought of as the reason behind the slight increase in the fraction of time the service process is in protected phases $FSP$.  

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The decrease in the expected number of customers as well as in the loss rate of customers points to a faster depletion of inventory in the system. However, the Table shows a narrow increase in the expected inventory level $EIL$. This can be thought of as due to an increase in the production by a narrow margin as indicated by the narrow increase in the fraction $FPon$.

**Table 2: Effect of the Service Repair Rate $\delta_2$ on Various Performance Measures**

$l = 3, l_1 = 3, n = 7, m = 7, \mu = 25, \mu_1 = 35, s = 3, S = 8, \lambda = 2, \delta_1 = 2, \delta_3 = 2, \delta_4 = 3$

<table>
<thead>
<tr>
<th>$\delta_2$</th>
<th>$PSB$</th>
<th>$PSI$</th>
<th>$EIL$</th>
<th>$ENCS$</th>
<th>$FSP$</th>
<th>$PSWO$</th>
<th>$PCOM$</th>
<th>$LZI$</th>
<th>$FPon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.363462</td>
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<td>4.269289</td>
<td>3.57877</td>
<td>0.15577</td>
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<td>0.087481</td>
<td>0.182685</td>
<td>0.764012</td>
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<td>0.365842</td>
<td>0.261316</td>
<td>4.28675</td>
<td>2.706066</td>
<td>0.15679</td>
<td>0.08778</td>
<td>0.087776</td>
<td>0.170786</td>
<td>0.769015</td>
</tr>
<tr>
<td>1.8</td>
<td>0.367506</td>
<td>0.233337</td>
<td>4.298384</td>
<td>2.196766</td>
<td>0.157502</td>
<td>0.08777</td>
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<td>0.772513</td>
</tr>
<tr>
<td>2</td>
<td>0.368702</td>
<td>0.210686</td>
<td>4.305717</td>
<td>1.862118</td>
<td>0.158015</td>
<td>0.08762</td>
<td>0.08762</td>
<td>0.156484</td>
<td>0.775026</td>
</tr>
<tr>
<td>2.2</td>
<td>0.369575</td>
<td>0.191987</td>
<td>4.310007</td>
<td>1.641239</td>
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<td>0.776863</td>
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</tr>
</tbody>
</table>

**6.5.3 Effect of the Production Interruption Rate $\delta_3$**

As the production becomes slower due to an increase in the interruption rate, we expect a decrease in the measures like expected inventory level $EIL$, production switch off rate $PSWO$ and the production commencement rate $PCOM$. Table 3 supports these intuitions. The increase in the fraction $FPon$ can be seen to be due to an increase in the length of the production process brought by the increase in the interruption rate. The decrease in the expected inventory level leads to an increase in the loss rate $LZI$. The possible decrease in the number of service completions due to an increased loss rate can be seen to be the reason behind the decrease in the server active probability $PSB$. The decrease in $PSB$ can
be thought of as the reason for the slight decrease in the server interruption probability $PSI$ and hence this decrease in $PSI$ is not in favor of the system under study. Again the decrease in $PSB$ can be thought of as the reason behind the narrow increase in the expected number of customers in the system.

### Table 3: Effect of the Production Interruption Rate $\delta_3$ on Various Performance Measures

$l = 3$, $l_1 = 3$, $n = 7$, $m = 7$, $\mu = 25$, $\mu_1 = 35$, $s = 3$, $S = 8$, $\lambda = 2$, $\delta_1 = 2$, $\delta_2 = 3$, $\delta_4 = 2$

<table>
<thead>
<tr>
<th>$\delta_3$</th>
<th>PSB</th>
<th>PSI</th>
<th>$EIL$</th>
<th>ENCS</th>
<th>FSP</th>
<th>PSWOF</th>
<th>PCOM</th>
<th>LZI</th>
<th>FPON</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.36876</td>
<td>0.14048</td>
<td>4.319611</td>
<td>1.177874</td>
<td>0.15804</td>
<td>0.09197</td>
<td>0.091972</td>
<td>0.156195</td>
<td>0.766493</td>
</tr>
<tr>
<td>1.6</td>
<td>0.362298</td>
<td>0.138018</td>
<td>4.14625</td>
<td>1.1784</td>
<td>0.155271</td>
<td>0.08233</td>
<td>0.082324</td>
<td>0.188505</td>
<td>0.791344</td>
</tr>
<tr>
<td>1.8</td>
<td>0.355355</td>
<td>0.135373</td>
<td>3.96877</td>
<td>1.17866</td>
<td>0.152295</td>
<td>0.07313</td>
<td>0.073134</td>
<td>0.223219</td>
<td>0.814541</td>
</tr>
<tr>
<td>2</td>
<td>0.347983</td>
<td>0.132565</td>
<td>3.788857</td>
<td>1.179261</td>
<td>0.149135</td>
<td>0.06471</td>
<td>0.064706</td>
<td>0.26008</td>
<td>0.836012</td>
</tr>
<tr>
<td>2.2</td>
<td>0.34024</td>
<td>0.129615</td>
<td>3.608022</td>
<td>1.179601</td>
<td>0.145817</td>
<td>0.05696</td>
<td>0.056963</td>
<td>0.298799</td>
<td>0.855723</td>
</tr>
<tr>
<td>2.4</td>
<td>0.332188</td>
<td>0.126547</td>
<td>3.42779</td>
<td>1.179879</td>
<td>0.142366</td>
<td>0.04991</td>
<td>0.049906</td>
<td>0.339057</td>
<td>0.873673</td>
</tr>
</tbody>
</table>

