CHAPTER 2

Parabolic Functional Differential
Equations of Neutral Type

2.1 Introduction

Consider the first order neutral partial functional differential equation of the form

\[
\begin{align*}
\frac{\partial}{\partial t} (u(x, t) + \lambda(t) u(x, t - \tau)) + q(x, t) f(u(x, t - \sigma)) & = a(t) \Delta u(x, t) + b(t) \Delta u(x, t - \rho), (x, t) \in \Omega \times [0, \infty) \equiv G,
\end{align*}
\]

(2.1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a piecewise smooth boundary \( \partial \Omega \) and \( \Delta u(x, t) = \sum_{r=1}^{n} \left( \frac{\partial^2 u(x, t)}{\partial x_r^2} \right) \). Equation (2.1.1) is supplemented by one of the following boundary conditions, namely

\[
\frac{\partial u(x, t)}{\partial \nu} + g(x, t) u(x, t) = 0, (x, t) \in \partial \Omega \times [0, \infty),
\]

(2.1.2)
where $\nu$ is the unit exterior normal vector to $\partial \Omega$, and $g(x, t)$ is a nonnegative continuous function on $\partial \Omega \times [0, \infty)$, and
\begin{equation}
  u(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, \infty). \tag{2.1.3}
\end{equation}

In what follows, we always assume that without further mention:

\( (H_1) \) $\lambda \in C^1([0, \infty); [0, \infty))$, $\tau, \sigma$ and $\rho$ are positive constants;

\( (H_2) \) $q \in C \left( \bar{G}, (0, \infty) \right)$, $q(t) = \min_{x \in \Omega} q(x, t)$;

\( (H_3) \) $a, b \in C ([0, \infty); [0, \infty))$;

\( (H_4) \) $f \in C \left( \mathbb{R}, \mathbb{R} \right)$ is convex in $[0, \infty)$ with $uf(u) > 0$ for $u \neq 0$ and there is a positive constant $M$ such that $\frac{f(u)}{u^\alpha} \geq M > 0$ where $\alpha$ is a ratio of odd positive integers.

A function $u \in C^2(G) \cap C^1(\bar{G})$ is called a solution of the problem (2.1.1), (2.1.2) ((2.1.1), (2.1.3)) if it satisfies (2.1.1) in the domain $G$ and the boundary condition (2.1.2)((2.1.3)). The solution $u(x, t)$ of the equation (2.1.1), (2.1.2)((2.1.1), (2.1.3)) is said to be oscillatory in the domain $G$ if for any positive number $\mu$ there exists a point $(x_0, t_0) \in \Omega \times [\mu, \infty)$ such that $u(x_0, t_0) = 0$ holds.

Most of the results obtained for the problem (2.1.1), (2.1.2)((2.1.1), (2.1.3)) in the literature are for the case $-1 < \lambda(t) < 0$. This motivated our interest in studying the oscillatory behavior of solutions of the problem (2.1.1) with the conditions different from the above mentioned one.

The purpose of this chapter is to establish some new oscillation criteria for the problem (2.1.1), (2.1.2)((2.1.1), (2.1.3)) when $\lambda(t) > 1$ or $\lambda(t) \equiv \lambda$ with $0 < \lambda < 1$. Examples are provided to illustrate the results.
In Section 2.2 we present some lemmas which are useful in establishing the main results. Also we derive some new oscillation criteria for the equation (2.1.1) under the boundary condition (2.1.2). In Section 2.3, we establish sufficient conditions for the oscillation of all solutions of equation (2.1.1) with the boundary condition (2.1.3). In Section 2.4, examples are inserted to illustrate our main results.

2.2 Oscillation of the Problem (2.1.1),(2.1.2)

In this section we present some new oscillation criteria for the problem (2.1.1),(2.1.2). With each solution \( u(x, t) \in C^2(G) \cap C^1(\overline{G}) \) of the problem (2.1.1), (2.1.2), we associate the function

\[
V(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx, \quad t \geq T \geq 0
\]  

(2.2.1)

where \(|\Omega| = \int_{\Omega} dx\).

**Lemma 2.2.1.** If \( u(x, t) \) is a solution of the problem (2.1.1), (2.1.2) for which \( u(x, t) > 0 \) in \( G_T = \Omega \times [T, \infty); T \geq 0 \), then the function \( V(t) \) defined by (2.2.1) satisfy the inequality

\[
\frac{d}{dt} (V(t) + \lambda(t)V(t - \tau)) + q(t)f(V(t - \sigma)) \leq 0
\]

(2.2.2)

with \( V(t) > 0, V(t - \tau) > 0 \) and \( V(t - \sigma) > 0 \) for \( t \geq T \).

