3.1 Introduction

In 1958, Juris Hartmanis [16] determined the automorphisms of the lattice $LT(X)$ of all topologies on a fixed set $X$ as follows: for $p \in S(X)$ and $\tau \in LT(X)$, define the mapping $A_p$ by $A_p(\tau) = \{p(U) : U \in \tau\}$. Then $A_p(\tau)$ is a topology on $X$ and $A_p$ is an automorphism of $LT(X)$. If $X$ is infinite or $X$ contains at most two elements, the set of all automorphisms of $LT(X)$ is precisely $\{A_p : p \in S(X)\}$. Otherwise, the set of all automorphisms of $LT(X)$ is $\{A_p : p \in S(X)\} \cup \{B_p : p \in S(X)\}$ where $B_p : LT(X) \rightarrow LT(X)$ is defined by $B_p(\tau) = \{X - p(U) : U \in \tau\}$ for $\tau \in LT(X)$. From this result, we can conclude that, if $X$ is an infinite set and $P$ is any topological property, then the set of
topologies in $LT(X)$ possessing the property $P$ may be identified simply from the lattice structure of $LT(X)$, since the only automorphisms of $LT(X)$ for infinite $X$ are those which simply permute elements of $X$. Therefore any automorphism of $LT(X)$ must map all the topologies in $LT(X)$ onto their homeomorphic images. Thus the topological properties of elements of $LT(X)$ must be determined by the position of the topologies in $LT(X)$ (see [24]). In this chapter, we try to investigate the group of automorphisms of the lattice $LFT(X, L)$ where $L$ is a complete, distributive lattice.

Let $X$ be any nonempty set and $L$ be a complete and distributive lattice.

**Definition 3.1.1.** [2] A t-homomorphism from a lattice $L$ into a lattice $M$ is a function $f : L \rightarrow M$ such that

(i) $h$ is a homomorphism

(ii) $h(0) = 0$ and $h(1) = 1$

(iii) $h(\bigvee k_i) = \bigvee h(k_i)$ where $\{k_i : i \in I\}$ is an arbitrary subset of $L$.

**Remark 3.1.2.** Obviously every t-homomorphism is a homomorphism. But the converse need not be true.

**Example 3.1.3.** Let $L = \{1, 2, 3\}$ and $M = \{1, 2, 3, 4, 5\}$ be lattices under usual order “$\leq$”. Define $f : L \rightarrow M$ by $f(1) = 2$, $f(2) = 3$, $f(3) = 4$. Then $f$ is a homomorphism, but not a t-homomorphism, since $f(0) \neq 1$ and $f(1) \neq 5$.

**Example 3.1.4.** Let $X$ be an infinite set and $L$ be a collection of subsets of $\wp(X)$ which are closed under finite union and $M$ be a collection of subsets of $\wp(X)$ which are closed under arbitrary union. Then $L$ and $M$ are lattices under set inclusion and $M \subseteq L$. The empty set $\phi$ is the smallest element “$0$” and $\wp(X)$ is the largest element “$1$” of both $L$ and $M$. If $f : M \rightarrow L$ is the inclusion map, then $f$ is a homomorphism such that $f(0) = 0$ and $f(1) = 1$, but is not a t-homomorphism. For every $x \in X$, the collection $\{\{x\}\}$ is in $M$. Also
\[ \bigvee_{x \in X} \{x\} \] in \( M \) is equal to \( \wp(X) \), where as \( \bigvee_{x \in X} f(\{x\}) \) is the collection of all finite subsets of \( X \). Thus \( f(\bigvee_{x \in X} \{x\}) = \wp(X) \neq \wp(\{x\}) \) is the collection of all finite subsets of \( X \) which is equal \( \bigvee_{x \in X} f(\{x\}) \).

### 3.2 Automorphisms of the lattice \( LFT(X, L) \)

Let \( X \) and \( Y \) be nonempty sets and \( L, M \) be complete, distributive lattices. For functions \( f: X \rightarrow Y \) and \( h: M \rightarrow L \), Babusundar [2] defined the induced function \( H: M^Y \rightarrow L^X \) as follows: for \( D \in M^Y \) and \( x \in X \), \( H(D)(x) = h(D(f(x))) \). He observed that \( H \) is a t-isomorphism if and only if \( f \) is a bijection and \( h \) is a t-isomorphism. He also defined the induced function \( H': LFT(Y, M) \rightarrow LFT(X, L) \) by \( H'(\delta) = \{ H(U) : U \in \delta \} \) for \( \delta \) in \( LFT(Y, M) \).