### 6.5.4 Effect of the Production Repair Rate $\delta_4$

As the production repair rate $\delta_4$ increases, the average span of production process being interrupted decreases and hence the production rate increases. The increase in the production switch off rate $PSWOF$ as well as the expected inventory level $EIL$ and the decrease in the fraction of time that the production process is in on mode $FPon$ can be seen to be due to an increase in the overall production rate. The same reasoning can be applied to explain the increase in the production commencement rate $PCOM$. Since the expected inventory level increases, the expected loss rate $LZI$ of customers decreases. This decrease in $LZI$ can be thought of as the reason for the slight increase in the expected number of customers in the system $ENCS$. Note that in the case of increase in $\delta_2$, which was discussed earlier, the decrease in the loss rate does not lead to an increase in
ENCS. Here, one can notice that in the case of increase in $\delta_2$, it is the faster service completion, that leads to a decrease in the number of customers and an increase in the service completion rate can’t be expected in the case of increase in $\delta_4$. The increase in the fraction of time $PSB$ as well as $PSI$ may be attributed to the increase in ENCS.

**Table 4: Effect of the Production Repair Rate $\delta_4$ on Various Performance Measures**

$l = 3, l_1 = 3, n = 7, m = 7, \mu = 25, \mu_i = 35, s = 3, S = 8, \lambda = 2, \delta_1 = 2, \delta_2 = 3, \delta_3 = 2$

<table>
<thead>
<tr>
<th>$\delta_4$</th>
<th>PSB</th>
<th>PSI</th>
<th>EIL</th>
<th>ENCS</th>
<th>FSP</th>
<th>PSWOF</th>
<th>PCOM</th>
<th>LZI</th>
<th>FPON</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>0.314368</td>
<td>0.179639</td>
<td>3.259093</td>
<td>1.85844</td>
<td>0.134729</td>
<td>0.04832</td>
<td>0.048318</td>
<td>0.363696</td>
<td>0.876677</td>
</tr>
<tr>
<td>1.6</td>
<td>0.327259</td>
<td>0.187006</td>
<td>3.476104</td>
<td>1.860745</td>
<td>0.140254</td>
<td>0.05482</td>
<td>0.05482</td>
<td>0.312789</td>
<td>0.859958</td>
</tr>
<tr>
<td>1.8</td>
<td>0.337441</td>
<td>0.192824</td>
<td>3.658436</td>
<td>1.862656</td>
<td>0.144618</td>
<td>0.06083</td>
<td>0.060828</td>
<td>0.312789</td>
<td>0.844474</td>
</tr>
<tr>
<td>2</td>
<td>0.345574</td>
<td>0.197471</td>
<td>3.812373</td>
<td>1.864207</td>
<td>0.148103</td>
<td>0.06635</td>
<td>0.066347</td>
<td>0.272124</td>
<td>0.830223</td>
</tr>
<tr>
<td>2.2</td>
<td>0.352143</td>
<td>0.201224</td>
<td>3.943064</td>
<td>1.865446</td>
<td>0.150918</td>
<td>0.0714</td>
<td>0.0714</td>
<td>0.239281</td>
<td>0.817153</td>
</tr>
<tr>
<td>2.4</td>
<td>0.357505</td>
<td>0.204288</td>
<td>4.054683</td>
<td>1.866421</td>
<td>0.153216</td>
<td>0.07602</td>
<td>0.076017</td>
<td>0.21247</td>
<td>0.805187</td>
</tr>
</tbody>
</table>

**6.5.5 Effect of the Maximum Inventory Level $S$**

When the maximum inventory level $S$ increases, it takes more time for switching off the production process, once it is switched on. This is reflected as the decrease in the production switch off rate in Table 5. The decrease in the production commencement rate $PCOM$ and the increase in the fraction $FPon$ have the same reasoning. Since the production remains in on mode for a longer time, the expected inventory level $EIL$ increases with increase in $S$. The presence of sufficient inventory in the system leads to a decrease in the loss rate $LZI$ and also to an increase in the server active probability $PSB$. The increase in the server interruption probability $PSI$ is due an increase in $PSB$. In Table 5, it can be seen that the expected number of customers is decreasing in spite of a decrease in the loss rate. This may seem contradictory with what we said in the case of increase in the production repair rate, $\delta_4$. We point out the significant increase in the
expected inventory level with an increase in the maximum inventory level, compared to the increase when \( \delta_i \) increases as the reason for this contradiction.