**Proof.** Let \( t \geq T \). Integrating (2.1.1) with respect to \( x \) over \( \Omega \), we have

\[
\frac{d}{dt} \left( \int_{\Omega} u(x, t) \, dx + \lambda(t) \int_{\Omega} u(x, t - \tau) \, dx \right) + \int_{\Omega} q(x, t)f(u(x, t - \sigma)) \, dx
\]
\[
    a(t) \int_{\Omega} \Delta u(x, t) \, dx + b(t) \int_{\Omega} \Delta u(x, t - \rho) \, dx. \tag{2.2.3}
\]

From Green's formula and boundary condition (2.1.2), it follows that
\[
    \int_{\Omega} \Delta u(x, t) \, dx = \int_{\partial \Omega} \frac{\partial u(x, t)}{\partial \nu} \, dS = -\int_{\partial \Omega} g(x, t) u(x, t) \, dS \leq 0, \quad t \geq T \tag{2.2.4}
\]
and
\[
    \int_{\Omega} \Delta u(x, t - \rho) \, dx = \int_{\partial \Omega} \frac{\partial u(x, t - \rho)}{\partial \nu} \, dS
    = -\int_{\partial \Omega} g(x, t - \rho) u(x, t - \rho) \, dS \leq 0, \quad t \geq T, \tag{2.2.5}
\]
where \(dS\) is the surface element on \(\partial \Omega\). Also, from \((H_3), (H_5)\) and using Jensen's inequality[72], we obtain
\[
    \int_{\Omega} q(x, t) f(u(x, t - \sigma)) \, dx
    \geq q(t) \int_{\Omega} f(u(x, t - \sigma)) \, dx,
    \geq q(t) \int_{\Omega} dx f \left( \int_{\Omega} u(x, t - \sigma) \, dx \left( \int_{\Omega} dx \right)^{-1} \right), \text{ for } t \geq T, \tag{2.2.6}
\]
In view of (2.2.1), (2.2.4) - (2.2.6), (2.2.3) yield
\[
    \frac{d}{dt} (V(t) + \lambda(t)V(t - \tau)) + q(t) f(V(t - \sigma)) \leq 0, \quad t \geq T.
\]
This proves the lemma. \(\square\)
Lemma 2.2.2. Let \( u(x, t) \) be a positive solution of the problem (2.1.1), (2.1.2) defined on \( G_T \). Then the function \( Z(t) = V(t) + \lambda(t)V(t - \tau) \) where \( V(t) \) is defined by (2.2.1), satisfies the following condition:

\[
V(t - \tau) \geq \frac{(\lambda(t + \tau) - 1)}{\lambda(t) \lambda(t + \tau)} Z(t) \tag{2.2.7}
\]

where \( \lambda(t) > 1 \) for all \( t \geq T \).

Proof. From Lemma 2.2.1, the function \( V(t) \) satisfies the inequality (2.2.2) with \( V(t) > 0, V(t - \tau) > 0 \) and \( V(t - \sigma) > 0 \) for \( t \geq T \). From (2.2.2) and the hypothesis we have \( Z(t) > 0 \) and \( Z'(t) \leq 0 \) for \( t \geq T \). Hence from the definition of \( Z(t) \), we have

\[
Z(t) \geq \lambda(t)V(t - \tau) \tag{2.2.8}
\]

and

\[
\lambda(t)V(t - \tau) = Z(t) - V(t). \tag{2.2.9}
\]

Using (2.2.8) and decreasing behavior of \( Z(t) \) in (2.2.2), we obtain

\[
\lambda(t)V(t - \tau) \geq Z(t) - \frac{Z(t)}{\lambda(t + \tau)}, \quad t \geq T.
\]

This implies (2.2.7).

\[\square\]

Remark 2.2.1. Let \( \lambda(t) \equiv \lambda > 1 \) in Lemma 2.2.2. Then from (2.2.7), we obtain

\[
V(t - \tau) \geq \left(\frac{\lambda - 1}{\lambda^2}\right) Z(t), \tag{2.2.10}
\]

for all \( t \geq T \).

Lemma 2.2.3. Let \( u(x, t) \) be a positive solution of the problem (2.1.1), (2.1.2) defined on \( G_T \) and assume that \( \lambda(t) \equiv \lambda \) with \( 0 < \lambda < 1 \). Then the function \( Z(t) = V(t) + \lambda V(t - \tau) \) satisfy the condition

\[
V(t - \tau) \geq \lambda(1 - \lambda) Z(t), \tag{2.2.11}
\]
for all $t \geq T$.