We use the notations \( H_{f, h} \) for \( H \) and \( H'_f \) for \( H' \) and proceed further.

As a particular case of the above result, that is, when \( X = Y \) and \( L = M \), we obtain the following result.

**Theorem 3.2.1.** [2] Let \( X \) be any nonempty set and \( L \) be a complete and distributive lattice. If \( p: X \rightarrow X \) is a bijection and \( g: L \rightarrow L \) is an isomorphism, then, \( H_{p, g}: L^X \rightarrow L^X \) defined by \( H_{p, g}(C)(x) = g(C(p^{-1}(x))) \); \( C \in L^X \), \( x \in X \) is an automorphism of \( L^X \). Further, if \( \delta \) is a fuzzy topology on \( X \), then the collection \( H_{p, g}(\delta) = \{ H_{p, g}(A) : A \in \delta \} \) is also a fuzzy topology on \( X \) and \( H_{p, g}^* \) is an automorphism of \( LFT(X, L) \).

**Theorem 3.2.2.** Let \( X \) be a finite set and \( L \) be a finite F-lattice. Let \( p: X \rightarrow X \) be a bijection and \( g: L \rightarrow L \) be an isomorphism. Define \( H_{p, g}: L^X \rightarrow L^X \) by \( H_{p, g}(C)(x) = g(C(p^{-1}(x))) \); \( C \in L^X \), \( x \in X \). For \( \delta \in LFT(X, L) \), define \( F_{p, g}^* \) by \( F_{p, g}^*(\delta) = \{ \text{comp}(H_{p, g}(C)) : C \in \delta \} \) where \( \text{comp}(H_{p, g}(C)) \) denotes the pseudo-complement of \( H_{p, g}(C) \) in \( L^X \). Then \( F_{p, g}^* \) is an automorphism of \( LFT(X, L) \).
3.2. Automorphisms of the lattice $LFT(X, L)$

**Proof:** We have $0 = \text{comp}(1) = \text{comp}(H_{p,g}(1))$ and $1 = \text{comp}(0) = \text{comp}(H_{p,g}(0))$. Since $0, 1 \in \delta$, it follows that $0, 1 \in F^*_p(g)(\delta)$.

Let $f_1, f_2 \in F^*_p(g)(\delta)$. Then $f_1 = \text{comp}(H_{p,g}(C))$ and $f_2 = \text{comp}(H_{p,g}(D))$ for some $C, D \in \delta$. We have $C, D \in \delta \implies C \land D \in \delta$. Now,

$$f_1 \lor f_2 = \text{comp}(H_{p,g}(C)) \lor \text{comp}(H_{p,g}(D))$$

$$= \text{comp}[H_{p,g}(C) \land H_{p,g}(D)]$$

$$= \text{comp}[H_{p,g}(C \land D)] \in F^*_p(g)(\delta).$$

Similarly, $f_1 \land f_2 \in F^*_p(g)(\delta)$.

Thus $F^*_p(g)(\delta)$ is a fuzzy topology on $X$.

For $\delta_1, \delta_2 \in LFT(X, L)$, let $F^*_p(g)(\delta_1) = F^*_p(g)(\delta_2)$. This implies,

$$\{\text{comp}(H_{p,g}(C)) : C \in \delta_1\} = \{\text{comp}(H_{p,g}(D)) : D \in \delta_2\} \implies \{C : C \in \delta_1\}$$

$$= \{D : D \in \delta_2\}$$

$$\implies \delta_1 = \delta_2.$$

Therefore $F^*_p(g)$ is one to one.

For $\tau \in LFT(X, L)$, consider the collection $\delta = \{H^{-1}_{p,g}(\text{comp}(C)) : C \in \tau\}$. Then $\delta$ is a fuzzy topology on $X$ and

$$F^*_p(g)(\delta) = \{\text{comp}(H_{p,g}(H^{-1}_{p,g}(\text{comp}(C)))) : C \in \tau\}$$

$$= \{\text{comp}(\text{comp}(C)) : C \in \tau\}$$

$$= \{C : C \in \tau\}$$

$$= \tau.$$
3.2. Automorphisms of the lattice $LFT(X,L)$

Therefore $F_{p,g}^*$ is onto. Also,

$$\delta_1 \subseteq \delta_2 \iff \{C : C \in \delta_1\} \subseteq \{D : D \in \delta_2\}$$

$$\iff \{H_{p,g}(C) : C \in \delta_1\} \subseteq \{H_{p,g}(D) : D \in \delta_2\}$$

