1.1 The theory of orthogonal series is not always a
classical section of the analysis. It has its origin since
last 200 years, which originated during the discussion of
the problem of vibrating string considered by Euler in 1753
in connection with the work by Daniel Bernoulli. During their
discussion they had advanced the theory of vibrating strings
to the stage where the partial differential equation
\[ y_{tt} = a^2 y_{xx} \]
was known and the solution of the boundary value
problem had been found from the general solution of that
equation. Thus, they have led to the possibility of
representing an arbitrary function by a trigonometric series.
The problem of what functions can be represented by
trigonometric series arose again later during the researches
by French mathematical physicist J. B. Fourier.
The last several years have been a period of intensive development in the theory of Fourier series. Advances have also been made in the theory of Fourier series with respect to general orthogonal systems, during the last thirty years. But less attention has been paid to the theory of orthogonal series. The present work is concerning the theory of orthogonal series. During the first half of present century, some of the mathematicians like Fejér, Hardy, Hilbert, Hobson, Lebesgue, F. Riesz, M. Riesz, Weyl, Alexits, Kaczmarz, Steinhaus, Menchoff, Zygmund, Lorentz, Meder and Tandori were mainly working in the subject of convergence, summability and approximation problems of orthogonal series. We would like to discuss some of the problems connected with the convergence and summability of orthogonal series. We begin with number of definitions and concept relevant to the body work of our thesis.

1.2 Throughout the thesis we shall make use of either Stieltjes-Lebesgue integral or the Lebesgue integral. The notion of orthogonality is introduced by means of the Stieltjes-Lebesgue integral. Let \( \mu(x) \) be a positive bounded and monotone increasing function in the closed interval \([a, b]\). Such a function is called distribution function\(^{1}\).

A real function \( f(x) \) is called \( L_\mu \)-integrable, if it is

\(^{1}\) Freud [21]
μ-measurable and

\[ \int_{a}^{b} |f(x)| \, d\mu(x) < \infty. \]  

(1.2.1)

If \( \mu(x) \) is absolutely continuous and \( \rho(x) = \mu'(x) \), then for any \( L_\mu \)-integrable function \( f(x) \), the relation

\[ \int_{a}^{b} f(x) \, d\mu(x) = \int_{a}^{b} f(x) \, \rho(x) \, dx \]  

(1.2.2)

is valid. In this case we shall say that \( f(x) \) is \( L_{\rho(x)} \)-integrable function and we call \( \rho(x) \) a covering function or weight function. If, in particular \( \rho(x) = 1 \), then we shall say in accordance with the usual terminology that \( f(x) \) is \( L \)-integrable.

A function \( f(x) \) is called \( L_{\mu}^2 \) or \( L_{\rho(x)}^2 \)-integrable, if it is \( L_{\mu} \) or \( L_{\rho(x)} \)-integrable respectively and if, furthermore,

\[ \int_{a}^{b} f^2(x) \, d\mu(x) < \infty \quad \text{or} \quad \int_{a}^{b} f^2(x) \, \rho(x) \, dx < \infty \]  

holds. We shall talk about an \( L^2 \)-integrable function, if \( \rho(x) = 1 \).

**ORTHOGONALITY** :- A finite or denumerably infinite system \( \{ \phi_n(x) \} \) of \( L_{\mu}^2 \)-integrable functions is said to be orthogonal with respect to the distribution \( d\mu(x) \) in the interval \([a,b]\),
if

\[(1.2.3) \quad \int_{a}^{b} \phi_n(x) \phi_m(x) \, d\mu(x) = 0, \quad m \neq n\]

holds and none of the functions \(\phi_n(x)\) vanishes a.e., that is almost everywhere.

A system \(\{\phi_n(x)\}\) is said to be orthonormal, if in addition to the condition (1.2.3) the condition

\[\int_{a}^{b} \phi_n(x)^2 \, d\mu(x) = 1, \quad n=0,1,2,\ldots\]

is also satisfied. Every orthogonal system \(\{\psi_n(x)\}\) can be converted into an orthonormal system (ONS) by means of multiplying everyone of its members by a suitably chosen factor. For, since none of the functions \(\psi_n(x)\) can vanish a.e., the functions

\[\phi_n(x) = \frac{\psi_n(x)}{\left(\int_{a}^{b} \psi_n(x)^2 \, d\mu(x)\right)^{\frac{1}{2}}}\]

exist and is immediately evident that they constitute an ONS with respect to \(d\mu(x)\). If, in particular \(\mu(x) = x\) i.e. \(\mu'(x) = \phi(x) = 1\), then \(\{\phi_n(x)\}\) is simply an ONS in the ordinary sense.
ORTHOGONALIZATION: A system of functions \( \{f_n(x)\} \) is called linearly independent in \([a,b]\), if the validity of the relation of the form

\[
\sum_{k=0}^{n} a_k f_k(x) = 0
\]

for \( \mu \)-almost every \( x \in [a,b] \) necessarily implies the relation

\[
a_0 = a_1 = \ldots = a_n = 0 , \text{ for all } n \in \mathbb{N} .
\]

Every orthogonal system \( \{\phi_n(x)\} \) is linearly independent. 1)

Conversely any linearly independent system of functions \( \{f_n(x)\} \) can be converted into an ONS \( \{\phi_n(x)\} \) such that for each \( n, \phi_n(x) \) are linear combinations of the functions \( f_0(x), f_1(x), \ldots, f_n(x) \). The process of constructing an ONS from a linearly independent functions is known as Gram-Schmidt 2) process of orthogonalization.