Proof. From (2.2.2), we see that $Z(t)$ is positive and decreasing for all $t \geq T$. Therefore we may assume without loss of generality that $W(t) = V(t) + \mu V(t - \tau)$ where $\mu = \frac{1}{\lambda} > 1$ is also positive and decreasing for all $t \geq T$. From Remark 2.2.1, we have

$$V(t - \tau) \geq \left(\frac{\mu - 1}{\mu^2}\right) W(t), \quad t \geq T. \tag{2.2.12}$$

From the definition of $Z(t)$ and $W(t)$, we have $W(t) \geq Z(t)$ and hence from (2.2.12) we obtain

$$V(t - \tau) \geq \left(\frac{\mu - 1}{\mu^2}\right) Z(t), \quad t \geq T. \tag{2.2.13}$$

The result (2.2.11) follows by substituting $\mu$ by $\frac{1}{\lambda}$ in (2.2.13). This completes the proof. \qed

Lemma 2.2.4. Assume that $\sigma > \tau$, $\lambda(t) > 1$ or $\lambda(t) \equiv \lambda$ with $0 < \lambda < 1$ for $\alpha = 1$, and

$$\limsup_{t \to \infty} \int_{t}^{t+\sigma-\tau} q(s)\mu(s)ds > 0, \tag{2.2.14}$$

where

$$\mu(t) = \begin{cases} \frac{\lambda(t + 2\tau - \sigma) - 1}{\lambda(t + \tau - \sigma)} \frac{\lambda(t + 2\tau - \sigma)}{\lambda(t + \tau - \sigma)} & \text{if } \lambda(t) > 1 \\ \lambda(1 - \lambda) & \text{if } \lambda(t) \equiv \lambda \text{ with } 0 < \lambda < 1. \end{cases}$$

If $u(x, t)$ is an eventually positive solution of the problem (2.1.1), (2.1.2), then

$$\liminf_{t \to \infty} \frac{Z(t + \tau - \sigma)}{Z(t)} < \infty, \tag{2.2.15}$$

where $Z(t) = V(t) + \lambda(t) V(t - \tau)$. 
Proof. From Lemma 2.2.1, the function $V(t)$ satisfies the inequality (2.2.2) with $V(t) > 0$, $V(t - \tau) > 0$ and $V(t - \sigma) > 0$ for $t \geq T$. From (2.2.2), we see that $Z(t)$ is positive and decreasing for all $t \geq T$. Hence from Lemmas 2.2.2 and 2.2.3, we have

$$V(t - \sigma) \geq \mu(t)Z(t + \tau - \sigma). \tag{2.2.16}$$

From (2.2.2) and (2.2.16), we obtain

$$Z'(t) + M\mu(t)q(t)Z(t + \tau - \sigma) \leq 0. \tag{2.2.17}$$

Now apply Lemma 1 of [47] to (2.2.17) to obtain the desired result (2.2.15).

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Lemma 2.2.5. Assume that $\sigma > \tau$, $\lambda(t) > 1$ or $\lambda(t) = \lambda$ with $0 < \lambda < 1$ for $t \geq 0$ and $\alpha = 1$. If $u(x, t)$ is an eventually positive solution of the problem (2.1.1),(2.1.2), then

$$t + \sigma - \tau \int_t^{t + \sigma - \tau} q(s)\mu(s)ds \leq \frac{1}{M} \tag{2.2.18}$$

for sufficiently large $t$.

Proof. Proceeding as in the proof of Lemma 2.2.4, we again obtain (2.2.17). Integrating (2.2.17) from $t$ to $t + \sigma - \tau$ and using the decreasing behavior of $Z(t)$, we obtain

$$Z(t + \sigma - \tau) + \left( M \int_t^{t + \sigma - \tau} q(s)\mu(s)ds - 1 \right) Z(t) \leq 0. \tag{2.2.19}$$

Since $Z(t) > 0$ eventually, (2.2.19) implies

$$M \int_t^{t + \sigma - \tau} q(s)\mu(s)ds - 1 \leq 0 \tag{2.2.20}$$
for large $t$, and the desired result (2.2.18) follows from (2.2.20). \hfill \Box

In the following, we obtain integral conditions for all solutions of the problem (2.1.1), (2.1.2). We first consider the case $\alpha = 1$.

