A CONJECTURE OF KOTZIG ON SELF-COMPLEMENTARY GRAPHS

This chapter deals with one of the main aims of the thesis, to discuss a conjecture of Kotzig on self-complementary graphs. Some of the results are reported in [45] and [46].

4.1 KOTZIG’S CONJECTURE

Recall that, a vertex in a self-complementary graph is a fixed vertex if it is mapped onto itself by a complementing permutation. The set of all fixed vertices in a self-complementary graph is denoted by \( F(G) \) and the set of all vertices with triangle number \( k(k-1) \) in a regular self-complementary graph of order \( 4k+1 \) is denoted by \( 
\hat{F}(G) \). Two vertices \( u \) and \( v \) are said to be similar, written as \( u \sim v \), if there exists an automorphism of \( G \) that maps \( u \) onto \( v \). Clearly \( \sim \) is an equivalence relation on \( V(G) \). The equivalence classes under \( \sim \) are called \( G \)-orbits. A vertex-symmetric graph has only one \( G \)-orbit.
Kotzig [41] observed that \( F(G) \leq \hat{F}(G) \) and asked about the possible characterization of \( F(G) \) and gave the following:

**KOTZIG'S CONJECTURE**

\[ F(G) = \hat{F}(G) \text{ for any regular self complementary graph } G. \]

In the subsequent sections, we recall the significant contribution made by Rao [54], characterize \( \hat{F}(G) \) which motivates its definition being extended to any graph \( G \) and construct more counterexamples to the conjecture.

### 4.2 EARLIER ATTEMPT.

Rao has characterized \( F(G) \) and constructed counterexamples to the conjecture in [54]. For convenience, we reproduce some of his results and a figure.

**Theorem 4.1** (part of the lemma 2.1 in [54]) If \( G \) is a self-complementary graph of order \( 4k+1 \), then exactly one of the \( G \)-orbits of \( V(G) \) is of odd cardinality. \( \quad \Box \)

**Theorem 4.2** (part of the theorem 2.2 in [54]) If \( G \) is a regular self-complementary graph of order \( 4k+1 \), then \( F(G) \) is the unique \( G \)-orbit of odd cardinality. \( \quad \Box \)

**Theorem 4.3** (theorem 4.1 in [54]) The following are equivalent for a self-complementary graph \( G \) of order \( \geq 5 \).

1. \( G \) is vertex-symmetric;
2. \( F(G) = V(G) \);
3. \( Z(G) = E(G) \).

(\( \Box \))
Theorem 4.4 (part of the theorem 4.2 of [54]) Let $G_1$, $G_2$ be two graphs and $G = G_1(G_2)$. Then the following hold.

1. If $G_1$, $G_2$ are regular, then so is $G$;
2. If $G_1$, $G_2$ are self-complementary, then so is $G$;
3. If $G_1$, $G_2$ are vertex-symmetric, then so is $G$.  

Theorem 4.5 (theorem 2.3 in [54]) For every integer $k \geq 2$, there is a regular self-complementary graph $G$ of order $4k+1$, such that $|F(G)| = 1$ but $|\hat{F}(G)| \geq 2k+1$.

Proof: Define a graph $G = G(4k+1)$ with $V(G) = \{0, 1, 2, ..., 4k+1\}$ and $E(G) = \bigcup_{i=1}^{k} A_i$, where $A_i$, $1 \leq i \leq 4$ is given below:

$A_1 = \{0, 2i+1\}, \{2i+1, 2i+2\}$, for every $i$, $0 \leq i \leq 2k-1$;
$\{4j+2, 4j+4\}$ for every $j$, $0 \leq j \leq k-1$,

$A_2 = \{4i+1, 4j+2\}, \{4i+3, 4j+4\}$, for every $i, j$,
$0 \leq i, j \leq k-1, i \neq j$,

$A_3 = \{4i+1, 4j+3\}$; for every $i, j$, $0 \leq i, j \leq k-1, i \neq j$,

$A_4 = \{4i+2, 4j+2\}, \{4i+4, 4j+4\}$ for every $i, j$,
$0 \leq i, j \leq k-1, i \neq j$.