ORTHOGONAL SERIES AND ORTHOGONAL EXPANSION: Let \( \{\psi_n(x)\} \) be an ONS. A series of the form

\[
(1.2.4) \quad \sum_{n=0}^{\infty} c_n \psi_n(x)
\]

where \( c_0, c_1, c_2, \ldots, c_n, \ldots \) are arbitrary real numbers, is called orthogonal series. However, if the coefficients \( c_n \) in the series (1.2.4) are representable in the form

1) Alexits ([4], p.4) 2) Schmidt [73]
\[ C_n = \frac{\int_a^b f(x) \psi_n(x) \, d\mu(x)}{\int_a^b \psi^2_n(x) \, d\mu(x)} , \quad n = 0, 1, 2, \ldots \]

according to Fourier's manner, then we say that the series (1.2.4) is the orthogonal expansion of the function \( f(x) \) and we shall express the relation by

\[ f(x) \sim \sum_{n=0}^{\infty} C_n \psi_n(x) . \]

In this case we shall call the numbers \( C_0, C_1, \ldots \) the expansion coefficients of the function \( f(x) \).

The orthogonal expansion and orthogonal series differ from each other due to following minimum property established by Gram\(^1\).

Let \( f(x) \in L_\mu^2[a, b] \) and \( \{\phi_n(x)\} \) be an arbitrary ONS. Among all the expansions of the form

\[ S_n(x) = \sum_{k=0}^{n} a_k \phi_k(x) \]

the integral

\[ I(S_n) = \int_a^b [f(x) - S_n(x)]^2 \, d\mu(x) \]

attains the least value for \( S_n(x) = s_n(x) \), where

\[ s_n(x) = \sum_{k=0}^{n} C_k \phi_k(x) , \quad C_k = \int_a^b f(t) \phi_k(t) \, d\mu(t) . \]

\(^1\) Gram [23]
The above result of Gram gives rise to the important property of expansion coefficients known as Bessel's inequality. \[ \sum_{n=0}^{\infty} c_n^2 \leq \int_{a}^{b} f(x)^2 \, d\mu(x). \]

Bessel's inequality implies that the expansion coefficients \( c_n \) of an \( L^2_{\mu} \) - integrable function converge to zero as \( n \to \infty \).

The most fundamental theorem in the theory of orthogonal series is the Riesz-Fischer theorem proved nearly simultaneously and independently by Riesz and Fischer. The theorem reads as follows:

"A necessary and sufficient condition that \( \{a_n\} \) be the sequence of expansion coefficients of a function \( f(x) \in L^2_{\mu}(a,b) \) is
\[ \sum_{n=0}^{\infty} a_n^2 < \infty. \]

The partial sums \( S_n(x) \) defined by (1.2.5) of the expansion of \( f(x) \) then converge in the mean to the generating function \( f(x) \)."

The above theorem was later on generalized by Fomin as follows:

1) Tricomi [89] 3) Fischer [15]
2) Riesz [70] 4) Fomin [17]
"Let \{\phi_k(\infty)\} be an ONS on the interval \([a,b]\), \(\phi_k \in L^q[a,b]\), \(k=0,1,2,..., 1 < q < \infty\) and let \(p\) be given by the relation
\[
\frac{1}{p} + \frac{1}{q} = 1, \text{ if } q < \infty \text{ and } p = 1 \text{ if } q = \infty.
\]
If there exists an increasing sequence \(v_k \to \infty\) such that
\[
\sum_{k=0}^{\infty} \left( \frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \int_a^b \sum_{m=0}^{k} a_m \phi_m(\infty) \phi_k(\infty) \right) dx < \infty
\]
with \(a_k\) real, then there exists a function \(f \in L^p[a,b]\) such that
\[
a_k = \int_a^b f(x) \phi_k(\infty) dx \quad \text{for } k=0,1,2,... .
\]

**ORTHOGONAL POLYNOMIALS:** The system of functions \(\{x^n\}\), for integral \(n\), is a linearly independent system. On orthogonalization of this system by the Schmidt's process gives an orthogonal system \(\{p_n(\infty)\}\), where \(p_n(\infty)\) is a polynomial of degree exactly equal to \(n\). The sign of this polynomial may be determined in such a way that the sign of the coefficient of highest power of \(x\) be taken as positive. The system of these polynomials \(\{p_n(\infty)\}\) is known as the system of orthogonal polynomials.