### Table 5: Effect of the Maximum Inventory Level \( S \) on Various Performance Measures

\( l = 3, l_1 = 3, n = 7, m = 7, \mu = 25, \mu_i = 35, s = 3, \lambda = 2, \delta_1 = 2, \delta_2 = 3, \delta_3 = 2, \delta_4 = 3 \)

<table>
<thead>
<tr>
<th>( S )</th>
<th>( PSB )</th>
<th>( PSI )</th>
<th>( EIL )</th>
<th>( ENCS )</th>
<th>( FSP )</th>
<th>( PSWOF )</th>
<th>( PCOM )</th>
<th>( LZI )</th>
<th>( FPon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.374236</td>
<td>0.142566</td>
<td>4.861727</td>
<td>1.177649</td>
<td>0.160387</td>
<td>0.070329</td>
<td>0.070329</td>
<td>0.128815</td>
<td>0.78666</td>
</tr>
<tr>
<td>10</td>
<td>0.37667</td>
<td>0.143493</td>
<td>5.4196</td>
<td>1.176987</td>
<td>0.161429</td>
<td>0.05898</td>
<td>0.05898</td>
<td>0.11665</td>
<td>0.791774</td>
</tr>
<tr>
<td>11</td>
<td>0.378755</td>
<td>0.144288</td>
<td>5.97946</td>
<td>1.176425</td>
<td>0.16323</td>
<td>0.050608</td>
<td>0.050608</td>
<td>0.106222</td>
<td>0.79616</td>
</tr>
<tr>
<td>12</td>
<td>0.380555</td>
<td>0.144973</td>
<td>6.53631</td>
<td>1.175947</td>
<td>0.163901</td>
<td>0.044205</td>
<td>0.044205</td>
<td>0.09722</td>
<td>0.799944</td>
</tr>
<tr>
<td>13</td>
<td>0.38212</td>
<td>0.145569</td>
<td>7.08564</td>
<td>1.175538</td>
<td>5.163765</td>
<td>0.039168</td>
<td>0.039167</td>
<td>0.089399</td>
<td>0.803232</td>
</tr>
<tr>
<td>14</td>
<td>0.383487</td>
<td>0.14609</td>
<td>7.629048</td>
<td>1.175189</td>
<td>0.164351</td>
<td>0.035112</td>
<td>0.035112</td>
<td>0.082562</td>
<td>0.806107</td>
</tr>
</tbody>
</table>

### 6.5.6 Effect of the Re-Order Level \( s \)

As the re-order level \( s \) increases, production switch on’s and switch off’s occur more frequently and hence the increase in the measures PCOM, PSWOF, FPon and EIL are expected. Increase in the expected inventory level can be thought of as the reason for decrease in the loss rate LZI as well as the increase in the server busy probability PSB. As in the case of the increase in the maximum inventory level \( S \), here also the expected number of customers is decreasing despite a decrease in the loss rate. The same reasoning as in the case of \( S \) can be given in this case also.
Table 6: Effect of the Re-Order Level \( s \) on Various Performance Measures

\( l = 3, l_1 = 3, n = 7, m = 7, \mu = 25, \mu_1 = 35, S = 15, \lambda = 2, \delta_1 = 2, \delta_2 = 3, \delta_3 = 2, \delta_4 = 3 \)

<table>
<thead>
<tr>
<th>( s )</th>
<th>PSB</th>
<th>PSI</th>
<th>EIL</th>
<th>ENCS</th>
<th>FSP</th>
<th>PSWOF</th>
<th>PCOM</th>
<th>LZI</th>
<th>FPon</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.146548</td>
<td>8.16732</td>
<td>1.174887</td>
<td>0.164866</td>
<td>0.031784</td>
<td>0.031784</td>
<td>0.076552</td>
<td>0.808633</td>
</tr>
<tr>
<td>4</td>
<td>0.386787</td>
<td>0.147349</td>
<td>8.432331</td>
<td>1.174407</td>
<td>0.165765</td>
<td>0.033856</td>
<td>0.033856</td>
<td>0.066061</td>
<td>0.813045</td>
</tr>
<tr>
<td>5</td>
<td>0.388528</td>
<td>0.14801</td>
<td>8.712697</td>
<td>1.174057</td>
<td>0.166512</td>
<td>0.036487</td>
<td>0.036486</td>
<td>0.057358</td>
<td>0.816704</td>
</tr>
<tr>
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<td>8.999142</td>
<td>1.173733</td>
<td>0.167136</td>
<td>0.039826</td>
<td>0.039826</td>
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<tr>
<td>7</td>
<td>0.391212</td>
<td>0.149033</td>
<td>9.280952</td>
<td>1.173458</td>
<td>0.167662</td>
<td>0.044108</td>
<td>0.044108</td>
<td>0.043936</td>
<td>0.822347</td>
</tr>
</tbody>
</table>