It can be checked that $G$ is a self-complementary graph of order $4k+1$ under $\sigma = (0) \prod_{i=0}^{k-1} (4i+1, 4i+2, 4i+3, 4i+4)$. Further the neighbourhood of 0 induces a regular graph of order $2k$ and degree $k-1$ and $0 \in F(G)$. It can be also checked that the neighbourhood of 2 induces a complete bipartite graph with bipartition $\{1, 5, 9, 13, ..., 4k-3\}$; $6, 10, 14, ..., 4k-2$ together with the isolated vertex 4, which clearly has $k(k-1)$
edges and is not regular. Further, for any $i, 1 \leq i \leq 2k$, the induced subgraph on the neighbourhood of $2i$ is isomorphic to that on the neighbourhood of 2. Therefore $F(G)$ contains the set \{ 0, 2, 4, \ldots, 4k \}. By Theorem 2.2 (theorem 4.2 here) and the fact that $0 \in F(G)$, it follows that for no $i, 1 \leq i \leq 2k$, the vertex $2i \in F(G)$. The set $F(G)$ being a $G$-orbit (namely the unique $G$-orbit of odd length) it is the union of some cycles of the above $\sigma$. This implies that $F(G) = \{0\}$.

Note that in case $k = 2$ for the graph $G(9)$, $F(G) = \{0\}$ and $\hat{F}(G) = V(G)$. However, for $k \geq 3$ and $G = G(4k+1)$, $F(G) = \{0\}$ and $\hat{F}(G) = \{0, 2, 4, \ldots, 4k\}$.

4.3 THE SET $\hat{F}(G)$

Recall that $\hat{F}(G)$ is the set of vertices in a regular self-complementary graph $G$ of order $4k+1$ with triangle number $k(k-1)$.
Theorem 4.6 A vertex $u$ in a regular self-complementary graph $G$ is in $\hat{F}(G)$ if and only if $t(u) = \overline{t}(u)$.

Proof: Let $G$ be a regular self-complementary graph of order $p = 4k+1$, $k \in \mathbb{N}$ and let $u \in \hat{F}(G)$. Then $t(u) = k(k-1)$ and hence, by (3.8), $\overline{t}(u) = k(k-1)$.

Conversely, let $t(u) = \overline{t}(u)$ for some $u \in V(G)$. Then $t(u) = \overline{t}(u) = k(k-1)$ by (3.8). So, $u \in \hat{F}(G)$. ■

An important and natural consequence of theorem 4.6 is that, $\hat{F}(G)$, which was defined only for regular self-complementary graphs can be extended to any graph.

Definition: Let $G$ be a simple graph. Then the set $\hat{F}(G)$ is defined as $\hat{F}(G) = \{ u \in V(G) / t(u) = \overline{t}(u) \}$.

The graph $G$ in fig. 4.2 is not self-complementary. The triangle number of each of the vertices labelled $u$ and $v$ is 3 in both $G$ and $\overline{G}$ and that of other vertices are different in $G$ and $\overline{G}$. Hence $\hat{F}(G) = \{ u, v \}$.

\[ G \]

\textbf{Figure 4.2}
Theorem 4.7 \( \tilde{F}(G) = \tilde{F}(\overline{G}) \) for any graph \( G \). \hfill (4.1)

Proof: \( u \in F(G) \iff \overline{t}(u) = t(u) \)
\[ \iff u \in \tilde{F}(\overline{G}) \]

Theorem 4.8 A vertex \( u \) in a \((p,q)\)-graph \( G \) is in \( \tilde{F}(G) \) if and only if the size of \( <N(u)> \) in \( G \) is
\[ \frac{1}{2} \left[ \frac{(p-d(u)-1)}{2} - q + \sum_{v \in N(u)} d(u) \right]. \]

Proof: Let \( G \) be a graph and \( u \in \tilde{F}(G) \). Then \( t(u) = \overline{t}(u) \).
But, we have, \( t(u) + \overline{t}(u) = \left( \frac{p-d(u)-1}{2} \right) - q + \sum_{v \in N(u)} d(u) \). So,
\[ 2t(u) = \left( \frac{p-d(u)-1}{2} \right) - q + \sum_{v \in N(u)} d(u) \]. Hence the necessary part.

Conversely, let \( u \in V(G) \) be such that
\[ t(u) = \frac{1}{2} \left[ \frac{(p-d(u)-1)}{2} - q + \sum_{v \in N(u)} d(u) \right]. \] Then \( \overline{t}(u) \) is also
\[ \frac{1}{2} \left[ \frac{(p-d(u)-1)}{2} - q + \sum_{v \in N(u)} d(u) \right] \text{ by (3.1). Hence } t(u) = \overline{t}(u). \]

Corollary 4.9 Let \( G \) be a regular graph of order \( p \) and degree of regularity \( r \), then a vertex \( u \) is in \( \tilde{F}(G) \) if and only if
\[ t(u) = \frac{1}{2} \left( \frac{p-1}{2} \right) - \frac{3}{4} r(p-r-1). \]

The proof being a routine one is omitted.