1) Freud [21]
Supposing that the orthogonalization of the system \( \{x^n\} \) has been done under the different distributions \( d\mu(x) = \rho(x)dx \), we could assign special forms to \( \rho(x) \) in order to obtain some well known systems of orthogonal polynomials, like \(^1\) Jacobi polynomials, Leguerre polynomials and Hermite polynomials.

\( \lambda(n) \)-LACUNARY ORTHOGONAL SERIES: - Let \( \lambda(x) \) denote a positive function concave from below, defined for \( x \geq 1 \), such that \( \lambda(x) \leq x \), and is increasing monotonely to infinity. We shall call the orthogonal series \((1.2.4)\lambda(n)\)-lacunary, if the number of its non-vanishing coefficients \( c_k \) with \( n < k \leq 2n \) does not exceed \( \lambda(n) \). Furthermore, we shall say that the coefficients have the positive number sequence \( \{q_n\} \) as a majorant, if the relation

\[
\frac{c_n}{q_n} = O(q_n)
\]

holds.

1.3 We would like to diverge a little at this point from our main theme so as to define various summability methods which are to be used in the body work of the thesis. In each of the following definitions we take

\[
\sum_{n=0}^{\infty} u_n
\]

as an infinite series and \( S_n \) to be its \( n^{th} \)-partial sum.
CESÁRO SUMMABILITY:

Let $A_n^\alpha = \left\{ \binom{n+\alpha}{n} \right\}$, where $\alpha > -1$, be given by

$$
\sum_{n=0}^{\infty} A_n^\alpha x^n = \frac{1}{(1-x)^{1+\alpha}} \quad (\alpha \neq -1,-2,\ldots)
$$

We write

$$
S_0^0 = S_0^n = S_n = u_0 + u_1 + \ldots + u_n
$$

Then the quotient

$$
\sigma_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha}
$$

is called the $n^{th}$ Cesàro mean of order $\alpha$ of the sequence $\{S_n\}$ or simply $(C,\alpha)$-mean. The series (1.3.1) is said to be $(C,\alpha)$-summable to $S$, if $\sigma_n^\alpha \rightarrow S$ as $n \rightarrow \infty$.

The series (1.3.1) with partial sum $S_n$ is said to be strongly $(C,\alpha)$-summable with index $k$ to the sum $S$, if

$$
\frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} |S_v - S|^k \rightarrow 0 \text{ as } n \rightarrow \infty.
$$

For $\alpha=1$, this gives the definition of strong summability $(H,K)^2$.

1) Cesàro [13], Chapman [14], Knopp ([34], [35])
2) Zygmund ([96], p.180), Bary ([7], p.2), Moricz [54]
The series (1.3.1) is said to be absolutely summable \( (C, \alpha) \) with index \( k \) or simply summable \( |C, \alpha|_k \), \( k \geq 1 \), if
\[
\sum_{n=1}^{\infty} (n^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty
\]

**RIESZ SUMMABILITY:**

Let \( \{\lambda_n\} \) be a positive, strictly increasing sequence of real numbers with \( \lambda_0 = 0 \) and \( \lambda_n \to \infty \) as \( n \to \infty \). The series (1.3.1) is said to be summable by Riesz means \(^2\) of order unity or \((R, \lambda_n, 1)\) summable to the sum \( S \) if
\[
\sigma_{n+1}(\lambda) = \sigma_n(\lambda) = \sum_{k=0}^{n} \left( 1 - \frac{\lambda_k}{\lambda_{n+1}} \right) u_k
\]
\[
= \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) S_k \to S \text{ as } n \to \infty.
\]

Throughout in the thesis we consider \((R, \lambda_n, 1)\) summability as a Riesz summability.

Obviously the \((R, \lambda_n, 1)\) summation introduced by M. Riesz is a generalization of the \((C, 1)\) summation process which is obtained by putting \( \lambda_n = n \). For \( \lambda_n = \log(n+1) \), the Riesz summability is known as Riesz logarithmic summability.

Further we say that series (1.3.1) is very strongly summable \((R, \lambda_n, 1)\) to the sum \( S \), if

1) Flett [16]
2) Hardy ([23], p. 86)
for every strictly increasing sequence of indices \( \{ r_n \} \).

In particular if \( r_k = k \) (\( k = 0,1,2,\ldots \)) we shall say that series (1.3.1) is strongly summable \((R, \lambda, 1)\) to the sum \( S \).

**Euler summability:**

For series (1.3.1) with partial sum \( S_n \), a sequence to sequence transformation given by the equation

\[
E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k,
\]

\((n=0,1,2,\ldots)\)

defines the sequence \( \{ E_n^q \} \) of \((E,q)\)-means \((q>0)\), known as Euler mean of order \( q \).

