5.1 Let $p_0(x)$, $p_1(x)$, $p_2(x)$, \ldots be the system of normalized orthogonal polynomials in the interval $(a,b)$ corresponding to a positive, bounded and summable weight function $\omega(x)$. We denote the expansion of a function $f(x) \in L^2(a,b)$ in terms of orthogonal polynomials $\{p_n(x)\}$, $(n = 0,1,2, \ldots)$ is given by

\begin{equation}
(5.1.1) \quad f(x) \sim \sum_{n=0}^{\infty} C_n \, p_n(x),
\end{equation}

where

\begin{equation}
(5.1.2) \quad C_n = \frac{b}{a} \int_{a}^{b} f(x) \, p_n(x) \, \omega(x) \, dx.
\end{equation}

The $n^{th}$ partial sums of (5.1.1) are denoted by $S_n(x)$ that is
\[
S_n(x) = \sum_{k=0}^{n} p_k(x) = \sum_{k=0}^{b} p_k(x) \int_{a}^{b} f(t) \omega(t) \, dt
\]

Thus

\[
S_n(x) = \int_{a}^{b} K_n(t, x) f(t) \omega(t) \, dt,
\]

where

\[
K_n(t, x) = \sum_{k=0}^{n} p_k(x) \omega(t).
\]

(5.1.3)

Thus

\[
S_n(x) = \int_{a}^{b} K_n(t, x) f(t) \omega(t) \, dt.
\]

(5.1.4)

Let us set

\[
\phi(t) = \frac{f(t) - f(x)}{t - x}, \quad t \in (a, b).
\]

The \(n^{th}\) \((R, 1)\)-means, that is first logarithmic means, \(L_n(x)\) of series (5.1.1) is defined as

\[
L_n(x) = \frac{1}{\log n} \sum_{i=1}^{n} \frac{S_i(x)}{i}
\]

where \(S_i(x)\) denotes the \(i^{th}\) partial sum of the series (5.1.1).
The series (5.1.1) is said to be summable by first logarithmic means or (R,1) summable to a sum s(x) if

\[ \lim_{n \to \infty} \frac{1}{\log n} \sum_{i=0}^{n} \frac{|S_i(x) - f(x)|}{i} = 0 \]

then we say that the series (5.1.1) is strongly (R,1) summable to the sum f(x).

The system of orthogonal polynomials as well as the convergence and summability of the expansion into this system have been studied in great details by Jackson\(^1\). Convergence and summability of series (5.1.1) by various summability methods have been discussed by many authors such as Alexits\(^2\), Tandori\(^3\), Freud\(^4\), Patel\(^5\) and Bhatnagar\(^6\).

The theorems of Jackson (Theorem-A) and Bhatnagar (Theorem-B) are as under:

**Theorem A:** If the weight function \( \omega(t) \) is bounded function and if \( \omega(t)\varphi^2(t) \) is summable in the interval (-1,1) then the series (5.1.1) is summable (C,1) to a function \( f(x) \) in (-1,1).

---

1) Jackson ([27], [28])
2) Alexits [2]
3) Tandori ([80], [81])
4) Freud [18]
5) Patel [S9]
6) Bhatnagar [8]
Theorem B: If $\omega(t)$ is a positive, bounded weight function in $(-1,1)$ and $\omega(t)\phi^2(t)$ is summable in $(-1,1)$ and if

$$|p_k(x)| \leq M,$$

$M$ being an absolute constant, then

$$\sigma_n(a,x) - f(x) = o(1)$$

a.e. in $(-1,1)$.