**Theorem 2.2.6.** Assume that $\sigma > \tau$, $\lambda(t) > 1$ or $\lambda(t) \equiv \lambda$ with $0 < \lambda < 1$ for $t \geq 0$ and $\alpha = 1$, and (2.2.14) holds. If

$$
\int_{t_0}^{\infty} \mu(t)q(t) \log \left( eM \int_t^{t+\sigma-\tau} \mu(s)q(s)ds \right) dt = \infty,
$$

(2.2.21)

then every solution $u(x, t)$ of the problem (2.1.1), (2.1.2) is oscillatory.

**Proof.** Assume to the contrary that there is a nonoscillatory solution $u(x, t)$ of the problem (2.1.1), (2.1.2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality we may assume that $u(x, t) > 0$, $u(x, t-\tau) > 0$ and $u(x, t-\sigma) > 0$ in $\Omega \times [T, \infty)$, $T \geq t_0$, where $T$ is chosen so large that Lemmas 2.2.1 to 2.2.4 hold for $t \geq T$. From Lemma 2.2.1 the function $V(t)$ defined by (2.2.1) satisfies the inequality (2.2.2). Then $Z(t)$ is eventually positive and decreasing and satisfies the inequality

$$
Z'(t) + M\mu(t)q(t)Z(t+\tau-\sigma) \leq 0.
$$

(2.2.22)

Let $r(t) = -\frac{Z'(t)}{Z(t)}$; then $r(t)$ is continuous and nonnegative, so there exists $t_1 \geq T$ with $Z(t_1) > 0$ such that

$$
Z(t) = Z(t_1) \exp \left( -\int_{t_1}^{t} r(s)ds \right).
$$

Moreover, $r(t)$ satisfies

$$
r(t) \geq Mq(t)\mu(t) \exp \left( \int_{t+\tau-\sigma}^{t} r(s)ds \right).
$$

(2.2.23)
Applying the inequality
\[ e^{yx} \geq x + \frac{\log (e^y)}{y} \text{ for } x > 0 \text{ and } y > 0 \]
to (2.2.23), we have
\[
\begin{align*}
\mu (t) & \geq M \mu (t) q(t) \exp \left( A(t) \frac{1}{A(t)} \int_{t+\tau-\sigma}^{t} r(s) \, ds \right) \\
& \geq M \mu (t) q(t) \left( \frac{1}{A(t)} \int_{t+\tau-\sigma}^{t} r(s) \, ds + \frac{\log (eA(t))}{A(t)} \right)
\end{align*}
\]
where we take
\[ A(t) = M \int_{t}^{t+\sigma-\tau} q(s) \mu(s) \, ds. \]

It follows that
\[
\begin{align*}
\mu (t) q(t) \int_{t}^{t+\sigma-\tau} \mu(s) q(s) \, ds & - \mu(t) q(t) \int_{t+\tau-\sigma}^{t} r(s) \, ds \\
& \geq \mu(t) q(t) \log \left( eM \int_{t}^{t+\tau+\sigma} \mu(s) q(s) \, ds \right)
\end{align*}
\]
Then, for \( u > t_2 + \sigma - \tau \), we have
\[
\begin{align*}
\int_{t_2}^{u} r(t) \left( \int_{t}^{t+\sigma-\tau} \mu(s) q(s) \, ds \right) \, dt & - \int_{t_2}^{u} \mu(t) q(t) \left( \int_{t+\tau-\sigma}^{t} r(s) \, ds \right) \, dt \\
& \geq \int_{t_2}^{u} \mu(t) q(t) \log \left( eM \int_{t}^{t+\sigma-\tau} \mu(s) q(s) \, ds \right) \, dt.
\end{align*}
\]
Interchanging the order of integration, we obtain
\[
\int_{t_2}^{u} \mu(t) q(t) \left( \int_{t+\tau-\sigma}^{t} r(s) \, ds \right) \, dt \geq \int_{t_2}^{u+\tau-\sigma} r(t) \left( \int_{t}^{t+\sigma-\tau} \mu(s) q(s) \, ds \right) \, dt.
\] (2.2.25)

From (2.2.24) and (2.2.25), it follows that
\[
\int_{u+\tau-\sigma}^{u} r(t) \left( \int_{t}^{t+\sigma-\tau} \mu(s) q(s) \, ds \right) \, dt \geq \int_{t_2}^{u} \mu(t) q(t) \log \left( eM \int_{t}^{t+\sigma-\tau} \mu(s) q(s) \, ds \right) \, dt.
\] (2.2.26)