If \( E_n^q \to S \) as \( n \to \infty \) then the series (1.3.1) is said to be \((E,q)\) summable to \( S \). In particular for \( q=1 \)

\[
E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_k
\]
defines the \((E,1)\)-means of the series (1.3.1).

**Logarithmic summability:**

The series (1.3.1) is said to be summable by first logarithmic means or \((R,1)\) summable\(^2\), if

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1) Hardy ([25], p.236), Knopp [36].
2) Meder [45]
The expression $L_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{S_k}{k}$ will be called $n^{\text{th}}$-logarithmic means of sequence $\{S_n\}$.

**Nörlund Summability:**

Let (1.3.1) be a given infinite series with partial sums $S_n$. Let $\{p_n\}$ be a sequence of constants such that

\[(1.3.2) \quad p_0 > 0; \quad p_n \geq 0, \quad (n=1,2,3,\ldots)\]

and let us write $P_n = p_0 + p_1 + \ldots + p_n$. The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k} S_k$$

defines the sequence $\{t_n\}$ of Nörlund means or $(N,p_n)$-means of the sequence $\{S_n\}$. If

$$\lim_{n \to \infty} t_n = S,$$

then we say that series (1.3.1) is said to be $(N,p_n)$ summable to the sum $S$.

**$(N,p_n)$-Summability:**

Let $\{p_n\}$ be a sequence of constants which satisfies condition (1.3.2) and

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1) Woroni [93]
If \( S_k \) denotes the \( k^{th} \)-partial sum of series (1.3.1) then this series is said to be \((\overline{N},p_n)\) summable to the value \( S \), if

\[
T_n = \frac{1}{p_n} \sum_{k=0}^{n} p_k S_k \rightarrow S \text{ as } n \rightarrow \infty.
\]

The expression \( T_n \) will be called the \((\overline{N},p_n)\)-means of the series (1.3.1). \(^1\)

When \( p_n = \frac{1}{\log n} \), \( p_n \sim \log n \) and in this case \((\overline{N},p_n)\) summability will be identical to the \((R,1)\) summability.

The series (1.3.1) is said to be summable \([\overline{N},p_n]_k\), \( k \geq 1 \), i.e. absolutely summable \((\overline{N},p_n)\) with index \( k \) if

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_{n+1}} \right)^{k-1} |T_n - T_{n-1}|^k < \infty.
\]

It should be noted that, if \( p_n = 1 \), then \([\overline{N},p_n]_k\) summability is identical with \([C,1]_k\) summability.

**MATRIX SUMMABILITY:**

Let \( \|T\| = (\lambda_{n,k}) \) be an infinite matrix which satisfies the Silverman-Toeplitz Conditions of regularity \(^3\).

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\(^1\) Hardy ([25], p. 57) \quad \(^3\) Hardy ([25], p. 43)
\(^2\) Bör ([11])
i.e. (1) For every $n$, $\sum_{k=0}^{\infty} |\lambda_{n,k}| \leq C$ where $C$ is a constant.

(2) For every $k$, $\lim_{n \to \infty} \lambda_{n,k} = 0$.

(3) $\lim_{n \to \infty} \sum_{k=0}^{\infty} \lambda_{n,k} = 1$.

The $n^{th}$ matrix mean or \( ||T|| = (\lambda_{n,k}) \) mean of the sequence of partial sums \( \{S_n\} \) of the series (1.3.1) is defined as

$$
\tau_n = \sum_{k=0}^{n} \lambda_{n,k} S_k, \quad (n = 0,1,2,...).
$$

If $\lim_{n \to \infty} \tau_n = S$, the series (1.3.1) is said to be summable \( (\lambda_{n,k}) \) or simply \( ||T|| \)-summable to $S$. 1)

Also, a series (1.3.1) is said to be absolutely summable by $T$-methods or simply summable \( ||T|| \), if the corresponding auxiliary sequence \( \{\tau_n\} \) is of bounded variation i.e to say,

$$
\sum_{n=1}^{\infty} |\tau_n - \tau_{n-1}| < \infty.
$$

The series (1.3.1) is called strongly $T$-summable with order $\gamma > 0$ or \( ||T||^\gamma \) summable to the sum $S$ if the strong means

$$
\tau_n^{(\gamma)} = \sum_{k=0}^{n} |\lambda_{n,k}| |S_k - S|^{\gamma} = o(1) \text{ as } n \to \infty. \quad 2)
$$

1) Bolgov [10]
2) Moricz [54]
Definition: A sequence \( \{n_k\} \) is said to satisfy condition (L) if

\[
\sum_{k=0}^{\infty} \frac{1}{n_k} < \infty \quad \text{and} \quad \sum_{k=m}^{\infty} \frac{1}{n_k} = O\left(\frac{1}{n_m}\right)
\]

The sequence \( \{n_k\} \) is, however, named lacunary if

\[
\frac{n_{k+1}}{n_k} \geq q > 1
\]

It can easily be verified that a lacunary sequence satisfies condition (L) but the converse is not true, that is the condition (L) is lighter than lacunary condition\(^1\).