Remark 4.10 It follows from lemma 3.31 that, if \( G \) is a regular self-complementary graph then, \( \tilde{F}(G) = V(G) \) if and only if \( G \) is SVTR.
4.4 PRESENT ATTEMPT.

Here, we mention a fallacy in the proof of theorem 4.5 and identify a class of counterexamples. A construction of such graphs of order $p$, for an infinite number of values of $p$, is also carried out.

While analyzing the counterexamples of Rao, we came to know that they are wrong except for $k = 2$. Because, the claim in the proof of theorem 4.5 "the neighbourhood of 2 induces the complete bipartite graph with bipartition \{ 1, 5, 9, \ldots, 4k-3 ; 6, 10, \ldots, 4k-2 \} together with the isolated vertex 4" is wrong for $k \geq 3$. In fact \{ 6, 10, \ldots, 4k-2 \} induces a complete subgraph due to the edges $(4i+1, 4j+2)$, $0 \leq i, j \leq k-1, i \neq j$. So $t(2) = k(k-1) + \frac{1}{2} - (k-1)(k-2)$ and $t(1) = (k-1) + \frac{1}{2} (k-1) = k(k-1) - \frac{1}{2} (k-1)(k-2)$ for every $k \geq 2$ and consequently $\hat{F}(G) = \{ 0 \}$ for $k \geq 3$.

Thus the conjecture was made open for $p = 4k+1$, $k \geq 2$

**Theorem 4.11 (A class of counterexamples)** If $G$ is a self-complementary graph which is strongly vertex triangle regular and not vertex symmetric, then it is a counterexample to the conjecture.

**Proof**: Let $G$ be a self-complementary graph. Then $F(G) = V(G)$ if and only if $G$ is vertex-symmetric ( theorem 4.3 ) and $\hat{F}(G) = V(G)$ if $G$ is strongly vertex triangle regular ( remark 4.10 ). So if $G$ is SVTR and not vertex-symmetric, then $\hat{F}(G) = V(G) \neq F(G)$. Hence this class provides counterexamples to the conjecture.
Remark 4.12 It is interesting to see that the counterexample $G_y$, of Rao is also of the type specified in the theorem 4.11.

Theorem 4.13 Let $G$ be an SVTRSC graph which is not vertex-symmetric and and $H$ be a VSSC graph. If there are two vertices $u$ and $u'$ in $G$ such that $\langle N(u) \rangle$ is regular and $\langle N(u') \rangle$ is not regular, then $G(H)$ is SVTRSC but not vertex-symmetric.

Proof: Let $G$ be a SVTRSC graph which is not vertex-symmetric and $H$ be a VSSC graph. Then clearly $H$ is SVTR and hence $G(H)$ is SVTRSC.

Now, let $G_1 = \langle N(u) \rangle_G$ and $G_2 = \langle N(u') \rangle_G$ where $u$ and $u'$ are as in the hypothesis. Then $G_1$ is regular and $G_2$ is not regular. It is obvious that $\langle N(u,v) \rangle$ in $G(H)$ is $G_1(H)$ and $\langle N(u',v) \rangle$ is $G_2(H)$. Because of the regularity of $G$ and $H$, $G(H)$ is also regular, but $G_2(H)$ is not regular since $G_2$ is not. So $\langle N(u,v) \rangle \not\equiv \langle N(u',v) \rangle$ in $G(H)$. Hence $G(H)$ is not vertex-symmetric.

Remark 4.14 If $G$ is a counterexample to the conjecture and $H$ is a vertex-symmetric self-complementary graph. If there are vertices $u$ and $u'$ in $G$ such that $\langle N(u) \rangle$ is regular and $\langle N(u') \rangle$ is not regular, then, by theorem 4.12, $G(H)$ and $H(G)$ are also counterexamples.
CONSTRUCTION OF COUNTEREXAMPLES

(1) Counterexample of order 17

Take a single vertex 0, a copy of the circulant graph $C(8; 1, 4)$ with vertices labelled 0, 1, 2, ..., 7 and a copy of its complement $C(8; 2, 3)$ with vertices labelled 0', 1', 2', ..., 7'. Join each vertex $i$ to $0$, $i'$, $i'+1$, $i'+2$ and $i'+3$, addition being taken modulo 8 and $i'+j$ is to mean $(i+j)'$. The graph $G_{17}$ so obtained is self-complementary, a complementing permutations is $(0 0' 1 1' 2 2' 3 3' 4 4' 5 5' 6 6' 7 7')$. From the figure of $G_{17}$, it's strong vertex triangle regularity is clear.