6.5.7 Effect of the Number \( l \) of Protected Phases of the Production Process

As the number \( l \) of protected phases in the production process increases, since the harm due to interruptions is reduced, production becomes faster. Recall that the production repair rate \( \delta_4 \), when increased, accelerates the production process by reducing the adverse effect of interruption. Hence, as \( l \) increases, we expect a similar impact on the performance measures as in the case of increase in the production repair rate \( \delta_4 \). Comparison of Tables 4 and 7 shows that, the expected number of customers, \( ENCS \) is the only measure, which shows an opposite behavior in the two tables. Precisely as \( \delta_4 \) increases, \( ENCS \) increases and when \( l \) increases, \( ENCS \) shows a narrow decreasing nature. As we pointed out in the case of the parameters \( S \) and \( \delta_4 \), here also the comparatively high inventory level in the case of increase in \( l \) as compared to the increase in \( \delta_4 \) could be the reason for the above phenomenon.
Table 7: Effect of the Number $l$ of Protected Phases of the Production Process on Various Performance Measures

$l_1 = 3$, $n = 8$, $m = 7$, $\mu = 45$, $\mu_1 = 45$, $s = 3$, $S = 8$, $\lambda = 2$, $\delta_1 = 2$, $\delta_2 = 3$,
$\delta_3 = 2$, $\delta_4 = 2$

<table>
<thead>
<tr>
<th>$l$</th>
<th>PSB</th>
<th>PSI</th>
<th>EIL</th>
<th>ENCS</th>
<th>FSP</th>
<th>PSWO</th>
<th>PCOM</th>
<th>LZI</th>
<th>FPON</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.61743</td>
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<td>0.125668</td>
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<tr>
<td>3</td>
<td>0.297146</td>
<td>0.113198</td>
<td>4.864789</td>
<td>0.755864</td>
<td>0.127348</td>
<td>0.16162</td>
<td>0.161619</td>
<td>0.089762</td>
<td>0.591293</td>
</tr>
<tr>
<td>4</td>
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<td>0.114893</td>
<td>4.86764</td>
<td>0.755677</td>
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<td>0.181432</td>
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</tr>
<tr>
<td>5</td>
<td>0.305052</td>
<td>0.116211</td>
<td>5.23527</td>
<td>0.755422</td>
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<td>0.200782</td>
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</tr>
<tr>
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<td>0.219385</td>
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<tr>
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<td>0.309633</td>
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<td>0.237032</td>
<td>0.009491</td>
<td>0.398101</td>
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</tbody>
</table>

6.5.8 Effect of the Number $l_i$ of Protected Phases of the Service Process

As the number of protected phases of the service process increases, intuitively, service becomes faster, which leads to a decrease in the expected number of customers $ENCS$ as well as in the server interruption probability $PSI$. Table 8 supports the above intuition. Notice that in Table 8, the expected inventory level is close to 5.5, whereas the re-order level is 3. Since there is sufficient inventory, the customer loss rate is not significant. This could be thought of as the reason behind the almost unchanged values for the performance measures $PSB$, $PCOM$, $PSWO$ and $FPon$.

Table 8: Effect of the Number $l_i$ of Protected Phases of the Service Process on Various Performance Measures

$l = 3$, $n = 7$, $m = 11$, $\mu = 45$, $\mu_1 = 45$, $s = 3$, $S = 8$, $\lambda = 2$, $\delta_1 = 2$, $\delta_2 = 3$,
$\delta_3 = 2$, $\delta_4 = 3$
### 6.5 Numerical Illustration

<table>
<thead>
<tr>
<th>$l_i$</th>
<th>PSB</th>
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<th>EIL</th>
<th>ENCS</th>
<th>FSP</th>
<th>PSWOF</th>
<th>PCOM</th>
<th>L2I</th>
<th>FPON</th>
</tr>
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<tbody>
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<td>0.220385</td>
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<td>0.446394</td>
</tr>
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<td>0.176354</td>
<td>0.220061</td>
<td>0.22006</td>
<td>0.016011</td>
<td>0.446371</td>
</tr>
<tr>
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<td>0.176347</td>
<td>5.54865</td>
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<td>0.220433</td>
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</tr>
<tr>
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<td>0.264513</td>
<td>0.219569</td>
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<td>0.016152</td>
<td>0.44634</td>
</tr>
<tr>
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<td>0.117559</td>
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<tr>
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</tbody>
</table>

### 6.5.9 Cost Function

For studying the optimality of the parameters like the re-order level $s$, the maximum inventory level $S$, the number of protected phases of the service as well as the production processes, we construct the following cost function:

$$
Cost = CI \times EIL + CN \times ENCS + CIP \times EIRP + CIS \times EIRS + CPON \times FPON + CZ \times L2I + CPPR \times FPP + CSPR \times FSP + CPRR \times FPRS + CSRR \times FSR + CPCOM \times PCOM,
$$

where $CI$ is the inventory holding cost per item per unit time, $CN$ is the holding cost per unit time per customer, $CIP$ is the cost per unit time per interruption, $CIS$ is the unit time cost incurred due to server interruptions, $CPON$ is the unit time cost incurred for running the production process, $CZ$ is the cost incurred for the loss of customers due to shortage of inventory, $CPPR$ is the unit time cost incurred due to protection of the production process, $CSPR$ is the unit time cost incurred due to the protection of the service process, $CPRR$ is the unit time cost incurred due to repair of the production process, $CSRR$ is the unit time cost incurred due to the repair of the service process and $CPCOM$ is the unit time cost for switching on the production process. Using this cost function, the optimality of the parameters $s, S, l$ and $l_1$ has been studied. Following are a few illustrations.
6.5.9.1 Optimality of the Re-Order Level $s$