1.4 In this section we mention some of the important and general results of real analysis which are to be used quite frequently during the course of proofs of our theorems.

B. LEVY'S THEOREM\(^2\): If \( \{f_n(x)\} \) is a monotone increasing sequence of \( L_\mu \)-integrable functions and furthermore

\[
\left| \int_a^b f_n(x) \, d\mu(x) \right| \leq C, \quad (n=0,1,2,\ldots)
\]

then the limiting function

\[
f(x) = \lim_{n \to \infty} f_n(x)
\]

---

\(^1\) Bary [6] (Introductory material p. 8)
\(^2\) Alexits [14], p. 11
is also $L_\mu$-integrable and the relation
\[
\lim_{n \to \infty} \int_a^b f_n(x) \, d\mu(x) = \int_a^b f(x) \, d\mu(x)
\]
holds.

In particular, if $u_0(x), u_1(x), \ldots, \ldots$ are $L_\mu$-integrable functions such that
\[
\sum_{n=0}^{\infty} \int_a^b |u_n(x)| \, d\mu(x) < \infty,
\]
then the series
\[
\sum_{n=0}^{\infty} u_n(x)
\]
is (absolutely) convergent a.e. in $[a,b]$.

**Kronecker's Lemma** \(^1\): If $\{\lambda_n\}$ is a positive, monotone increasing sequence tending to infinity, then the convergence of the series
\[
\sum_{n=0}^{\infty} \frac{u_n}{\lambda_n}
\]
implies the estimate
\[
\sum_{k=0}^{n} u_n = o(\lambda_n).
\]

**Christoffel-Darboux Formula** \(^2\): Let $c_n$ is the leading coefficient (coefficient of highest power of $x$) in the

---

1) Alexits ([4], p.72).
2) Szego ([78], p.43).
polynomial $p_k(t)$ of degree $k$ then

$$
\sum_{k=0}^{n} p_k(x) p_k(t) = \frac{\alpha_n}{\alpha_{n+1}} \frac{p_{n+1}(t)p_n(x) - p_{n+1}(x)p_n(t)}{(t-x)}.
$$

The formula was proved for Legendre polynomial by Christoffel in 1858 and in the general case by Darboux in 1878.

**ABLE'S TRANSFORMATION** 1): If $u_0, u_1, \ldots, v_0, v_1, \ldots, v_n, \ldots$ are any real numbers, let us assume that

$$
V_n = v_0 + v_1 + \ldots + v_n.
$$

Then for any values of $m$ and $n$ we find that

$$
\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} (u_k - u_{k+1})v_k + u_n v_n - u_m v_{m-1}
$$

(under the condition that if $m=0$, $V_{-1} = 0$).

**1.5 CONVERGENCE OF ORTHOGONAL SERIES:**

Returning to the main theme of the work of the present thesis we would first of all discuss the convergence of the orthogonal series

$$(1.5.1) \quad \sum_{n=0}^{\infty} C_n \phi(x),$$

where $\{C_n\}$ is an arbitrary sequence of real numbers. It can easily be verified by the application of Schwarz inequality

1) Bary ([6], introductory material p.1)
that

\[ (1.5.2) \quad \sum_{n=0}^{\infty} |c_n| < \infty \]

implies the absolute convergence\(^1\) of the series (1.5.1) a.e. in the interval of orthogonality. At the same time the consideration of some particular orthogonal series e.g. Rademacher series\(^2\) shows that the condition

\[ (1.5.3) \quad \sum_{n=0}^{\infty} c_n^2 < \infty \]

is necessary for the convergence of the series (1.5.1) a.e. in the interval of orthogonality. From this we conclude that a useful condition for convergence of series (1.5.1) lies somewhere between the conditions (1.5.2) and (1.5.3).

The first result of this nature was obtained by Jerosch and Weyl\(^3\) who established that the condition

\[ c_n = 0 \left( n - \frac{3}{4} - \varepsilon \right), \quad \varepsilon > 0 \]

is sufficient for the convergence of the series (1.5.1) a.e. in the interval of orthogonality. Weyl\(^4\) himself improved this condition by showing that, the condition

\[ \sum_{n=0}^{\infty} c_n^2 \sqrt{n} < \infty \]

is sufficient for the convergence of the series (1.5.1). Later on Hobson\(^5\) and Plancherel\(^6\) had modified the above condition

\[ \begin{align*}
1) & \text{ Alexits [44], p. 533} & 4) & \text{ Weyl [92]} \\
2) & \text{ Alexits [44], p. 522} & 5) & \text{ Hobson [26]} \\
3) & \text{ Jerosch and Weyl [29]} & 6) & \text{ Plancherel [67]} 
\end{align*} \]
to the form