$G_{17}$ : a counterexample of order 17

figure 4.3
It is not vertex-symmetric because the subgraph induced by the neighbourhood of 0 is the circulant graph $C(8; 1, 4)$ which is not isomorphic to the subgraph induced by the neighbourhood of any of the other vertices. Further $<N(i)>$ and $<N(j')>$ are also non-isomorphic for every $i$ and $j'$ (see fig. 4.4)

Counterexample of order 33

Take a single vertex labelled 0, a copy of the circulant graph $C(16; 1, 2, 6, 7)$ with vertices labelled 0, 1, 2, ..., 15 and a copy of its complement $C(16; 3, 4, 5, 8)$ with vertices labelled 0', 1', 2', ..., 15'. Join each vertex $i$ to $i'$, $i'+1$, $i'+2$, ..., $i'+7$ and each $i'$ to 0. Additions being taken modulo 16. The resulting graph $G_{33}$ is self-complementary.
under the complementing permutation \((0)(0' 1 1' 2 2' \ldots \cdot \cdot J5 15')\) and strongly vertex triangle regular. But it is not vertex-symmetric, since the subgraph induced by the neighbourhood of 0 is the circulant graph \(C(16; 3, 4, 5, 8)\) which is not isomorphic to the subgraph induced by the neighbourhood of any of the other vertices (see fig. 4.5) and \(<N(j')>\) are also non-isomorphic for every \(i\) and \(j'\).

**Present Status of the Conjecture.**

The conjecture is trivially true for \(p = 5\). We have seen that strongly vertex triangle regular self-complementary graphs which are not vertex-symmetric form counterexamples. We have one such graph is \(G(9)\) (fig. 4.1) and of order 17 (fig. 4.3) and 33 by the above construction. Hence by theorem 4.12, counterexamples of order \(p = 9^a17^b33^\gamma p_1^\delta\) where \(p_1\) is an integer for which VSSC graph of order \(p_1\) exists and \(a, b, \gamma\) and \(\delta\) are integers such that at least one of \(a, b\) and \(\gamma\) is non-zero. Thus, the conjecture is false for \(p = 9^a17^b33^\gamma p_1^\delta\) where \(p_1, a, b, \gamma\) and \(\delta\) are integers as above. We are examining the conjecture for other orders also and expect that our construction can be extended to graphs of order \(p = 4k+1\) where \(k = 2^n, n \in \mathbb{N}\). Then theorem 4.13 can be applied to get still more counterexamples.

* * * *
5

ISOMORPHIC FACTORIZATION
OF
COMPLETE GRAPHS

5.1 ISOMORPHIC FACTORIZATION

A factorization of a graph is a partition of it into edge disjoint spanning subgraphs. A factorization in which any two factors are isomorphic is called an isomorphic factorization. A graph $G$ is said to be divisible by an integer $m$ if it can be factored into exactly $m$ isomorphic factors and we write $m/G$. If $G$ is divisible by $m$, then the set of all graphs $H$ such that $G$ can be factored into $m$ isomorphic copies of $H$ is denoted by $G/m$.

\[ T \]

Figure 5.1
If $G$ has $q$ edges, $G/m$ will be empty unless $m/q$. This necessary condition is not in general sufficient as in the case of the tree $T$ in fig 5.1, which has six edges, yet $T/2$ is empty.

5.2 ISOMORPHIC FACTORIZATION OF COMPLETE GRAPHS.

Isomorphic factorization of complete graphs into $m$ factors is a generalization of self-complementation. If $K_p$ is divisible by two, then the members of $K_p/2$ are the self-complementary graphs of order $p$. Even though self-complementary graphs are connected, the elements of $K_p/m$ need not be so for $m \geq 3$. For example see fig. 5.2. If the members of $K_p/m$ are of size $q$, then $mq = \frac{p(p-1)}{2}$ and so $\frac{p(p-1)}{2m}$ is an integer. The result in the converse direction was independently proved by Guidotti [33] and Harary et al. [35].

Here, we give a simpler proof by generalizing a method of constructing self complementary graphs given by Gibbs [30]. The proof given in [35] essentially involves permutations of the $p$ vertices and the $\frac{p(p-1)}{2}$ edges of $K_p$, while we use permutations of the vertices only.