Table 6 shows that loss rate $LZI$ and the expected number of customers, $ENCS$ are the only measures that shows a decreasing nature, which are involved in the cost function, as the re-order level increases. Therefore unless we select the cost $CZ$ and $CN$ so as to capture this decrease, the cost function will be linearly increasing. Figure 1(a) shows that by taking a comparatively high cost $CZ$, optimal value of $s$ is 5 and in Figure 1(b) the cost curve shows a linearly increasing nature suggesting $s=2$ as the optimal value.

**Figure 1(a): Effect of the Re-Order Level $s$ on the Cost Function**

$l = 3, n = 7, l_1=3, m = 7, k = 25, k_1 = 35, S = 17, \lambda = 1.2, \delta_1 = 2, \delta_2 = 1, \delta_3 = 2.5, \delta_4 = 1.5$

$CN=1000, CI=100, CPRR=750, CPPR=750, CSPR=750, CSRR=750, CPON=500, CPCOM=500, CIP=500, CIS=500, CZ=17000$

<table>
<thead>
<tr>
<th>$s$</th>
<th>COST</th>
</tr>
</thead>
<tbody>
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<td>3504.491</td>
</tr>
<tr>
<td>3</td>
<td>3429.888</td>
</tr>
<tr>
<td>4</td>
<td>3389.797</td>
</tr>
<tr>
<td>5</td>
<td><strong>3374.475</strong></td>
</tr>
<tr>
<td>6</td>
<td>3377.203</td>
</tr>
<tr>
<td>7</td>
<td>3393.082</td>
</tr>
<tr>
<td>8</td>
<td>3418.647</td>
</tr>
<tr>
<td>9</td>
<td>3451.228</td>
</tr>
<tr>
<td>10</td>
<td>3489.021</td>
</tr>
<tr>
<td>11</td>
<td>3531.185</td>
</tr>
<tr>
<td>12</td>
<td>3576.578</td>
</tr>
</tbody>
</table>

**Figure 1(b): Effect of the Re-Order Level $s$ on the Cost Function**

$l = 3, n = 7, l_1=3, m = 7, k = 25, k_1 = 35, S = 17, \lambda = 1.2, \delta_1 = 2, \delta_2 = 1, \delta_3 = 2.5, \delta_4 = 1.5$
6.5 Numerical Illustration

\[ CN=200, CI=300, CPPR=200, CPRR=400, CSPR=100, CSRR=200, CPON=250, CPCOM=100, CIP=250, CIS=300, CZ=1350 \]

<table>
<thead>
<tr>
<th>s</th>
<th>COST</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3287.48</td>
</tr>
<tr>
<td>3</td>
<td>3413.14</td>
</tr>
<tr>
<td>4</td>
<td>3545.43</td>
</tr>
<tr>
<td>5</td>
<td>3683.73</td>
</tr>
<tr>
<td>6</td>
<td>3826.6</td>
</tr>
<tr>
<td>7</td>
<td>3972.22</td>
</tr>
<tr>
<td>8</td>
<td>4119.32</td>
</tr>
<tr>
<td>9</td>
<td>4266.22</td>
</tr>
<tr>
<td>10</td>
<td>4411.81</td>
</tr>
<tr>
<td>11</td>
<td>4555.92</td>
</tr>
<tr>
<td>12</td>
<td>4695.78</td>
</tr>
</tbody>
</table>

6.5.9.2 Optimality of the Maximum Inventory Level \( S \)

Table 5 shows that as the maximum inventory level \( S \) increases, the measures which show a decreasing nature are \( ENCS, PSWOF, PCOM \) and \( LZI \). But the magnitude of the above decrease is small as compared to the increase in the measures like \( EIL \). Figure 2(a) shows that particular choice of high costs corresponding to the measures which show a decreasing nature, an optimal value for the maximum inventory level is attainable. However, if we use the same costs as for Figure 1(b), we get a linearly increasing cost function as in Figure 2(b).

**Figure 2(a): Effect of the Maximum Inventory Level \( S \) on the Cost Function**

\( l = 3, n = 7, l_1=3, m = 7, k = 25, k_1 = 35, s = 3, \lambda = 1.2, \delta_1 = 2, \delta_2 = 1, \delta_3 = 2.5, \delta_4 = 1.5 \)

\[ CN=1000, CI=200, CPPR=500, CPRR=1000, CSPR=500, CSRR=500, CPON=5000, CPCOM=5000, CIP=500, CIS=1000, CZ=10000 \]
6.5 Numerical Illustration

**Figure 2(b): Effect of the Maximum Inventory Level \( S \) on the Cost Function**

\( l = 3, n = 7, l_1 = 3, m = 7, k = 25, k_1 = 35, s = 3, \lambda = 1.2, \delta_1 = 2, \delta_2 = 1, \delta_3 = 2.5, \delta_4 = 1.5 \)