$$\iff \{\text{comp}(H_{p,g}(C)) : C \in \delta_1\} \subseteq \{\text{comp}(H_{p,g}(D)) : D \in \delta_2\}$$

$$\iff F_{p,g}^*(\delta_1) \subseteq F_{p,g}^*(\delta_2).$$

Hence $F_{p,g}^*$ is an automorphism of $LFT(X,L)$. \qed

Example 3.2.3. Let $X = \{a, b\}$, $L = \{0, 1/2, 1\}$. Then the lattice $L^X = \{a^0b^0, a^1b^1, a^{1/2}b^{1/2}, a^0b^{1/2}, a^{1/2}b^0, a^1b^0, a^{1/2}b^{1/2}\}$ where $a^ib^j$; $i, j = 0, 1/2, 1$ is the map $a \longrightarrow i$ and $b \longrightarrow j$. Let $S(X)$ denote the group of bijections on $X$ and $A(L)$ denote the group of automorphisms of $L$. Then $S(X) = \{p_1, p_2\}$ where $p_1$ is the identity map on $X$ and $p_2$ is the map on $X$ which sends $a \longrightarrow b$ and $b \longrightarrow a$. $A(L)$ consists only one member $g$ which is the identity map on $L$. Thus by Theorem 3.2.1, and Theorem 3.2.2, $H_{p_1, g}^*, H_{p_2, g}^*, F_{p_1, g}^*$ and $F_{p_2, g}^*$ are the automorphisms of the lattice $LFT(X,L)$.

Example 3.2.4. Let $X = \{x, y\}$, $L = \{0, 1/2, 1\}$ where $a$ and $b$ are not comparable. Then the lattice $L^X = \{x^0y^0, x^ay^a, x^by^b, x^0y^1, x^0y^a, x^0y^b, x^1y^0, x^1y^a, x^1y^b, x^0y^b, x^1y^1\}$ where $x^iy^j$; $i, j = 0, a, b, 1$ is the map on $X$ which sends $x \longrightarrow i$ and $y \longrightarrow j$. Here, $S(X) = \{p_1, p_2\}$ where $p_1$ is the identity map on $X$ and $p_2$ is the map which sends $x \longrightarrow y$ and $y \longrightarrow x$. $A(L) = \{g_1, g_2\}$ where $g_1$ is the identity map on $L$ and $g_2$ is the map which sends $0 \longrightarrow 0$, $a \longrightarrow b$, $b \longrightarrow a$, $1 \longrightarrow 1$. Thus by Theorem 3.2.1, and Theorem 3.2.2, $H_{p_1, g_1}^*, H_{p_1, g_2}^*, H_{p_2, g_1}^*, H_{p_2, g_2}^*, F_{p_1, g_1}^*, F_{p_1, g_2}^*, F_{p_2, g_1}^*$ and $F_{p_2, g_2}^*$ are automorphisms of $LFT(X,L)$.

Remark 3.2.5. When $g : L \longrightarrow L$ is the identity map, then $H_{p,g}$ is denoted by $H_p$, $H_{p,g}^*$ is denoted by $H_p^*$ and $F_{p,g}^*$ is denoted by $F_p^*$. 


3.3. Automorphisms of $LFT(X, L)$ when $L = \{0, 1/2, 1\}$

Example 3.2.6. When $X$ contains only one element, we can identify $L^X = \{0, 1/2, 1\}$ with $L = \{0, 1/2, 1\}$. Then the only fuzzy topologies on $X$ are the discrete fuzzy topology $L^X$ and the indiscrete fuzzy topology $\{0, 1\}$. Thus $LFT(X, L) = \{\{0, 1\}, L^X\}$ whose only automorphism is the identity map which corresponds to $H_p^*$ where $p$ is the identity map on $X$.

Remark 3.2.7. When $L = \{0, 1\}$, $LFT(X, L)$ coincides with $LT(X)$, $H_{p,g}^*$ coincides with $A_p$ and $F_{p,g}^*$ coincides with $B_p$ where $A_p$ and $B_p$ are as defined in the beginning of this chapter. Note that we are identifying the subsets of $X$ as characteristic functions.

3.3 Automorphisms of $LFT(X, L)$ when $L = \{0, 1/2, 1\}$

In this section, we determine the set of all automorphisms of the lattice $LFT(X, L)$ of all fuzzy topologies on $X$, when the membership lattice $L = \{0, 1/2, 1\}$ with usual order. Note that when $L = \{0, 1/2, 1\}$, $A(L)$, the group of automorphisms of $L$ contains only one element, the identity map of $L$. Therefore as in the Remark 3.2.5, $H_{p,g} = H_p^*$, $F_{p,g}^* = F_p^*$ and $p$ is the identity map on $X$.