When $m/K_p$, there are isomorphisms, that is permutations of $V(K_p)$, that maps between the factors. We call such a permutation $\sigma$ as factorizing permutation. We label the $m$ factors in an isomorphic factorization of $K_p$ by $G_0, G_1, G_2, \ldots, G_{m-1}$ so that a factorizing permutation $\sigma$ of $V(K_p)$ maps $G_i$ onto $G_{i+1(mod m)}$, $i = 1, 2, \ldots, m-1$. 
The nine members of $K^3_{6}$

Factorizing permutation for each factorization is $(123)(456)$

Figure 5.2
Theorem 5.1 ([35]) If \( m \mid \frac{P(p-1)}{2} \) and \((p,m) = 1\) or \((p-1,m) = 1\), then \( K_p \) is divisible by \( m \).

Proof (by construction): Let \( m \) and \( p \) be such that \( m \mid \frac{P(p-1)}{2} \) and either \((p,m) = 1\) or \((p-1,m) = 1\). We have to find \( m \) isomorphic factors of \( K_p \).

CONSTRUCTION

Case (i) \( m \) is odd.

If there is an \( m \)-factorization, the edges in the subgraph of \( K_p \) spanned by each cycle of a factorizing permutation \( \sigma \) is to be distributed equally in the factors, every cycle of \( \sigma \) should be of length multiple of \( m \) except in the case of a 1-cycle when \( p \equiv 1 \pmod{m} \). But it is sufficient to consider the permutations with cycle length power of \( m \). Because, if there is a cycle of \( \sigma \) not of this form, that is of length \( \alpha m \) where \( \alpha \) is not a multiple of \( m \) then the permutation \( \sigma^\alpha \) will be of the required form and will be a factorizing permutation (not necessarily in the same order in which \( \sigma \) acts).

Consider a permutation \( \sigma \) of \( p \) symbols with each of its cycles is of length power of \( m \), except one cycle of length one when \( p \equiv 1 \pmod{m} \). Assume without loss of generality that the symbols in \( \sigma \) are numbered consecutively from 1 to \( p \) and that the cycles are of non-decreasing length \( k_1, k_2, k_3, \ldots \) except the 1-cycle \((p)\), if exists, at the end. It is to be noted that each \( k_i \) is a power of \( m \). Now, the symbols 2, 3, \ldots, \( \frac{k_1+1}{2} \) of the first cycle, the first \( k_1 \) symbols of each of the other cycle
and the symbol $p$ if $(p)$ is a 1-cycle constitute the range of the symbol 1. We shall construct the graphs $G_0$, $G_1$, $G_2$, ..., $G_{m-1}$ with vertices labelled 1, 2, 3, ..., $p$ and hence identify the symbols in $\sigma$ with the vertices in $G_j$; $j = 0, 1, 2, ..., m-1$.

For each unordered pair $\{1, j\}$, where $j$ is in the range of 1, arbitrarily decide the graph $G_i$ in which 1 and $j$ are adjacent. Once these choices have been made, the symbols $\sigma^k(1)$ and $\sigma^k(j)$ are adjacent in $G_{i+k(modm)}$, $k = 1, 2, ..., k_j$ where $j$ belongs to a cycle of length $k_j$. A table of the following form is helpful. In the first column of the table, the symbols 1, 2, $\ldots$, $k_1$ is to be repeated $\frac{k_1}{k_1}$ times where $k_1$ is the maximum cycle length of $\sigma$.

<table>
<thead>
<tr>
<th>vertex $u$</th>
<th>neighbours of $u$ in the factor $G_0$</th>
<th>$G_1$</th>
<th>$\ldots$</th>
<th>$G_{m-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v_{01}$, $v_{02}$, $\ldots$</td>
<td>$v_{11}$, $v_{12}$, $\ldots$</td>
<td>$\ldots$</td>
<td>$v_{m-1,1}$, $v_{m-1,2}$, $\ldots$</td>
</tr>
<tr>
<td>2</td>
<td>$\sigma(v_{m-1,1})$, $\ldots$</td>
<td>$\sigma(v_{01})$, $\ldots$</td>
<td>$\ldots$</td>
<td>$\sigma(v_{m-2,1})$, $\ldots$</td>
</tr>
<tr>
<td>3</td>
<td>$\sigma(v_{m-2,1})$, $\ldots$</td>
<td>$\sigma(v_{m-1,1})$, $\ldots$</td>
<td>$\ldots$</td>
<td>$\sigma(v_{m-2,1})$, $\ldots$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$k_1$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

Adjacency table for the isomorphic factorization of complete graphs

Table 5.1

This completes the first stage of the algorithm. In the next stage, reduce the permutation $\sigma$ to $\sigma_1$ on $p-k_1$ symbols.
by deleting the first cycle and do the process for the symbol $k_1 + 1$. Continue the process till all the cycles of non-unit length has been considered.

Case (ii) $m$ is even.

