CHAPTER IV

THE TOPOLOGICAL DUAL AND REFLEXIVITY

§1. INTRODUCTION.

In Chapter II and III we dealt with a class of properties of Hilbert-spaces which are extendible to complete s.d.i.p. spaces. However, on the other hand, there are a number of properties of Hilbert (Banach) spaces which either fail to get extended, or sometimes carry no meaning at all in the context of s.d.i.p. (semi-normed) spaces. It is in fact this class of properties which makes the independent study of such spaces meaningful. The nature of topological dual and bidual of s.d.i.p. spaces is studied in this chapter, which shall be seen to differ significantly from those of Hilbert spaces.

It has been shown here that the topological dual $E^*$ of a semi-normed space $(E, p)$ is a Banach space and that it is isometrically isomorphic to the topological dual of the quotient space $E_p = E/Ker p$. This appears to be a very significant result, and one of its obvious consequences is that $E$ cannot be reflexive. In case of s.d.i.p. spaces, contrary to Hilbert spaces, it has been shown that not only they are not, in general, reflexive but in fact, where the
semi-norm is generated by a linear functional, even the relation $E \subseteq E^{**}$ does not hold. As a matter of fact, the notion of reflexivity does not seem to make sense in this setting.

§2. TOPOLOGICAL DUAL.

Let $E$ be a linear space and $g$ a s.d.i.p. defined on $E$. Let $p$ be the semi-norm on $E$ induced by $g$. The topological dual (i.e. the set of all continuous linear functionals $f : E \to \mathbb{F}$) is denoted by $E^*$. 

**Proposition (2.1).** $E^*$ is a Banach space with norm $p'(f) = \sup \{|f(x)| : p(x) \leq 1\}$.

**Proof.** It is easy to show that $p'$ is a norm on $E^*$. To establish the completeness of the normed space $(E^*, p')$ take a Cauchy sequence $(f_n)$ in $E^*$. Then for each $x \in E$, $(f_n(x))$ is a Cauchy sequence of scalars; for

$$|f_m(x) - f_n(x)| = |(f_m - f_n)(x)| \leq p(f_m - f_n)p(x). \quad (IV.2.1)$$

Define $f$ on $E$ by taking $f(x) = \lim_{n \to \infty} f_n(x)$, $\forall x \in E$.

Then $f(x+y) = \lim_{n \to \infty} f_n(x+y) = \lim_{n \to \infty} \left\{f_n(x) + f_n(y)\right\} = f(x) + f(y)$, and $\lim_{n \to \infty} \left\{\lambda f_n(x)\right\} = \lambda \lim_{n \to \infty} f_n(x) = \lambda f(x)$. That is to say...
that \( f \) is linear. Also \( f \) is continuous; for \((f_n)\),
being a Cauchy sequence, is norm bounded, viz. \( \exists \) a number
\( M > 0 \) \( \exists p'(f_n) < M, \forall n \). Then \( |f_n(x)| \leq p'(f_n) \) \( p(x) < M p(x) \) for all \( n \) and all \( x \Rightarrow |f(x)| \leq M p(x) \) for all \( x \).

Finally \( f_n \rightarrow f \). For, if \( p(x) \leq 1 \), then \( \exists \) a
positive integer \( N \geq r, s \geq N \) imply that \( |f_r(x) - f_s(x)| < \varepsilon \). Letting \( r \rightarrow \infty \) we get \( |f(x) - f_s(x)| < \varepsilon \), for
all \( s \geq N \), and therefore \( p'(f - f_s) < \varepsilon \), for all \( s \geq N \).

We know that the quotient space \( E_p \) is a normed
space with norm \( \hat{p} \) given by \( \hat{p} (\hat{x}) = p(x) \), \( \hat{x} \in E_p \), and
\( x \) is any member of \( \hat{x} \). Let \( E_p^* \) be the topological dual
of \( E_p \), then the relation between \( E^* \) and \( E_p^* \) is given
by:

**PROPOSITION (2.2).** The topological dual of a s.d.i.p.
space \((E, g)\) is isometrically isomorphic to the topological
dual of the quotient space \( E_p \), i.e. \( E^* \cong E_p^* \), \( \exists p \) is
the semi-norm induced by \( g \).