Definition 3.3.1. Let $X$ be a finite set and $L = \{0, 1/2, 1\}$. For a bijection $p$ on $X$, define the mappings $H_p, F_p : L^X \to L^X$ by $H_p(C)(x) = C(p^{-1}(x))$; $x \in X, C \in L^X$ and $F_p(C) = \text{pseudo-complement of } H_p(C)$ in $L^X$. [Note that $H_p$ can also be defined by $H_p(C) = \bigvee \{(p(x))_l : x_l \in C\}; C \in L^X\]}. Also define the mappings $H_p^*, F_p^* : LFT(X, L) \to LFT(X, L)$ by $H_p^*(\tau) = \{H_p(C) : C \in \tau\}$ and $F_p^*(\tau) = \{F_p(C) : C \in \tau\}$. Then $H_p^*$ and $F_p^*$ are automorphisms of $LFT(X, L)$ (refer Theorems 3.2.1 and 3.2.2).

Notation 3.3.2. Let $X$ be any non-empty set and $L$ be the lattice $\{0, 1/2, 1\}$. 
3.3. Automorphisms of $LFT(X, L)$ when $L = \{0, 1/2, 1\}$

An atom of the lattice $LFT(X, L)$ is of the form $\{0, C, 1\}$ where $0 \neq C \neq 1 \in L^X$.

Let us denote the pseudo-complement of the fuzzy point $x_{1/2}$ by $x^{1/2}$ and that of $x_1$ by $x^0$. We shall denote the atoms of the type $\{0, x_{1/2}, 1\}$ by $I_{x_{1/2}}$, atoms of the type $\{0, x^{1/2}, 1\}$ by $\bar{I}_{x_{1/2}}$, atoms of the type $\{0, x_1, 1\}$ by $I_{x_1}$ and atoms of the type $\{0, x^0, 1\}$ by $\bar{I}_{x_1}$. Also let $\eta = \{I_{x_{1/2}} : x \in X\} = \{\{0, x_{1/2}, 1\} : x \in X\}$ and $\xi = \{\bar{I}_{x_{1/2}} : x \in X\} = \{\{0, x^{1/2}, 1\} : x \in X\}$.

**Remark 3.3.3.** Note that an automorphism of a lattice maps atoms to atoms and dual atoms to dual atoms.

**Lemma 3.3.4.** An automorphism of the lattice $LFT(X, L)$ maps a fuzzy topology to a fuzzy topology having the same cardinality.

**Proof:** When $T$ is a fuzzy topology consisting of a finite number $n$ of open sets, then $T$ is larger than precisely $n - 2$ atoms. Therefore the image of $T$ under an automorphism should also be larger than precisely $n - 2$ atoms. Hence the image of $T$ consists of $n$ open sets. When $T$ is a fuzzy topology whose cardinality is infinite, say, $\alpha$, then it contains $\alpha$ atoms. Hence the image of $T$ also contains $\alpha$ atoms. Thus the cardinality of the image is also $\alpha$. \hfill $\Box$

**Lemma 3.3.5.** The join of any atom from $\eta \cup \xi$ with any atom of $LFT(X, L)$ consists at most 5 open sets.

**Proof:** For any $0 \neq B \neq 1 \in L^X$,

$$\{0, x_{1/2}, 1\} \vee \{0, B, 1\} = \{0, x_{1/2}, B, 1\} \text{ if } B(x) = 1/2 \text{ or } 1,$$

$$= \{0, x_{1/2}, B, B \vee x_{1/2}, 1\} \text{ otherwise.}$$
3.3. Automorphisms of \( LFT(X, L) \) when \( L = \{0, 1/2, 1\} \)

\[
\{0, x^{1/2}, 1\} \lor \{0, B, 1\} = \{0, x^{1/2}, B, 1\} \text{ if } B(x) = 0 \text{ or } 1/2, \\
= \{0, x^{1/2}, B, B \land x^{1/2}, 1\} \text{ if } B(x) = 1.
\]

\( \square \)

**Definition 3.3.6.** Let \( a \) and \( b \) are elements of a complete lattice, then \( a \) and \( b \) are said to be complementary if \( a \land b = 0 \) and \( a \lor b = 1 \).