In this case it is sufficient to consider permutations $\sigma$ with cycle length powers of $2m$ only. Arrange $\sigma$ so that the cycles are in the order of non-decreasing length except the one 1-cycle $(p)$, if exists, at the end. Let $k_1$, $k_2$, $\ldots$ be the cycle lengths and 1, 2, $\ldots$, $p$ be the symbols in the permutation. The range of 1 consists of the symbols 2, 3, $\ldots$, $\frac{k_1 - 1}{2} + 1$, the first $k_1$ symbols of the remaining cycles and the symbol $p$, if $(p)$ is a 1-cycle. The rest of the algorithm is same as the first case.

Now we have to prove that the algorithm will produce a well defined isomorphic factorization.

Claim: As a result of performing the construction algorithm, (1) the adjacency relation between vertices is well-defined (2) every pair of vertices is assigned an adjacency relation and (3) the graphs $G_0$, $G_1$, $\ldots$, $G_{m-1}$ thus obtained are isomorphic.

Proof of (1) The pair $\{1, j\}$ cannot be sent to itself by $\sigma^k$ when $k$ is not a multiple of $m$, because $\sigma^k(j) \neq j$ for $k \neq M(m)$, except for the trivial case of the 1-cycle $(p)$ and if $\sigma^k(1) = j$, then $j$ is the symbol 1+$k$ in the first cycle of $\sigma$ and $\sigma^k(j) = 1+2k \neq 1$ since $k \neq M(m)$. Thus the pair $\{1, j\}$ can never be assigned simultaneous adjacency and non-adjacency.
The same argument applies to \( \{ \sigma^1(1), \sigma^1(j) \} \) and carries over for all stages of the algorithm.

**Proof of (2)** Here we have to consider the two cases separately.

**Case (i)** \( m \) is odd

From the definition of range of the symbol 1, we have assigned adjacency to each pair \( 1, i \) when \( 2 \leq i \leq \frac{k_1 + 1}{2} \). For every \( j \) in the first cycle, symbols in its range from the first cycle are \( j+1, j+2, \ldots, j+\frac{k_1 - 1}{2} \). Now \( \frac{k_1 + 3}{2} = \sigma^2(1) \) and so the range of \( \frac{k_1 + 3}{2} \) contains the symbols \( \frac{k_1 + 5}{2}, \ldots, \frac{k_1 + 3}{2}, \frac{k_1 - 1}{2} = k_1 + 1 \) of the first cycle. Hence 1 is in the range of \( \frac{k_1 + 3}{2}, \ldots, k_1 \). Thus the adjacency between 1 and every other symbol in the first cycle are defined if we fix the adjacency of 1 and those symbols in its range. This argument carries to all other symbols in the first cycle and for the adjacencies of the other symbols with those in the same cycle. Consider the cycle \( \sigma_j \) of length \( k_j \), \( j \neq 1 \). We initially fix the adjacencies of the first \( k_1 \) symbols. But \( \sigma^k(1) = 1 \) and if \( k_j > k_1 \), then \( \sigma^k \) will give the adjacencies of next \( k_1 \) symbols in \( \sigma_j \) with 1. Our construction algorithm insists on continuing the process at least \( \frac{k_j}{k_1} \) times. Thus the adjacency of 1 with each symbol in the cycle \( \sigma_j \) is defined. This is also applicable to all symbols in the first cycle and to all steps of the algorithm.
Case (ii) \( m \) is even.

Here the range of 1 is 2, 3, \( \ldots \ldots \), \( \frac{k}{2}+1 \) and that of any \( j \) in the first cycle is \( j+1, j+2, \ldots \ldots \), \( j+\frac{k}{2}+1 \). Now,

\[
\frac{k}{2}+1 = \sigma^2 (1) \quad \text{and hence the range of} \quad \frac{k}{2}+2 \quad \text{is} \quad \frac{k}{2}+3, \frac{k}{2}+4, \ldots
\]

\[
\vdots, \quad \frac{k}{2}+\frac{k}{2}+1 = k+1 = 1 \quad \text{and the remaining arguments are similar to that in the first case.}
\]

Proof of (3) Now, we have shown that all the adjacencies are well defined and all possible adjacencies are determined. Clearly \( \sigma, \sigma^2, \ldots \ldots, \sigma^{m-1} \) are isomorphism from \( G_0 \) to \( G_1, G_2, \ldots, G_{m-1} \), respectively.

\[ \blacksquare \]

**ILLUSTRATIONS**

(i) **ISOMORPHIC FACTORIZATION OF \( K_7 \) INTO THREE FACTORS**

 corresponding to the permutation \( \sigma = (123)(456)(7) \)

Stage 1: The range of 1 is \( \{ 2, 4, 5, 6, 7 \} \). Let the vertex labelled 1 be adjacent to 2 and 7 in \( G_0 \), to 4 in \( G_1 \) and to 5 and 6 in \( G_2 \). The corresponding adjacency table is given in table 5.2.