As we have discussed in earlier chapter, the strong summability of general orthogonal series has been studied by Sunouchi, Tandori, Alexits and Zygmund. Freud, at the same time, has investigated the strong $(C,1)$ summability of series (5.1.1) and then Bhatnagar has extended this result to the strong $(V_n, \lambda)$ summability, that is De La Vallee Poussin summability of (5.1.1). Freud's theorem states that:

Theorem C: If

$$\sum_{v=0}^{n} p^2_v(x) = O(n)$$

for every $x \in (a,b)$ and if for every $f(x) \in L^2_{\omega(x)}$

$$\phi_x(h) = \int \frac{[f(t) - f(x)]^2 \omega(t) \ dt}{x} = o(|h|)$$

then

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{v=0}^{n} |S_v(x) - f(x)| = 0.$$

1) Freud [19]
2) Bhatnagar [8]
In this chapter we intend to discuss the $(R,1)$ summability of orthogonal polynomials. Firstly, we extend Jackson's theorem to $(R,1)$ summability and then we prove a theorem on strong $(R,1)$ summability of $(S.1.1)$.

We propose to prove the following theorems:

**THEOREM 1:** If the weight function $\omega(t)$ is positive, bounded and if $\omega(t)\phi^2(t)$ is summable in $(a,b)$ then the series $(S.1.1)$ is $(R,1)$ summable to $f(x)$ a.e. in $(a,b)$.

**THEOREM 2:** If

$$
U_{x}(h) = \sum_{x}^{x+h} [f(t) - f(\omega)] \omega(t) dt = o(|h|),
$$

for every $a < x < b$ and for every $f(x) \in L_2(\omega,x)$ then series $(S.1.1)$ is strongly $(R,1)$ summable to the sum $f(x)$.

During the course of proof of theorem-2 we shall require the following lemma:

**Lemma 1:** If $\{\lambda_n\}$ denotes a monotone increasing sequence of numbers for which

$$
\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty
$$

1) Alexits ([4], p.38)
then the estimate
\[ \sum_{k=0}^{n} p_k^2(x) = o(\lambda_n) \]
is valid for every orthonormal system \( \{p_n(x)\} \) a.e.

Using this lemma we have

\[ \sum_{k=0}^{n} p_k^2(x) = o(\log^2 n) \]

since

\[ \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < \infty. \]

5.2 PROOF OF THEOREM 1:-

Let \( \alpha_k \) denote the coefficient of \( x^k \) in \( p_k(x) \) and we assume it to be positive. Since \( 1 \) is a polynomial of degree 0, we have

\[ \int_{a}^{b} p_k(t) \omega(t) dt = \int_{a}^{b} 1 p_k(t) \omega(t) dt = 0, \quad \text{for} \ k \geq 1, \]

and \( p_0(x) = p_0(t) = \alpha_0 \), so

\[ \int_{a}^{b} p_0(x) p_0(t) \omega(t) dt = \int_{a}^{b} \alpha_0^2(t) \omega(t) dt = 1, \]

by orthonormality property.
Therefore

\[ b \int_a^b K(t,x) \omega(t) dt \]

\[ = \int_a^b p_0(x) p_0(t) \omega(t) dt + \int_a^b p_1(x) p_1(t) \omega(t) dt + \ldots \]

\[ = \int_a^b p_n(x) p_n(t) \omega(t) dt \]

= 1.

Consequently,

\[(5.2.1) \quad 1 = \int_a^b K(t,x) \omega(t) dt.\]

Multiplying \((5.2.1)\) by \(f(x)\) we get

\[(5.2.2) \quad f(x) = \int_a^b f(t) K(t,x) \omega(t) dt.\]

From \((5.1.4)\) and \((5.2.2)\) we get

\[(5.2.3) \quad S_n(x) - f(x) = \int_a^b \left[ f(t) - f(x) \right] K(t,x) \omega(t) dt.\]

Applying Christoffel Darboux formula\(^1\) to \((5.1.3)\).