$$\sum_{n=0}^{\infty} c_n^2 n^\varepsilon < \infty, \ v>0$$

and

$$\sum_{n=2}^{\infty} c_n^2 (\log n)^3 < \infty$$

as being sufficient for the convergence a.e. of the series (1.5.1) respectively. The chain of ideas continued and ultimately Rademacher\textsuperscript{1)} and Menchoff\textsuperscript{2)} nearly simultaneously but independently of each other succeeded in obtaining a condition known to be the best of its kind. They have shown that the series (1.5.1) is convergent a.e. in the interval of orthogonality if the condition

$$\sum_{n=2}^{\infty} c_n^2 \log^2 n < \infty$$

Several generalization of this theorem given by Walfisz\textsuperscript{3)}, Salem\textsuperscript{4)}, Talalyan\textsuperscript{5)}, Tandori\textsuperscript{6)} and Gaposkin\textsuperscript{7)}. The theorem of Rademacher and Menchoff is the best of its kind is obvious from the following fundamental theorem of convergence theory given by Menchoff\textsuperscript{2)}

If $\omega(n)$ is an arbitrary positive monotone increasing sequence of numbers with $\omega(n)=o(\log n)$, then there exists an everywhere divergent orthogonal series

\begin{tabular}{llll}
\end{tabular}
whose coefficients satisfy the condition
\[ \sum_{n=1}^{\infty} c_{\omega(n)}^2 < \infty. \]

Another result that has been mentioned in this chain is due to Tandori\(^{1}\), who proved that if \( \{c_n\} \) is a positive monotone decreasing sequence of numbers for which
\[ \sum_{n=2}^{\infty} c_n^2 \log^2 n = \infty, \]
holds, then there exists in \([a, b]\), an ONS \( \{\psi_n(x)\} \) dependent on \( \{c_n\} \) such that the orthogonal series (1.5.4) is divergent everywhere in \([a, b]\).

Absolute convergence of orthogonal series (1.5.1) has been discussed by Stetchkin\(^2\), Alexits\(^3\), and Freud\(^4\).

For many special orthogonal series the condition for convergence has still better form. Kolmogoroff-Silverstoff\(^5\) and Plessener\(^6\) independently of each other proved that
\[ \sum_{n=2}^{\infty} (a_n^2 + b_n^2) \log n < \infty, \]
is sufficient for the convergence of the Fourier series
\[ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \]

1) Tandori [63] 4) Freud [20]
2) Stetchkin ([74], [75]) 5) Kolmogoroff-Silverstoff [40]
1.6 **SUMMABILITY OF ORTHOGONAL SERIES:**

It was first time shown by Kaczmarz, for Cesàro summability of order unity, that under the condition

\[ \sum_{n=0}^{\infty} c_n^2 < \infty \]

the necessary and sufficient condition for the series (1.5.1) to be summable \( (C,1) \) a.e. is that there exists a sequence of partial sums \( \{S_{\nu}(x)\} \), \( 1 < q \leq \frac{\nu_{n+1}}{\nu_n} < r \) convergent a.e. in the interval of orthogonality.

Later on Menchoff, Kaczmarz and Borgen had established the analogue of Rademacher-Menchoff theorem, for \( (C,\omega) \) summability, which states that

\[ \sum_{n=3}^{\infty} c_n^2 (\log n \log n)^2 < \infty \]

implies \( (C,\omega_0) \) summability of the series (1.5.1).

Zygmund and Lorentz has proved the similar type of results for \( (R,\lambda_n,1) \) summability of series (1.5.1) under the condition

\[ \sum_{n=1}^{\infty} c_n^2 (\log n \log \lambda_n)^2 < \infty . \]

---

1) Kaczmarz [30]  
2) Menchoff [51, 52]  
3) Kaczmarz [31]  
4) Borgen [12]  
5) Zygmund [95]  
6) Lorentz [43]
1.7 By considering the sequence \( \{ S_v(x) - f(x) \}^2 \), Zygmund \(^1\) has proved the following theorem.

**Theorem A:** If the orthogonal series (1.5.1) with the condition (1.5.3) is \((C,1)\) summable to the function \( f(x) \) a.e. in the interval \([a,b]\) then

\[
\lim_{n \to \infty} \frac{1}{n} \left( \sum_{n} |c_n| \right)^2 - f(x)^2, x = 0 \text{ is valid a.e. in } [a,b].
\]

With the same assertion Tandori \(^2\) has generalized the above result for \((C,\alpha)\)-mean \(0<\alpha<1\) of the same sequence.