Stage 2 The reduced permutation to be considered is \( \sigma_1 = (456)(7) \). The range of 4 is \( \{ 5, 7 \} \). Let the vertex labelled 4 be adjacent to 5 and 7 in \( G_1 \). The adjacency table is given in table 5.3.
<table>
<thead>
<tr>
<th>vertex $u$</th>
<th>neighbours of $u$ in the factor $G_0$</th>
<th>neighbours of $u$ in the factor $G_1$</th>
<th>neighbours of $u$ in the factor $G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2, 7</td>
<td>4, 5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3, 7</td>
<td>5, 6</td>
</tr>
<tr>
<td>3</td>
<td>6, 4</td>
<td>5</td>
<td>1, 7</td>
</tr>
</tbody>
</table>

Adjacency table at stage 1 for an isomorphic factorization of $K_7$ into three factors corresponding to the permutation $(123)(456)(7)$

table 5.2

The factor $G_0$ of $K_7$ resulting from the above construction

figure 5.3
(ii) ISOMORPHIC FACTORIZATION OF $K_{20}$ INTO THREE FACTORS

corresponding to the permutation

$$\sigma = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20).$$

Stage 1: The range of 1 is $\{2, 3, 5, 6, 7, 8\}$. Let the vertex labelled 1 be adjacent to 3 in $G_0$, to 6 and 7 in $G_1$ to 2, 5 and 8 in $G_2$, and none in $G_3$. The adjacency table is given in table 5.4.

Stage 2: The reduced permutation to be considered in this stage is $\sigma_1 = (5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20)$. The range of 5 is $\{6, 7, 8, 9, 10, 11, 12, 13\}$. Let the vertex labelled 5 be adjacent to 6 and 13 in $G_0$, 7 and 12 in $G_1$, 8 and 11 in $G_2$ and 9 and 10 in $G_3$. The adjacency table is given in table 5.5.
<table>
<thead>
<tr>
<th>vertex $u$</th>
<th>$G_0$</th>
<th>neighbours of $u$ in the factor</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>3</td>
<td>6, 7</td>
<td>2, 5, 8</td>
<td>3, 6, 9</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>7, 8</td>
<td>1</td>
<td>8, 9</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4, 7, 10</td>
<td>1, 8, 11</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9, 10</td>
<td>1, 8, 11</td>
<td>2, 9, 12</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>10, 11</td>
<td>2, 9, 12</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>11, 12</td>
<td>3, 10, 13</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4, 11, 14</td>
<td>1, 12, 15</td>
<td>12, 13</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>13, 14</td>
<td>1, 12, 15</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>3</td>
<td>14, 15</td>
<td>2, 13, 16</td>
<td>2</td>
<td></td>
</tr>
<tr>
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<td>4</td>
<td>15, 16</td>
<td>3, 14, 17</td>
<td>2</td>
<td></td>
</tr>
<tr>
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<td>4, 15, 18</td>
<td>1, 16, 19</td>
<td>16, 17</td>
<td>2</td>
<td></td>
</tr>
<tr>
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<td>17, 18</td>
<td>1, 16, 19</td>
<td>2</td>
<td></td>
<td></td>
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<tr>
<td>1</td>
<td>3</td>
<td>18, 19</td>
<td>2, 17, 20</td>
<td>2</td>
<td></td>
</tr>
<tr>
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<td>4</td>
<td>19, 20</td>
<td>3, 18, 5</td>
<td>2</td>
<td></td>
</tr>
<tr>
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<td>4, 19, 6</td>
<td>1, 20, 7</td>
<td>20, 5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5, 6</td>
<td>1, 20, 7</td>
<td></td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Adjacency table at stage 1 for an isomorphic factorization of $K_{20}$ into four factors corresponding to the permutation $(1 2 3 4)(5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20)$