---

\(^1\) Alexits ([4], p. 26)
Substituting this in (5.2.3) we have

\[ S_n(x) - f(x) = \int_a^b \left[ \frac{f(t) - f(x)}{t - x} \right] \left[ \frac{\alpha_n}{\alpha_{n+1}} \right] \, \omega(t) \, dt. \]

Now,

\[ L_m(x) - f(x) = \frac{1}{\log m} \sum_{n=1}^{m} \frac{S_n(x) - f(x)}{n} \]

\[ = \frac{1}{\log m} \sum_{n=1}^{m} \frac{1}{n} \left[ S_n(x) - f(x) \right] \text{ as } m \to \infty. \]

\[ \left( \frac{1}{\log m} \left[ 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} \right] \to 1 \text{ as } m \to \infty \right) \]

Using this estimation together with (5.2.4) we get

\[ \log m \left[ L_m(x) - f(x) \right] \]

\[ = \sum_{n=1}^{m} \frac{1}{n} \int_a^b \frac{\phi(t)}{\alpha_n} \left[ P_{n+1}(t)p_n(x) - P_{n+1}(x)p_n(t) \right] \omega(t) \, dt. \]

Hence
\[(5.2.5) \quad \log m \left[ L_m(x) - f(x) \right] = \]
\[= \int_a^b H_m(n; t, x) \phi(t) \omega(t) \, dt - \int_a^b H_m(n; x, t) \phi(t) \omega(t) \, dt \]
\[= I_1 - I_2, \]

where
\[H_m(n; t, x) = \sum_{n=1}^m \frac{1}{n} \frac{\alpha_n}{\alpha_{n+1}} p_{n+1}(t) p_n(x).\]

Using Schwartz's inequality we can write
\[I_1^2 = \left( \int_a^b H_m(n; t, x) \phi(t) \omega(t) \, dt \right)^2 \]
\[\leq \left( \int_a^b \phi^2(t) \omega(t) \, dt \right) \left( \int_a^b H_m^2(n; t, x) \omega(t) \, dt \right) \]
\[= O(1) \sum_{n=1}^m \frac{\alpha_n^2}{\alpha_{n+1}^2} \frac{p_n^2(x)}{n^2}, \]

from the hypothesis that \(\omega(t)\phi^2(t)\) is summable.

But \(\frac{\alpha_n}{\alpha_{n+1}}\), for large \(n\), are bounded.\(^1\)

Therefore,

---

\(^1\) Alexits ([4], p.28)
Similarly

\[ I_2^2 = O(1) \sum_{n=1}^{m} \frac{p_n^{2(x)}}{n^2}. \]

In order to establish theorem-1 now it remains only to show that

\[ \sum_{n=1}^{m} \frac{p_n^{2(x)}}{n^2} = o(\log^2 m). \]

As \( \omega(t) \) is bounded, let \( \nu > 0 \) be its lower bound then

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 \log^2 n} \int_{a}^{b} p_n^{2(x)} dx
\]

\[
\leq \frac{1}{\nu} \sum_{n=1}^{\infty} \frac{1}{n^2 \log^2 n} \int_{a}^{b} p_n^{2(x)} \omega(x) dx
\]

\[
= \frac{1}{\nu} \sum_{n=1}^{\infty} \frac{1}{n^2 \log^2 n} < \infty.
\]

Hence the series

\[
\sum_{n=1}^{\infty} \frac{p_n^{2(x)}}{n^2 \log^2 n}
\]
converges by Levy's theorem. Since \( \{\log^2 n\} \) is a positive, strictly increasing sequence, (5.2.6) follows immediately by the Kronecker's lemma.

This implies from (5.2.4) that

\[
\lim_{n \to \infty} L_n(x) = f(x)
\]

or that (5.1.1) is \((R,1)\) summable. This proves theorem-1.

**PROOF OF THEOREM 2:**

As shown before in (5.2.4) we have for any \( n \) and \( k \leq n \)

\[
S_k(x) - f(x) = \int_a^b \left[ \frac{f(t) - f(x)}{t-x} \right] \left[ \frac{a}{a_{k+1}} \right]
\]

\[
\cdot \left[ p_{k+1}(t)p_k(x) - p_{k+1}(x)p_k(t) \right] \omega(t) dt.
\]

This can be written as

(5.2.7) \( S_k(x) - f(x) = A_k - \gamma_k p_{k+1}(t) B_k + \gamma_k p_k(x) B_{k+1} \)

where \( \frac{a_k}{a_{k+1}} = \gamma_k \),

\[
A_k = \int_x^{x + \frac{c}{\log^2 n}} K_k(t,x) \left[ f(t) - f(x) \right] \omega(t) dt
\]

\[
B_k = \int_x^{x - \frac{c}{\log^2 n}} K_k(t,x) \left[ f(t) - f(x) \right] \omega(t) dt
\]

and
\[ B_k = \int_a^b \frac{f(t) - f(x)}{t - x} p_k(t) \omega(t) dt + \int_{x + \frac{c}{\log^2 n}}^b \frac{f(t) - f(x)}{t - x} p_k(t) \omega(t) dt. \]

\( c \) being any number such that \( c > 0 \), and \( a < x - c < x + c < b \).