Using (2.2.18) and (2.2.26), we have
\[
\int_{u+\tau-\sigma}^{u} r(t) \, dt \geq M \int_{t_2}^{u} \mu(t) q(t) \log \left( eM \int_{t}^{t+\sigma-\tau} \mu(s) q(s) \, ds \right) \, dt
\]
or
\[
\log \frac{Z(u+\tau-\sigma)}{Z(u)} \geq M \int_{t_2}^{u} \mu(t) q(t) \log \left( eM \int_{t}^{t+\sigma-\tau} \mu(s) q(s) \, ds \right) \, dt.
\]

In view of (2.2.21), we must have
\[
\lim_{u \to \infty} \frac{Z(u+\tau-\sigma)}{Z(u)} = \infty
\]
which contradicts (2.2.15) and completes the proof of the theorem. \(\square\)

In our next theorem, we again consider the case \(\alpha = 1\) and obtain a different type of sufficient condition for the oscillation of all solutions of the problem (2.1.1),(2.1.2).
Theorem 2.2.7. Assume that \( \sigma > \tau, \lambda(t) > 1 \) or \( \lambda(t) \equiv \lambda \) with \( 0 < \lambda < 1 \) for \( t \geq 0 \) and \( \alpha = 1 \), and there exists a constant \( k > 0 \) such that

\[
\frac{1}{e} \leq \int_{t-\sigma+\tau}^{t} \mu(s)q(s) \, ds < k,
\]

and

\[
\int_{t_0}^{\infty} \mu(t)q(t) \exp \left( e \int_{t-\sigma+\tau}^{t} \mu(s)q(s) \, ds \right) \, dt = \infty. \tag{2.2.27}
\]

Then every solution \( u(x,t) \) of the problem (2.1.1),(2.1.2) is oscillatory in \( G \).

Proof. Proceeding as in the proof of Theorem 2.2.6, we see that \( Z(t) \) is eventually positive, decreasing, and satisfies (2.2.22). Moreover, the generalized characteristic equation for (2.2.22) is given by

\[
r(t) \geq M \mu(t)q(t) \exp \left( \int_{t+\tau-\sigma}^{t} r(s) \, ds \right). \tag{2.2.28}
\]

If we let \( A(t) = MB(t) \) where \( B(t) = \exp \left( e \int_{t+\tau-\sigma}^{t} \mu(s)q(s) \, ds \right) \), then we have from (2.2.28)

\[
r(t) \geq M \mu(t)q(t) \exp \left( \frac{A(t)}{A(t)} \int_{t+\tau-\sigma}^{t} r(s) \, ds \right).
\]

Applying the inequality

\[
e^{x/y} \geq 1 + \frac{x}{y^2} \text{ for } x > 0 \text{ and } y > 1,
\]

we obtain

\[
A(t)r(t) - M \mu(t)q(t) \int_{t+\tau-\sigma}^{t} r(s) \, ds \geq Mq(t)\mu(t)A(t).
\]
Then for \( u > t_2 + \sigma - \tau \),

\[
\int_{t_2}^{u} A(t) r(t) \, dt - M \int_{t_2}^{u} \mu(t) q(t) \left( \int_{t+\tau-\sigma}^{t} r(s) \, ds \right) \, dt \geq M \int_{t_2}^{u} q(t) \mu(t) A(t) \, dt. 
\]  
(2.2.29)

Interchanging the order of integration and simplifying, we obtain

\[
\int_{t_2}^{u} \mu(t) q(t) \int_{t+\tau-\sigma}^{t} r(s) \, ds \, dt \geq \int_{t_2}^{u} r(t) \left( \int_{t+\tau-\sigma}^{t} \mu(s) q(s) \, ds \right) \, dt.
\]  
(2.2.30)

From (2.2.29) and (2.2.30), it follows that

\[
\int_{t_2}^{u} r(t) A(t) \, dt - M \int_{t_2}^{u} r(t) \left( \int_{t+\tau-\sigma}^{t} \mu(s) q(s) \, ds \right) \, dt \geq M \int_{t_2}^{u} q(t) \mu(t) A(t) \, dt
\]

and so, since \( B(t) = \exp \left( e \int_{t+\tau-\sigma}^{t} \mu(s) q(s) \, ds \right) \geq \int_{t+\tau-\sigma}^{t} \mu(s) q(s) \, ds \).