Table 5.4
<table>
<thead>
<tr>
<th>vertex $u$</th>
<th>$G_0$</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6, 13</td>
<td>7, 12</td>
<td>8, 11</td>
<td>9, 10</td>
</tr>
<tr>
<td>6</td>
<td>10, 11</td>
<td>7, 14</td>
<td>8, 13</td>
<td>9, 12</td>
</tr>
<tr>
<td>7</td>
<td>10, 13</td>
<td>11, 12</td>
<td>8, 15</td>
<td>9, 14</td>
</tr>
<tr>
<td>8</td>
<td>10, 15</td>
<td>11, 14</td>
<td>12, 13</td>
<td>9, 16</td>
</tr>
<tr>
<td>9</td>
<td>10, 17</td>
<td>11, 16</td>
<td>12, 15</td>
<td>13, 14</td>
</tr>
<tr>
<td>10</td>
<td>14, 15</td>
<td>11, 18</td>
<td>12, 17</td>
<td>13, 16</td>
</tr>
<tr>
<td>11</td>
<td>14, 17</td>
<td>15, 16</td>
<td>12, 19</td>
<td>13, 18</td>
</tr>
<tr>
<td>12</td>
<td>14, 19</td>
<td>15, 18</td>
<td>16, 17</td>
<td>13, 20</td>
</tr>
<tr>
<td>13</td>
<td>14, 15</td>
<td>15, 20</td>
<td>16, 19</td>
<td>17, 18</td>
</tr>
<tr>
<td>14</td>
<td>18, 19</td>
<td>15, 6</td>
<td>16, 5</td>
<td>17, 20</td>
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<tr>
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<td>19, 20</td>
<td>16, 7</td>
<td>17, 6</td>
</tr>
<tr>
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<td>18, 7</td>
<td>19, 6</td>
<td>20, 5</td>
<td>17, 8</td>
</tr>
<tr>
<td>17</td>
<td>18, 9</td>
<td>19, 8</td>
<td>20, 7</td>
<td>5, 6</td>
</tr>
<tr>
<td>18</td>
<td>6, 7</td>
<td>19, 10</td>
<td>20, 9</td>
<td>5, 8</td>
</tr>
<tr>
<td>19</td>
<td>6, 9</td>
<td>7, 8</td>
<td>20, 11</td>
<td>5, 10</td>
</tr>
<tr>
<td>20</td>
<td>6, 11</td>
<td>7, 10</td>
<td>8, 9</td>
<td>5, 12</td>
</tr>
</tbody>
</table>

Adjacency table at stage 2 for an isomorphic factorization of $K_{20}$ into four factors corresponding to the permutation $(1 2 3 4) (5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20)$

Table 5.5
5.3 CONCLUDING REMARK AND SUGGESTIONS FOR FURTHER STUDY

This thesis is an attempt to shed more light on a conjecture of Anton Kotzig on self-complementary graphs. During this process, we have obtained several results relating the concepts of triangle and self-complementation, spread over the different chapters of this thesis. The survey of earlier results have been done to the extent possible and any serious omission due to oversight may kindly be pointed out.

Results of the thesis are far from complete. We list below some of the problems which we have either not attempted or found the answers to be difficult.

1. ANTIPODAL ITERATION NUMBER (ain.)

Consider a graph $G$ and its antipodal graph $A(G)$. Let $G_0 = G$ and $G_{i+1}$ be the graph obtained by superimposing $A(G_i)$ on $G_i$, for $i = 0, 1, 2, \ldots$. If $G$ is not complete, this process ultimately results in a complete graph since $E(A(G)) \subseteq E(G^\overline{G})$. The minimum value of $i$ for which $G_i$ is complete is called the antipodal iteration number (ain.) of $G$. It is obvious that $\text{ain}(K_p) = 0$ and $\text{ain}(G) = 1$ if $\text{diam}(G) = 2$. If $G$ is disconnected, then its ain. is 1 if every component of $G$ is complete and 2 otherwise. A formula for $\text{ain}(G)$ can be attempted.
2. S-ANTIPODAL GRAPH OF GRAPHS WITH A GIVEN PROPERTY

We have characterized $A^*(G)$ when $G$ is a tree. Similar analysis can be done for a graph $G$ with a given property $P$, where $P$ could be maximal outer planar, hamiltonian, eulerian, chordal, etc. The question whether eulerian graph of odd order is the S-antipodal graph of some eulerian graph remains to be settled. We have answered (theorem 2.20) a similar question for even order.

3. TRIANGLE SEQUENCE

Similar to the results on degree sequences [63], the concept of triangle sequence could be investigated and characterization of an integer sequence being the triangle sequence of a graph may be attempted.

4. TRIANGLE NUMBER IN THE G-JOIN

Expression for the triangle number of a vertex / edge in the G-join of a family of graphs in the general setting is worth studying. See theorem 3.27 for our observation.

5. COUNTER EXAMPLES TO KOTZIG'S CONJECTURE

Our method of construction of counterexamples of order 17 and 33 could be extended to that of order $p = 4k+1$, where $k = 2^n$, $n \in \mathbb{N}$.