\( CN = 200, \ CI = 300, \ CPPR = 200, \ CPRR = 400, \ CSPR = 100, \ CSRR = 200, \ CPON = 250, \ CPCOM = 100, \ CIP = 250, \ CIS = 300, \ CZ = 1350 \)

<table>
<thead>
<tr>
<th>( S )</th>
<th>( \text{COST} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>6702.57</td>
</tr>
<tr>
<td>8</td>
<td>6622.267</td>
</tr>
<tr>
<td>9</td>
<td>6590.386</td>
</tr>
<tr>
<td>10</td>
<td>6590.016</td>
</tr>
<tr>
<td>11</td>
<td>6611.220</td>
</tr>
<tr>
<td>12</td>
<td>6647.06</td>
</tr>
<tr>
<td>13</td>
<td>6693.465</td>
</tr>
<tr>
<td>14</td>
<td>6747.94</td>
</tr>
<tr>
<td>15</td>
<td>6808.712</td>
</tr>
<tr>
<td>16</td>
<td>6842.117</td>
</tr>
</tbody>
</table>

\[ \text{Cost} =\]

\[ 0 \quad 5 \quad 10 \quad 15 \quad 20 \]

\[ 6500 \quad 6600 \quad 6700 \quad 6800 \quad 6900 \]

**6.5.9.3 Optimality of the Number \( l_1 \) of Protected Phases of the Service Process**

Figure 3(a) shows an optimal value for the number of protected phases of the service process for the selected costs, whereas if we use the costs as for Figure

<table>
<thead>
<tr>
<th>( S )</th>
<th>( \text{COST} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1982.47</td>
</tr>
<tr>
<td>8</td>
<td>2123.93</td>
</tr>
<tr>
<td>9</td>
<td>2265.64</td>
</tr>
<tr>
<td>10</td>
<td>2408.41</td>
</tr>
<tr>
<td>11</td>
<td>2552.34</td>
</tr>
<tr>
<td>12</td>
<td>2696.15</td>
</tr>
<tr>
<td>13</td>
<td>2839.48</td>
</tr>
<tr>
<td>14</td>
<td>2982.56</td>
</tr>
<tr>
<td>15</td>
<td>3125.48</td>
</tr>
<tr>
<td>16</td>
<td>3264.49</td>
</tr>
</tbody>
</table>
1(b), the cost curve shows a decreasing nature suggesting that to attain the optimal cost, we have to protect all the service phases.

**Figure 3(a): Effect of $l_1$ on the Cost Function**

$l = 3, n = 6, m = 14, K = 35, K_1 = 55, s = 3, S = 8, \lambda = 1.2, \delta_1 = 2, \delta_2 = 3, \delta_3 = 2.5, \delta_4 = 1.5$;

$CN=200, CI=100, CPPR=700, CPRR=5000, CSPR=5000, CSRR=5000, CPON=1000, CPCOM=500, CIP=500, CIS=800, CZ=200$

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>COST</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4024.602</td>
</tr>
<tr>
<td>3</td>
<td>3913.678</td>
</tr>
<tr>
<td>4</td>
<td>3866.162</td>
</tr>
<tr>
<td>5</td>
<td>3841.629</td>
</tr>
<tr>
<td>6</td>
<td><strong>3839.682</strong></td>
</tr>
<tr>
<td>7</td>
<td>3854.263</td>
</tr>
<tr>
<td>8</td>
<td>3875.564</td>
</tr>
<tr>
<td>9</td>
<td>3902.444</td>
</tr>
<tr>
<td>10</td>
<td>3934.289</td>
</tr>
</tbody>
</table>

**Figure 3(b): Effect of $l_1$ on the Cost Function**

$l = 3, n = 6, m = 14, K = 35, K_1 = 55, s = 3, S = 8, \lambda = 1.2, \delta_1 = 2, \delta_2 = 3, \delta_3 = 2.5, \delta_4 = 1.5$

$CN=200, CI=300, CPPR=200, CPRR=400, CSPR=100, CSRR=200, CPON=250, CPCOM=100, CIP=250, CIS=300, CZ=1350$
6.5 Numerical Illustration

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>COST</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3569.37</td>
</tr>
<tr>
<td>3</td>
<td>3206.14</td>
</tr>
<tr>
<td>4</td>
<td>2977.84</td>
</tr>
<tr>
<td>5</td>
<td>2784.9</td>
</tr>
<tr>
<td>6</td>
<td>2643.32</td>
</tr>
<tr>
<td>7</td>
<td>2535.52</td>
</tr>
<tr>
<td>8</td>
<td>2438.76</td>
</tr>
<tr>
<td>9</td>
<td>2350.96</td>
</tr>
<tr>
<td>10</td>
<td>2272.2</td>
</tr>
</tbody>
</table>

5.9.4 Optimality of the Number $l$ of Protected Phases of the Production Process

Figure 4(a) shows an optimal value, for the costs selected, for the number of phases to be protected for the production process; however, if we take the costs as for Figure 2(a), the cost function decreases, which suggests that protecting all the production phases leads to the optimal value of the cost function.