Hence for any \( k \leq n \) we have by Schwarz's inequality

\[ |A_k| \leq \left[ \frac{x + \frac{c}{\log^2 n}}{x - \frac{c}{\log^2 n}} \int_x^{b} [f(t) - f(x)]^2 \omega(t) dt \right]^{\frac{1}{2}}. \]

\[ \times \left[ \int_a^b [K_k(t, x)]^2 \omega(t) dt \right]^{\frac{1}{2}} \]

\[ = o \left( \frac{1}{\log^2 n} \right) \frac{1}{2} \left( \sum_{i=0}^{k} p_i^2(x) \right) \frac{1}{2} \left( \sum_{i=0}^{k} p_i^2(x) \right) \frac{1}{2} \text{ (using (5.1.5))} \]

\[ = o \left( \frac{\log^2 k}{\log^2 n} \right) \frac{1}{2} \frac{1}{2} \text{ (using (5.1.6))} \]

\[ = o(1) \quad \text{as} \quad k \leq n. \]

Hence

\[ (5.2.8) \quad \sum_{k=1}^{n} \frac{1}{k} |A_k| = o(\log n) \quad \text{as} \quad n \to \infty. \]
Next, let us set

\[ \phi(t) = \begin{cases} 
0 & \text{for } x - \frac{c}{n+1} < t < x + \frac{c}{n+1} \\
\frac{f(t) - f(x)}{t - x} & \text{for } t \in \left\{ a, x - \frac{c}{n+1}, x + \frac{c}{n+1}, b \right\} 
\end{cases} \]

then we observe from (5.2.9) that are nothing but expansion coefficients of the function \( \phi(t) \) in terms of the polynomial system \( \{p_n(x)\} \), that is

\[ B_k = \int_a^b \phi(t) p_k(t) \omega(t) \, dt \text{ as } \phi(t) \in L^2 \omega(t). \]

Then by Bessel's inequality, we obtain

\[ \sum_{k=0}^{\infty} B_k^2 \leq \int_a^b \phi^2(t) \omega(t) \, dt \leq C', \]

where \( C' \) is an absolute constant. Hence by Schwartz's inequality we have

\[ \sum_{k=1}^{n} \frac{1}{k} |p_k(x)| |B_{k+1}| \leq \left( \sum_{k=1}^{n} p_k^2(x) \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n+1} \frac{B_k^2}{k^2} \right)^{\frac{1}{2}} \]

\[ \leq \left( \sum_{k=1}^{n} p_k^2(x) \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} \frac{B_k^2}{k^2} \right)^{\frac{1}{2}} \]

\[ = o(\log^2 n) \frac{1}{C'} \quad \text{from (5.1.6))} \]
Thus

\[(5.2.9) \quad \sum_{k=1}^{n} \frac{1}{k} \left| \mathbf{P}_k \mathbf{c} \right| \left| \mathbf{B}_{k+1} \right| = o(\log n) .\]

We can similarly show that

\[(5.2.10) \quad \sum_{k=1}^{n} \frac{1}{k} \left| \mathbf{P}_{k+1} \mathbf{c} \right| \left| \mathbf{B}_k \right| = o(\log n) .\]

From (5.2.7), (5.2.8), (5.2.9) and (5.2.10) it follows that

\[\sum_{k=1}^{n} \frac{1}{k} \left| \mathbf{S}_k \mathbf{c} - f(x) \right| = o(\log n)\]

which completes the proof of theorem-2.