Hence

\[
\int_{t_2}^{u} r(t) B(t) \, dt - M \int_{t_2}^{u} r(t) B(t) \, dt \geq M \int_{t_2}^{u} q(t) \mu(t) B(t) \, dt.
\]  
(2.2.31)

On the other hand, since \( e \leq B(t) < k_1 \) for some \( k_1 > 0 \), (2.2.31) implies

\[
\int_{u+\tau-\sigma}^{u} r(t) dt \geq \frac{1}{k_1} \int_{t_2}^{u} q(t) \mu(t) B(t) \, dt.
\]
Since (2.2.27) implies that the integral on the right hand side of the above inequality diverges as \( u \to \infty \), the remainder of the proof is similar to that of Theorem 2.2.6 and so we omit the details. This completes the proof of the theorem.

\[ \square \]

**Theorem 2.2.8.** Assume that \( \sigma > \tau \), \( \lambda(t) > 1 \) or \( \lambda(t) \equiv \lambda \) with \( 0 < \lambda < 1 \) and \( \alpha \in (0, 1) \). If

\[
\int_{t_0}^{\infty} \mu^\alpha(t) q(t) \, dt = \infty
\]  

(2.2.32)

then every solution \( u(x, t) \) of the problem (2.1.1), (2.1.2) is oscillatory in \( G \).

**Proof.** Proceeding as in the proof of Theorem 2.2.6, we see that \( Z(t) \) is eventually positive, decreasing, and satisfies

\[
Z'(t) + M \mu^\alpha(t) q(t) Z^\alpha(t + \tau - \sigma) \leq 0, \quad t \geq T.
\]  

(2.2.33)

Since \( t > t + \tau - \sigma \) and \( Z(t) \) is decreasing, we have from (2.2.33)

\[
Z'(t) + M \mu^\alpha(t) q(t) Z^\alpha(t) \leq 0, \quad t \geq T.
\]

Dividing the last inequality by \( Z^\alpha(t) \) and integrating we obtain

\[
M \int_{T}^{t} \mu^\alpha(s) q(s) \, ds \leq \frac{Z^{1-\alpha}(T)}{1-\alpha} < \infty,
\]

which is impossible in view of (2.2.32). This completes the proof of the theorem.

\[ \square \]

**Theorem 2.2.9.** Assume that \( \sigma > \tau \), \( \lambda(t) > 1 \) or \( \lambda(t) \equiv \lambda \) with \( 0 < \lambda < 1 \) for \( t \geq 0 \) and \( \alpha > 1 \). In addition, assume that there exists a continuously
differentiable function $\phi(t)$ such that

\[ \phi'(t) > 0, \quad \lim_{t \to \infty} \phi(t) = \infty \]  \hspace{1cm} (2.2.34)

\[ \limsup_{t \to \infty} \frac{\phi'(t + \tau - \sigma)}{\phi'(t)} < \frac{1}{\alpha} \]  \hspace{1cm} (2.2.35)

and

\[ \liminf_{t \to \infty} \left[ M \mu^\alpha(t) q(t) \frac{e^{-\phi(t)}}{\phi'(t)} \right] > 0. \]  \hspace{1cm} (2.2.36)

Then every solution $u(x, t)$ of the problem (2.1.1), (2.1.2) is oscillatory in $G$.

**Proof.** Proceeding as in the proof of Theorem 2.2.6, we see that $Z(t)$ is eventually positive, decreasing and satisfies inequality

\[ Z'(t) + M \mu^\alpha(t) q(t) Z^\alpha(t + \tau - \sigma) \leq 0, \]  \hspace{1cm} (2.2.37)

From (2.2.34) and (2.2.35), we see that

\[ \limsup_{t \to \infty} \frac{\alpha \phi(t + \tau - \sigma)}{\phi(t)} < 1. \]  \hspace{1cm} (2.2.38)

Now by (2.2.35) and (2.2.38), there exists $0 < \ell < 1, \epsilon > 0$ and $T \geq T_0$ such that

\[ \frac{(1 + \epsilon) \alpha \phi'(t + \tau - \sigma)}{\phi'(t)} \leq \ell \quad \text{and} \quad \frac{(1 + \epsilon) \alpha \phi(t + \tau - \sigma)}{\phi(t)} \leq \ell \]  \hspace{1cm} (2.2.39)

for $t \geq T$. In view of (2.2.36), we may choose $T_0 \geq T$ such that

\[ M \mu^\alpha(t) q(t) \geq \phi'(t) e^{\frac{\alpha \phi(t)}{1+\alpha}} \]  \hspace{1cm} (2.2.40)

for $t \geq T_0$. Next set $p(t) = \phi'(t) e^{\frac{\alpha \phi(t)}{1+\alpha}}$. By Lemma 2 in [88] it suffices to consider the inequality