Figure 4(a): Effect of $l$ on the Cost Function

$l_1=3$, $n=14$, $m=6$, $K=25$, $K_1=30$, $s=3$, $S=8$, $\lambda=1.2$, $\delta_1=2$, $\delta_2=1$, $\delta_3=2.5$, $\delta_4=1.5$

$CN=200$, $CI=300$, $CPPR=200$, $CPRR=400$, $CSPR=100$, $CSRR=200$, $CPON=250$, $CPCM=100$, $CIP=250$, $CIS=300$, $CZ=1350$
### Figure 4(b): Effect of $l$ on the Cost Function

$l_1=3$, $n=14$, $m=6$, $K_1=25$, $K_1=30$, $s=3$, $S=8$, $\lambda=1.2$, $\delta_1=2$, $\delta_2=1$, $\delta_3=2.5$, $\delta_4=1.5$

$CN=1000$, $CI=200$, $CPPR=500$, $CPRR=1000$, $CSPR=500$, $CSRR=500$, $CPON=5000$, $CPCOM=5000$, $CIP=500$, $CIS=1000$, $CZ=10000$

<table>
<thead>
<tr>
<th>$l$</th>
<th>COST</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>15351.15</td>
</tr>
<tr>
<td>3</td>
<td>14868.72</td>
</tr>
<tr>
<td>4</td>
<td>14305.12</td>
</tr>
<tr>
<td>5</td>
<td>13672.22</td>
</tr>
<tr>
<td>6</td>
<td>12911.49</td>
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<tr>
<td>7</td>
<td>12194.89</td>
</tr>
<tr>
<td>8</td>
<td>11370.66</td>
</tr>
<tr>
<td>9</td>
<td>10514.08</td>
</tr>
<tr>
<td>10</td>
<td>9665.41</td>
</tr>
</tbody>
</table>
6.6 Conclusion

We analyzed an \((s, S)\) production inventory system, where the processing of inventory requires some random amount time. Erlang distributions are used to model the service as well as the production processes, which are subject to multiple interruptions. For reducing the adverse effect of the interruptions, the concept of protecting certain phases of the service as well as the production process from interruption has been introduced. We further assumed that no customers would be allowed to join the system if there is no inventory in the system. This assumption lead us to derive an explicit expression for stability condition, which even holds if one assumes general PH in place of the assumed Erlang distributions.

In Studies like [3, 4] on inventory systems where customers are barred to join the system when there is shortage of inventory, the authors were able to show that the stability of such inventory systems is not affected by the inventory parameters and also that their steady state distributions can be obtained in product form. However, in these studies, the underlying distributions were all exponential. The proof, which we have given for the stability of the system, can be used to characterize the stability of the above said inventory systems with more general underlying distributions. However, we have not yet able to check whether some kind of product form expression for the steady state distribution is possible, which will be really interesting if one can do that.
6.7 Appendix I

I. Transitions leading to an increase in the level:
   Transitions due to arrival of customers

From $L(0,0,j_1) \rightarrow L(1,1,j_1,j_1)$ is governed by
\[ \beta \otimes \lambda I_{n-\delta_{j_1}}; \ 1 \leq j \leq S - 1; \ j_1 = 0,1 \]

From $L(0,0,j,0) \rightarrow L(1,1,j,0)$ is governed by $\lambda \beta; \ s + 1 \leq j \leq S$

From $L(i,1) \cup L(i,2) \rightarrow L(i+1,1) \cup L(i+1,2)$ is governed by $\lambda I$ (note that these are diagonal transitions)

II. Transitions leading to a decrease in the level:
   Transitions due to service completion

$L(1,1,j_1,j_1) \rightarrow L(0,0,j-1,1,j_1)$ is governed by $T^0 \otimes I_{n-\delta_{j_1}}; \ 1 \leq j \leq S - 1; \ j_1 = 0,1$

$L(1,1,s + 1,0,0) \rightarrow L(0,0,s,1,0)$ is governed by $T^0 \otimes \alpha$ (note that the production process needs to be switched to on mode the moment such a transition occurs)

$L(1,1,j,0,0) \rightarrow L(0,0,j-1,0,0)$ is governed by $T^0; \ s + 2 \leq j \leq S$

For $i \geq 2$,

$L(i,1,1,1,j_1) \rightarrow L(i-1,0,0,1,j_1)$ is governed by $T^0 \otimes I_{n-\delta_{j_1}}; \ j_1 = 0,1$

$L(i,1,j,1,j_1) \rightarrow L(i-1,1,j-1,1,j_1)$ is governed by

$T^0 \otimes \beta \otimes I_{n-\delta_{j_1}}; \ 2 \leq j \leq S - 1; \ j_1 = 0,1$

$L(i,1,s + 1,0,0) \rightarrow L(i-1,1,s,1,0)$ is governed by $T^0 \otimes \beta \otimes \alpha$
\[ L(i,1,j,0,0) \rightarrow L(i-1,1,j-1,0,0) \] is governed by \( T^0 \otimes \beta; \ s+2 \leq j \leq S \]