**PROOF.** To establish this result we show that there
exists a 1-1 linear map \( \Phi \), mapping \( E^* \) onto \( E_p^* \) which
is also an isometry. For any \( f \in E^* \) define
\[
\hat{\phi}(f) = \hat{f} \ \text{and} \ \hat{f}(x) = f(x). \quad (\text{IV.} \ 2. \ 2)
\]
Then \( \hat{\phi}(f) \) is well-defined. For, if \( y \neq x \) be another member of \( \hat{x} \), then \( y-x \in \text{Ker} \ p \). Thus \( f(y) = f(x) \)
\((= \hat{f}(x))\). Furthermore, linearity of \( \hat{f} \) follows from the
following: \( \hat{f}(ax+by) = f(ax+by) = af(x) + bf(y) = a\hat{f}(x) + b\hat{f}(y) \).

To establish the continuity, consider a sequence
\( (\hat{x}_n) \longrightarrow \hat{o} \) in \( E_p \). Then the corresponding sequence
\( (x_n) \) in \( E \) has \( o \) as one of its limit points, and
\( \hat{f}(\hat{x}_n) = f(x_n), \ \forall n \). As \( f(x_n) \longrightarrow o \), continuity of \( f \)
follows.

We have thus shown that \( \hat{\phi}(f) = \hat{f} \) is a continuous
linear functional on \( E_f \), i.e., \( \phi(f) = \hat{\phi} \in E^*_p \). The fact
that every continuous linear functional on \( E_p \) is obtainable
in this way follows from arguments similar to the above.
Thus \( \hat{\phi} : E^* \longrightarrow E^*_p \) is a 1-1 onto map whose linearity is
obvious.

Let \( \hat{p}' \) be the norm on \( E^*_p \). Then \( \hat{p}'(\hat{f}) = \)
\[\sup\{|\hat{f}(\hat{x})| : \hat{p}(\hat{x}) \leq 1\} = \sup\{ |f(x)| : p(x) \leq 1 \} = p(f). \]
\[\Rightarrow \hat{p}'(\hat{\phi}(f)) = p(f), \text{where} \ p' \text{is the norm on} \ E^*. \text{This}
\text{shows that the map} \ \hat{\phi} : E^* \longrightarrow E^*_p \ \ni \ \hat{\phi}(f) = \hat{f} \ \text{is an}
\text{isometry.} ///\]
§ 3. NON-REFLEXIVITY.

For the bidual $E^{**}$ of $E$ we establish the following result:

**Proposition (3.1).** For a semi-normed space $(E,p)$ the subspace $\text{Ker} \ p$ is of finite codimension if and only if $E^*$ (and $E^{**}$) is finite dimensional, and in this case $\text{Cod} (\text{Ker} \ p) = \dim E^* = \dim E^{**}.$

To prove it we establish the following:

**Lemma (3.2).** A closed subspace $M$ of a semi-normed space $(E,g)$ is of finite codimension, say $n,$ if and only if $\exists$ $n$ linearly independent functionals $f_1,f_2,\ldots,f_n$ in $E^*$ such that $M = \bigcap_{i=1}^n \text{Ker} \ f_i.$

**Proof. (I) Necessity.** If $\text{Cod} M = n < \infty,$ then we can write $E = M \oplus N$ where $N$ is an $n$-dimensional linear subspace of $E.$ Let $\{e_i\}_{i=1}^n$ be a Hamel basis for $N.$ Then for any $x \in E$ we can write $x = y + \sum_{i=1}^n \alpha_i e_i$ where $y \in M$ and $\alpha_i \in \mathcal{F}.$ Define linear functionals $\{f_i\}_{i=1}^n$ on $E$ by the relation $f_i(x) = \alpha_i$ for $1 \leq i \leq n,$ $x \in E.$
Their linear independence is obvious by the relation
\( f_i(e_j) = \delta_{ij} \). To establish their continuity consider the
maps \( \phi_M : E \rightarrow E/M \) and \( F_i : E/M \rightarrow \mathbb{F} \) defined by
\[
\phi_M(x) = \hat{x} \quad \text{and} \quad F_i(\hat{x}) = f_i(x), \quad 1 \leq i \leq n.
\]
It has already been shown that these mappings are well
defined, the quotient map \( \phi_M \) is continuous and, that the
continuity of \( F_i \) follows from the fact that every linear
functional on a finite dimensional normed linear space is
continuous. Thus \( f_i = F_i \circ \phi_M \) are \( n \) linearly indepen-
dent continuous linear functionals on \( E \).