\[ Z'(t) + p(t) Z^\alpha(t + \tau - \sigma) \leq 0 \]  \hspace{1cm} (2.2.41)
instead of (2.2.37). In order to see that \( Z(t) \to 0 \) as \( t \to \infty \), first observe that 
\[ Z(t + \tau - \sigma) \geq Z(t) \]. Hence

\[
Z'(t) + p(t) Z^\alpha(t) \leq Z'(t) + p(t) Z^\alpha(t + \tau - \sigma) \leq 0
\]
and so

\[
\frac{Z'(t)}{Z^\alpha(t)} \leq -p(t).
\]
Integrating, we have

\[
\frac{Z^{1-\alpha}(t) - Z^{1-\alpha}(T_0)}{1 - \alpha} \to -\infty
\]
as \( t \to \infty \). This implies that \( Z^{1-\alpha} \to +\infty \) so \( Z(t) \to 0 \). Thus, there exists a \( T_1 \geq T_0 \) such that

\[
0 < Z(t) < 1 \text{ and } Z'(t) \leq 0
\]
for \( t \geq T_2 \). Letting \( y(t) = -\log Z(t) \) for \( t \geq T_2 = T_1 + \sigma - \tau \), we see that \( y(t) > 0 \) for \( t \geq T_2 \) and (2.2.41) implies

\[
y'(t) \geq p(t) e^{y(t) - \alpha y(t + \tau - \sigma)}
\]
for \( t \geq T_2 \). The remainder of the proof is similar to that of Theorem 1 in [88] and hence the details are omitted.

\[\square\]

### 2.3 Oscillation of the Problem (2.1.1),(2.1.3)

In this section we establish sufficient conditions for the oscillation of all solutions of the problem (2.1.1), (2.1.3). For this we need the following result whose proof can be found in [10, 93].

The smallest eigen value \( \beta_0 \) of the Dirichlet problem

\[
\Delta \omega(x) + \beta \omega(x) = 0 \text{ in } \Omega
\]
\[ \omega(x) = 0 \text{ on } \partial \Omega, \]

is positive and the corresponding eigen function \( \phi(x) \) is positive in \( \Omega \).

**Theorem 2.3.1.** Let all conditions of Theorem 2.2.6 be hold. Then every solution \( u(x,t) \) of the problem (2.1.1), (2.1.3) oscillates in \( G \).

**Proof.** Assume the contrary: then there exists a nonoscillatory solution \( u(x,t) \) of the problem (2.1.1), (2.1.3) which has no zero in \( \Omega \times [t_0, \infty) \) for some \( t_0 > 0 \). Without loss of generality we may assume that \( u(x,t) > 0, u(x,t-\tau) > 0, u(x,t-\rho) > 0 \) and \( u(x,t-\sigma) > 0 \) in \( \Omega \times [t_1, \infty), t_1 \geq t_0 \).

Multiply both sides of equation (2.1.1) by \( \phi(x) > 0 \) and then integrating with respect to \( x \) over the domain \( \Omega \), we obtain for \( t \geq t_1 \),

\[
\frac{d}{dt} \left( \int_{\Omega} u(x,t) \phi(x) \, dx + \lambda(t) \int_{\Omega} u(x,t-\tau) \phi(x) \, dx \right)
+ \int_{\Omega} q(x,t) f(u(x,t-\sigma)) \phi(x) \, dx
= a(t) \int_{\Omega} \Delta u(x,t) \phi(x) \, dx + b(t) \int_{\Omega} \Delta u(x,t-\rho) \phi(x) \, dx.
\]  

(2.3.1)

From Green's formula and boundary condition (2.1.3), it follows that

\[
\int_{\Omega} \Delta u(x,t) \phi(x) \, dx = \int_{\Omega} u(x,t) \Delta \phi(x) \, dx
= -\beta_0 \int_{\Omega} u(x,t) \phi(x) \, dx \leq 0, \quad t \geq t_1.
\]  

(2.3.2)

and

\[
\int_{\Omega} \Delta u(x,t-\rho) \phi(x) \, dx = \int_{\Omega} u(x,t-\rho) \Delta \phi(x) \, dx.
\]
\[ = -\beta_0 \int_{\Omega} u(x, t - \rho) \phi(x) dx \leq 0, \quad t \geq t_1. \quad (2.3.3) \]