In chapter-II we extend the result of Tandori for the sequence \( \{ S_v(x) - f(x) \}^p \), \( p \geq 2 \) as follows:

If the orthogonal series (1.5.1) with the condition

\[
\sum_{n=1}^{\infty} |c_n|^p (n)^{p-2} < \infty, \quad (p \geq 2)
\]

is \((C,1)\) summable to the function \( f(x) \) a.e. in the interval \([a,b]\), where \( |\phi_n(x)| \leq M\), then for \( 0<\alpha<1 \) and \( p \geq 2 \)

\[
\phi_n^{\alpha} \left( \sum_{v \geq 0} |S_v(x) - f(x)|^p; x \right) \to 0 \text{ as } n \to \infty,
\]

\((v=0,1,\ldots)\) is valid a.e. in the interval \([a,b]\).

---

1) Zygmund [94]
2) Tandori [82]
The \( n^{th} \) Cesàro mean of order \( c > 0 \), i.e. \( \sigma_n^{c}(\times) \) of the orthogonal series (1.5.1) had been estimated by Tandori\(^1\).

Dealing with the estimation of \( \sigma_n^{c}(\times) \) of (1.5.1) Tandori has proved several theorems out of which one result read:

**Theorem B:** Under the condition (1.5.3), we have
\[ \sigma_n^{c}(\times) = o(\log \log n) \]
a.e. in the interval \([a, b]\) for any \( \alpha > 0 \).

These results have been extended for the first logarithmic mean i.e. \((R, 1)\)-mean and \((N, p_n)\)-mean of the orthogonal series (1.5.1) by Meder\(^2\) and Meder\(^3\) respectively. We prove the analogous results for \((R, \lambda_n, 1)\)-mean of the series (1.5.1) in the chapter-II out of which one result is as follows:

If the orthogonal series (1.5.1) satisfies the condition (1.5.3) then
\[ \sigma_n^{y}(\lambda, x) = o(\log \log \lambda_n) \] as \( n \to \infty \) a.e. in the interval \([a, b]\).

---

1) Tandori [84]  
2) Meder [45]  
3) Meder [47]
Strong (C,1)-summability of orthogonal series (1.5.1) with exponents \( k \leq 2 \) were first investigated by Zygmund\(^1\) and Borgen\(^2\). After that concerning the strong and very strong (C, \( a \geq 0 \)) summability of (1.5.1) Sunouchi\(^3\) proved two classical results.

In Chapter-III, we have generalized the theorem of Meder\(^4\) on the lines of Sunouchi. Here we propose to establish the following theorems:

(i) If the orthogonal series (1.5.1) with the condition

\((1.7.1)\) is \((R, \lambda_n, 1)\) summable to \( f(x) \) a.e. in \([a, b]\) and \( |\phi_n(x)| \leq M \) for \( a < x < b \), then

\[
\lim_{n \to \infty} \frac{1}{\lambda_{n+1}} \sum_{v=0}^{n} (\lambda_{v+1} - \lambda_{v}) |S_v(x) - f(x)|^k = 0
\]

a.e. in \([a, b]\) for any \( k > 0 \).

(ii) If the orthogonal series (1.5.1) with (1.7.1) satisfies the condition

\[
\sum_{n=1}^{\infty} c_n^2 \left( \log \log \lambda_n \right)^2 < \infty,
\]

and \( |\phi_n(x)| \leq M \) for \( a < x < b \), then there exists square integrable function \( f(x) \) such that

\[
\lim_{n \to \infty} \frac{1}{\lambda_{n+1}} \sum_{v=0}^{n} (\lambda_{v+1} - \lambda_{v}) |S_v(x) - f(x)|^k = 0, \quad (k > 0)
\]

---

1) Zygmund [95]  
2) Borgen [12]  
3) Sunouchi [77]  
4) Meder [46]
a.e. in [a, b] for every increasing sequence \( \{n_j\} \).

We have also discussed strong matrix summability of series (1.5.1) in chapter-IV.

Sunouchi\(^1\) has discussed the convergence of the series

\[
\sum_{n=1}^{\infty} \frac{|S_n(x) - \tau_n(x)|^k}{n}, \quad k \geq 1
\]

under the assumption of uniform boundedness of the functions \( \phi_n(x) \).

In chapter-IV we have established the convergence of the series

\[
\sum_{n=1}^{\infty} \frac{|S_n(x) - \tau_n(x)|^q}{n^{\frac{q}{n}}} \quad (q \geq 1)
\]

by proving the following result on Matrix summability of series (1.5.1)

Let \( \left\{\frac{\lambda_{n,k}}{n}\right\}_{k=0}^{n} \) is non-negative non-increasing sequence such that \( \lambda_{n,0} = O(\frac{1}{n}) \). If for \( q \geq 2 \) the condition (1.7.1) is satisfied then

\[
\sum_{m=0}^{n} |S_m(x) - \tau_m(x)|^q = o(n) \text{ as } n \to \infty \text{ a.e. in } (a, b)
\]

where \( q \geq 2 \) and \( |\phi_n(x)| \leq M \) everywhere in [a, b].

\(^1\) Sunouchi [76]
1.8 ABSOLUTE SUMMABILITIES OF ORTHOGONAL SERIES:

Absolute summability of Fourier-trigonometric series by Cesàro, Nörlund and Riesz means have been engaging the attention of a large number of workers in this line.