Now \( x \in M \Rightarrow f_i(x) = 0 \) for \( 1 \leq i \leq n \),
i.e.
\[
M \subseteq \bigcap_{i=1}^{n} \ker f_i. \quad \text{(*)}
\]
Again if \( x \in \bigcap_{i=1}^{n} \ker f_i \) then \( \alpha_i = f_i(x) = 0 \) for \( 1 \leq i \leq n \Rightarrow x \in M \),
i.e.
\[
\bigcap_{i=1}^{n} \ker f_i \subseteq M. \quad \text{(**)}
\]
Comparing (*) and (**) we get \( M = \bigcap_{i=1}^{n} \ker f_i \).

(II) **Sufficiency.** Next assume that \( M = \bigcap_{i=1}^{n} \ker f_i \),
where \( f_i \)'s \( (1 \leq i \leq n) \) are \( n \) linearly independent conti-
uous linear functionals on \( E \). Then \( \exists \) positive numbers
\( K_i \)'s \( \forall x \in E \) \( |f_i(x)| \leq K_i p(x) \), \( \forall x \in E \). Define a map
\( F : E/M \rightarrow \mathbb{F}^n \) by \( F(\hat{x}) = (f_1(x), f_2(x), \ldots, f_n(x)) \). It
is easy to see that \( F \) is well-defined, linear and 1-1.
Now if \( F(\hat{x}) = 0 \), then \( f_i(x) = 0 \) for \( i = 1, 2, \ldots, n \), and therefore \( x \in \bigcap_{i=1}^{n} \text{Ker } f_i = M \). From the properties of \( F \) it follows that \( \dim (E/M) \leq \dim \phi^n = n \). Suppose \( \dim (E/M) = m < n \).

It has been established in the necessity part that if a closed subspace \( M \) is of finite codimension, say \( m \), then \( M = \bigcap_{j=1}^{m} \text{Ker } g_j \), where \( g_j(1 \leq j \leq m) \) are \( n \) linearly independent functionals belonging to \( E^* \). But by hypothesis \( M = \bigcap_{i=1}^{n} \text{Ker } f_i \). Thus \( \bigcap_{j=1}^{m} \text{Ker } g_j = \bigcap_{i=1}^{n} \text{Ker } f_i \), for \( 1 \leq i \leq n \). It therefore follows that \( f_i \)'s belong to the span of \( \{g_j\}_{1}^{m} \), and consequently, span of \( \{f_i\}_{1}^{n} \subset \text{Span } \{g_j\}_{1}^{m} \), which in turn implies that \( n \leq m \). Interchanging the roles of \( f_i \)'s and \( g_j \)'s we get \( m \leq n \), and hence \( m = n \). \///

**Proof of the Theorem.** Assume that \( \text{cod}(\text{Ker } p) = n < \infty \). Replacing \( M \) by \( \text{Ker } p \), which is a closed subspace of \( E \), as in the lemma we get \( n \) linearly independent functionals \( f_i \in E^* \) \( (1 \leq i \leq n) \) \( M = \text{Ker } p = \bigcap_{i=1}^{n} \text{Ker } f_i \).