**III. Transitions where no level change occurs:**

**III(a) Transitions due to a production completion**

\[ L(0,0,j,1,0) \rightarrow L(0,0,j+1,1,0) \] is governed by \( U^0 \otimes \alpha; \ 0 \leq j \leq S-2 \]
\[ L(0,0,S-1,1,0) \rightarrow L(0,0,S,0,0) \] is governed by \( U^0 \)
\[ L(0,0,j,1,0) \rightarrow L(0,0,j+1,1,0) \] is governed by \( U^0 \otimes \alpha; \ 0 \leq j \leq S-2 \]
\[ L(i,0,0,1,0) \rightarrow L(i,1,1,1,0) \] is governed by \( \beta \otimes U^0 \otimes \alpha; \ i \geq 1 \)
\[ L(i,i',j,1,0) \rightarrow L(i,i',j+1,1,0) \] is governed by \( I_{m-\delta_{ij}} \otimes U^0 \otimes \alpha; \ i \geq 1; \ i' = 1,2; \ 1 \leq j \leq S-2 \)
\[ L(i,i',S-1,1,0) \rightarrow L(i,i',S,0,0) \] is governed by \( I_{m-\delta_{ij}} \otimes U^0; \ i \geq 1; \ i' = 1,2 \)

**III(b) Transitions due to a production interruption**

\[ L(0,0,j,1,0) \rightarrow L(0,0,j,1,1) \] is governed by \( \delta_j E; \ 0 \leq j \leq S-1, \) where \( E = \begin{bmatrix} I_{n-1} \\ 0 \end{bmatrix} \)
\[ L(i,0,0,1,0) \rightarrow L(i,0,0,1,1) \] is governed by \( \delta_j E; \ i \geq 1 \)
\[ L(i,i',j,1,0) \rightarrow L(i,i',j,1,1) \] is governed by \( I_{m-\delta_{ij}} \otimes \delta_j E; \ i \geq 1; \ i' = 1,2; \ 1 \leq j \leq S-1 \)

**III(c) Transitions due to completion of repair of an interrupted production process**

\[ L(0,0,j,1,1) \rightarrow L(0,0,j,1,0) \] is governed by \( \delta_j e \otimes \alpha; \ 0 \leq j \leq S-1 \)
Appendix I

$L(i,0,0,1,1) \rightarrow L(i,0,0,1,0)$ is governed by $\delta_i e \otimes \alpha; \ i \geq 1$

$L(i,l',j,1,1) \rightarrow L(i,l',j,1,0)$ is governed by $I_{m-\delta_j l'} \otimes \delta_i e \otimes \alpha; \ i \geq 1; \ l' = 1,2; \ 1 \leq j \leq S - 1$

III(d) Transitions due to a service interruption

$L(i,1,j,1,j_i) \rightarrow L(i,2,j,1,j_i)$ is governed by $\delta_i \tilde{E}_j^T; \ i \geq 1; \ 1 \leq j \leq S - 1; \ j_i = 0,1$,

where $\tilde{E}_j = \begin{bmatrix} I_{(m-l_i)(n-\delta_j l')} & 0 \end{bmatrix}$

$L(i,1,j,0) \rightarrow L(i,2,j,0)$ is governed by $\delta_i \left( \tilde{E}_0^T \right)^T; \ i \geq 1; \ s + 1 \leq j \leq S$, where $\tilde{E}_0 = \begin{bmatrix} I_{(m-l_i)} & 0 \end{bmatrix}$

III(e) Transitions due to completion of repair of an interrupted service

$L(i,2,j,1,j_i) \rightarrow L(i,1,j,1,j_i)$ is governed by $\delta_i \tilde{E}_j; \ i \geq 1; \ 1 \leq j \leq S - 1; \ j_i = 0,1$

$L(i,2,j,0) \rightarrow L(i,1,j,0)$ is governed by $\delta_i \tilde{E}_0; \ i \geq 1; \ s + 1 \leq j \leq S$
CONCLUSION

In this thesis we have developed a few inventory models in which items are served to the customers after a processing time. This leads to a queue of demand even when items are available. In chapter two we have discussed a problem involving search of orbital customers for providing inventory. Retrial of orbital customers was also considered in that chapter; in chapter 5 also we discussed retrial inventory model which is sans orbital search of customers. In the remaining chapters (3, 4 and 6) we did not consider retrial of customers, rather we assumed the waiting room capacity of the system to be arbitrarily large. Though the models in chapters 3 and 4 differ only in that in the former we consider positive lead time for replenishment of inventory and in the latter the same is assumed to be negligible, we arrived at sharper results in chapter 4. In chapter 6 we considered a production inventory model with production time distribution for a single item and that of service time of a customer following distinct Erlang distributions. We also introduced protection of production and service stages and investigated the optimal values of the number of stages to be protected. In all problems investigated closed form expressions for the system stability were derived. Our conclusions in each chapter are provided with numerous illustrations.