From (H_2), (H_4) and Jensen's inequality, it follows that
\[ \int_{\Omega} q(x, t)f(u(x, t - \sigma))\phi(x) dx \geq q(t) \int_{\Omega} f(u(x, t - \sigma))\phi(x) dx \]
\[ \geq q(t) \int_{\Omega} \phi(x) dx f \left( \int_{\Omega} u(x, t - \sigma)\phi(x) dx \left( \int_{\Omega} \phi(x) dx \right)^{-1} \right), \quad (2.3.4) \]
for \( t \geq t_1 \). Set
\[ V(t) = \int_{\Omega} u(x, t)\phi(x) dx \left( \int_{\Omega} \phi(x) dx \right)^{-1}, \quad t \geq t_1. \quad (2.3.5) \]

In view of (2.3.1) - (2.3.5), we obtain
\[ \frac{d}{dt} (V(t) + \lambda(t)V(t - \tau)) + q(t)f(V(t - \sigma)) \leq 0 \]
for \( t \geq t_1 \). Rest of the proof is similar to that of Theorem 2.2.6 and hence the details are omitted. \( \square \)

The following theorems can be proved analogously as that of in Section 2.2.

**Theorem 2.3.2.** Let the conditions of Theorem 2.2.7 hold; then every solution \( u(x, t) \) of the problem (2.1.1), (2.1.3) is oscillatory in \( G \).

**Theorem 2.3.3.** Let the conditions of Theorem 2.2.8 hold; then every solution \( u(x, t) \) of the problem (2.1.1), (2.1.3) is oscillatory in \( G \).

**Theorem 2.3.4.** Let the conditions of Theorem 2.2.9 hold; then every solution \( u(x, t) \) of the problem (2.1.1), (2.1.3) is oscillatory in \( G \).
2.4 Examples

In this section we give some examples to illustrate our results established in Sections 2.2 and 2.3.

Example 2.4.1. Consider the differential equation

$$
\frac{\partial}{\partial t} \left( u(x, t) + (t + \pi) u(x, t - 2\pi) \right) \\
+ (t + \pi + 1) u \left( x, t - \frac{5\pi}{2} \right) = u_{xx} \left( x, t - \pi \right)
$$

(2.4.1)

for \((x, t) \in (0, \pi) \times [0, \infty)\) with boundary condition

$$
u_x(0, t) = u_x(\pi, t) = 0, \quad t \geq 0.
$$

(2.4.2)

Here \(\lambda(t) = t + \pi > 1, \tau = 2\pi, \sigma = \frac{5\pi}{2}, \rho = \pi, q(x, t) = t + \pi + 1, a(t) = 0, b(t) = 1\) and \(f(u) = u\). It is easy to see that all conditions of 2.2.6 are satisfied. Hence every solution of the problem (2.4.1),(2.4.2) oscillates in \((0, \pi) \times [0, \infty)\). In fact \(u(x, t) = \cos x \sin t\) is such a solution.

Example 2.4.2. Consider the differential equation

$$
\frac{\partial}{\partial t} \left( u(x, t) + \frac{1}{2} u(x, t - \pi) \right) \\
+ 2u \left( x, t - \frac{3\pi}{2} \right) = \frac{5}{2} u_{xx} \left( x, t - \frac{\pi}{2} \right)
$$

(2.4.3)

for \((x, t) \in (0, \pi) \times [0, \infty)\) with boundary condition

$$
u(0, t) = u(\pi, t) = 0, \quad t \geq 0.
$$

(2.4.4)

Here \(\lambda(t) = \frac{1}{2}, \tau = \pi, \sigma = \frac{3\pi}{2}, \rho = \frac{\pi}{2}\), and \(q(x, t) = 2\). It is easy to see that all conditions of 2.3.2 are satisfied. Hence every solution of the problem (2.4.3),(2.4.4) oscillates in \((0, \pi) \times [0, \infty)\). In fact \(u(x, t) = \sin x \cos t\) is such a solution.
We conclude this chapter with the following remark.

**Remark 2.4.1.** The results obtained in this chapter can be easily extended to the equation of the form

\[
\frac{\partial}{\partial t}(u(x, t) + \lambda t)u(x, t - \tau)) + q(x, t)u(x, t) + \sum_{j=1}^{m} q_j(x, t)f_j(u(x, t - \sigma_j)) = a(t)\Delta u(x, t) + \sum_{i=1}^{n} a_i\Delta u(x, t - \rho_i), \quad (x, t) \in G
\]

with appropriate conditions assumed on the known functions and therefore the details are omitted.

3.1 Introduction

In this chapter, we consider the differential equation

\[
\frac{\partial}{\partial t}(u(x, t) - \lambda u)\Delta u(x, t) = a(t)\Delta u(x, t) + f(x, t)
\]

where \( \Omega \) is a bounded domain,

and \( \Delta u(x, t) \) is the Laplacian of \( u(x, t) \).