If \( f \in E^* \), \( \exists \) a number \( K > 0 \) \( \exists |f(x)| \leq K p(x) \) \( \forall x \in E \). Thus \( \bigcap_{i=1}^{n} \text{Ker } f_i = \text{Ker } p \subset \text{Ker } f \) [14]. It,
therefore, follows (see eg. [27], p.124) that \( f \) is a linear combination of \( f_i \)'s, i.e. \( f = \sum_{i=1}^{n} \alpha_i f_i \) \( \exists \alpha_i \in \Phi \)

\[ \implies \dim E^* = n. \] Hence \( \dim E^{**} (= \dim E^*) = n. \)

Conversely, let \( \dim E^* = n \), and let \( \{f_i\}_{i=1}^{n} \) be a Hamel basis for \( E^* \). For each \( f_i \), \( \exists K > 0 \exists \)

\[ |f_i(x)| \leq K p(x), \forall x \in E. \] Thus \( M \subseteq \ker f_i \) \( (1 \leq i \leq n) \), that is to say that

\[ M \subseteq \bigcap_{i=1}^{n} \ker f_i. \] \((IV.3.1)\)

To prove the reverse inclusion take any \( x \in \bigcap_{i=1}^{n} \ker f_i \), which implies that \( f_i(x) = 0. \) Since \( \{f_i\}_{i=1}^{n} \) is a Hamel basis, \( f(x) = 0, \forall f \in E^*. \) By Hahn-Banach Theorem, there exists \( F \in E^* \) such that \( F(x) = f(x). \) In particular

\[ F(x) = 0, \] so that \( p(x) = 0. \) Thus \( x \in M, \) and therefore

\[ \bigcap_{i=1}^{n} \ker f_i \subseteq M. \] \((IV.3.2)\)

Relations \((IV.3.1)\) and \((IV.3.2)\) combined together complete the proof. \///

It is well known that for a Hilbert space \( E \), one has \( E = E^{**} \), and for any normed linear space \( N, N \subseteq N^{**}. \)
However, the same can never be true in general semi-normed spaces. In fact no semi-normed space can sit inside its bidual as an isometric image for the simple reason that the bidual is always normed (Hausdorff) and hence cannot contain a semi-normed (non-Hausdorff) subspace. This leads us to state the following result:

**Proposition (3.4).** A semi-Hilbert space $E$ can never be reflexive. Further $E \not\subset E^{**}$.

A particular case of the above result, where the semi-norm is generated by linear functionals, is illustrated by the following example:

**Example (3.5).** Let $E$ be a linear space with $\dim E > 1$. Then for any $f \in E$, the semi-norm $p : E \rightarrow \mathbb{R}_+$ defined by $p(x) = |f(x)|$ is semi-Hilbertian [Chap. III prop. 6.2]. We now show that the topological dual $E^*$ of $E$ is $\{af : a \in \mathbb{R}\}$. Since $af$ is continuous on $(E, p)$ $\forall \alpha \in \mathbb{R}$, and if $F \in E^*$, $\exists$ a number $K > 0$ $\Rightarrow |F(x)| \leq K p(x) = K |f(x)|$, $\forall x \in E$. $\Rightarrow$ Ker $f \subset$ Ker $F$ [2] and therefore $F = af$ for some $\alpha \in \mathbb{R}$, i.e. every linear functional on $E$ is of the form $af$. This obviously implies that $\dim E^* = 1$. The algebraic dual $(E^*)'$ of $E^*$ is therefore one-dimensional. Since $E^{**}$ is a subspace...
of \((E^*)', \text{ dim } (E^{**}) \leq \text{ dim } (E^*)'\). Hahn-Banach Theorem implies that \(\text{ dim } E^* \neq 0\), thus \(\text{ dim } E^{**} = \text{ dim } (E^*)' = 1\).

If \(E\) is reflexive, the natural correspondence \(\Psi : E \rightarrow E^{**}\) is a surjective (and hence bijective) linear mapping. In particular, \(E\) and \(E^{**}\) have the same dimension, which contradicts the assumption that \(\text{ dim } E > 1\).

For a Hausdorff locally convex space \(E\) there is a 1-1 correspondence from \(E\) into \(E^{**}\), whereas the same does not hold true in the case of non-Hausdorff locally convex spaces. It turns out that in certain situations the bidual \(E^{**}\) is too 'small' to contain an isometric copy of \(E\). The above example demonstrates this phenomenon and shows in addition that the notion of reflexivity does not make sense in this setting.