In the case of Fourier-orthogonal expansion the earliest result on \(|C,\alpha|\) summability are due to Tsuchikura\(^1\) and Tandori\(^2\). Tandori\(^3\)’s theorem on \(|C,1|\) states that:

**Theorem A:** The condition

\[
\sum_{m=1}^{\infty} A_m < \infty
\]

where

\[
A_m = \left( \sum_{k=0}^{m} \frac{C_{2^{m+1}}^2 + C_{2^{m+2}}^2 + \ldots + C_{2^{m+1}}^2}{2^{m+2}} \right)^{\frac{1}{2}}
\]

(m=0,1,2,3,\ldots)

is necessary and sufficient for the \(|C,1|\) summability a.e. of (1.5.1).

The necessary part of the above theorem was later proved in a very easy way by Billard\(^4\). Leindler\(^5\), Grepacevskaja\(^6\) and Patel\(^7\) have extended Tandori’s theorem to \(|C,\alpha|\) summability. The same result has been extended for absolute Riesz summability, by Alexits & Kralik\(^8\) and Moricz\(^9\), while for absolute Euler summability by Patel\(^10\) and Bhatnagar\(^11\), of series (1.5.1).

---

1) Tsuchikura [90]  
2) Tandori [85],[86]  
3) " [86]  
4) Billard [9]  
5) Leindler [41]  
6) Grepacevskaja [24]  
7) Patel [58]  
8) Alexits and Kralik [5]  
9) Moricz [54]  
10) Patel [87]  
11) Bhatnagar [8]
Same theorem we extend for absolute Matrix summability, of series (1.5.1), in chapter-IV. We prove,

If the condition (1.8.1) is satisfied, then

\[ \sum_{m=1}^{n} |\tau_m(x) - \tau_{m-1}(x)| = O(n^p), \quad (p > 0) \quad \text{as} \quad n \to \infty \]

a.e. in \((a, b)\) where \(\{\lambda_n, k\}_{k=0}^{n}\) and \(\{\lambda_{n-1}, k - \lambda_n, k+1\}_{k=0}^{n}\) are nondecreasing sequences with respect to \(k\).

In chapter-VI we have discussed \(|C, \alpha|_k\) and \(|N, \alpha|_k| (k \geq 2)\) summabilities of orthogonal series (1.5.1). One of two representative results proved by us in chapter-VI read:

The condition (1.7.1) is sufficient for orthogonal series (1.5.1) to be, \(|N, \alpha|_k| (k \geq 2)\) summable where \(\{p_n\}\) be a sequence of positive real constants such that as \(n \to \infty\)

\[ n^2 p_n = O(p_n) \]

as well as \(|C, \alpha|_k| (k \geq 2)\) summable where \(\alpha \geq 1\), a.e. for every ONS \(\{\phi_n(x)\}\) in the interval \([a, b]\) where \(|\phi_n(x)| \leq M\) for \(a < x < b\).

1.9 SUMMABILITY OF SERIES OF ORTHOGONAL POLYNOMIALS:

The series of orthogonal polynomials i.e. the series

\[ \sum_{n=0}^{\infty} c_n p_n(x) \]
have their own importance. Jackson was first who discussed, in great details, the convergence and Cesàro summability of (1.9.1). His theorem concerning \((C,1)\) summability states that:

**Theorem A:** If the weight function \(\omega(t)\) is bounded function and if \(\omega(t)\phi^2(t)\) is summable in the interval \((-1,1)\) then the series (1.9.1) is summable \((C,1)\) to a function \(f(x)\) in \((-1,1)\), where \(\phi(t)\) being given by

\[
\phi(t) = \frac{f(t) - f(x)}{t - x}, \quad t \in (-1,1).
\]

Subsequent results for various summability methods in this line are due to Alexits\(^2\), Tandori\(^3\), Freud\(^4\), Patel\(^5\), Bhatnagar\(^6\) and Patel\(^7\).

In chapter-V of our thesis we prove the following two results regarding \((R,1)\) and strong \((R,1)\) summability of the series (1.9.1).

(i) If the weight function \(\omega(t)\) is positive, bounded and and if \(\omega(t)\phi^2(t)\) is summable in \((a,b)\) then the series (1.9.1) is \((R,1)\) summable to \(f(x)\) a.e. in \((a,b)\).

---

1) Jackson ([27], [28])  5) Patel [59]
3) Tandori ([80], [81])  7) Patel [65]
4) Freud [18]
(iii) If

$$V_x(h) = \int_x^{x+h} (f(t) - f(x))^2 \omega(t) \, dt = o(|h|),$$

for every $a < x < b$ and for every $f(x) \in L^2_{w(x)}$ then series (1.9.1)

is strongly $(R,1)$ summable to the sum $f(x)$. 