**Lemma 3.3.7.** For every atom \( u \notin \eta \cup \xi \), there is an atom \( v \) of \( LFT(X, L) \) such that \( u \lor v \) consists of 6 open sets.

**Proof:** Let \( u \notin \eta \cup \xi \). Then \( u = \{0, C, 1\} \) where \( 0 \neq C \neq 1 \in L^X \) is such that \( C \neq x_{1/2} \) and \( C \neq x^{1/2} \) for any \( x \in X \).

Now we prove that there exists an atom \( v = \{0, D, 1\} \) where \( D \in L^X \) such that \( C \) and \( D \) are not comparable and not complementary.

Since \( C \neq 0, C \neq 1, C \neq x_{1/2} \) and \( C \neq x^{1/2} \), for any \( x \in X \), there exists \( x, y \in X \) such that \( x_{1/2} < C < y^{1/2} \).

If \( x_{1/2} < C \), then there exists \( z_1 \in X \) such that \( x_{1/2}(z_1) < C(z_1) \). Therefore \( C(z_1) \neq 0 \) and \( C(z_1) = 1/2 \) or \( 1 \). If \( z_1 = x \), then \( 1/2 = x_{1/2}(x) < C(x) \) and therefore \( C(x) = C(z_1) = 1 \).

If \( C < y^{1/2} \), then there exists \( z_2 \in X \) such that \( C(z_2) < y^{1/2}(z_2) \). Therefore \( C(z_2) \neq 1 \) and therefore \( C(z_2) = 0 \) or \( 1/2 \).

If \( z_2 = y \), \( C(z_2) = C(y) < y^{1/2}(y) = 1/2 \). Therefore \( C(z_2) = C(y) = 0 \).

Now define \( D \in L^X \) such that

\[
D(w) = \begin{cases} 
  x_{1/2}(z_1) & \text{if } w = z_1, \\
  y^{1/2}(z_2) & \text{if } w = z_2, \\
  C(w) & \text{otherwise.}
\end{cases}
\]
3.3. Automorphisms of $LFT(X, L)$ when $L = \{0, 1/2, 1\}$

Then $D(z_1) = x_{1/2}(z_1) < C(z_1)$. Therefore $C \not< D$ and $D(z_2) = y_{1/2}(z_2) > C(z_2)$. Therefore $D \not< C$. Thus $C$ and $D$ are not comparable.

Now we are going to show that $C$ and $D$ are not complementary.

**Claim**: $x_{1/2} \leq D$. .................................................................(1)

For otherwise, $D(x) = 0$. But $D(x) = x_{1/2}(x) = x_{1/2}(z_1)$ or $D(x) = y_{1/2}(x) = y_{1/2}(z_2)$ or $D(x) = C(x)$ by definition of $D$. But $x_{1/2} < C$. Therefore $C(x) \neq 0$ and therefore $D(x) \neq C(x)$. Therefore $D(x) = x_{1/2}(x) = x_{1/2}(z_1)$ or $y_{1/2}(x) = y_{1/2}(z_2)$. $D(x) = 0$ and $x_{1/2}(x) = 1/2$. Therefore $D(x) \neq x_{1/2}(x)$. If $D(x) = y_{1/2}(x) = y_{1/2}(z_2) \geq 1/2$, a contradiction, since $D(x) = 0$. Therefore $x_{1/2} \leq D$.

But $x_{1/2} < C$ and therefore $C \wedge D \neq 0$.

**Claim**: $D \leq y_{1/2}$. .................................................................(2)

For otherwise $D(y) = 1$. But $D(y) = x_{1/2}(y) = x_{1/2}(z_1)$ or $D(y) = y_{1/2}(y) = y_{1/2}(z_2)$ or $D(y) = C(y)$, by definition of $D$. $C(y) \leq y_{1/2}(y) = 1/2$. Therefore $C(y) \neq 1$. Therefore $D(y) \neq C(y)$. If $D(y) = x_{1/2}(y) = x_{1/2}(z_1) \leq 1/2$, a contradiction. If $D(y) = y_{1/2}(y) = y_{1/2}(z_2)$, then $y_{1/2}(y) = 1/2$. Therefore $D(y) = 1/2$, a contradiction, since $D(y) = 1$. Therefore $D \leq y_{1/2}$. Also $C < y_{1/2}$.

Therefore $C \lor D \leq y_{1/2}$ and hence $C \lor D \neq 1$.

Hence $C$ and $D$ are neither comparable nor complementary. Therefore $\{0, C, 1\} \lor \{0, D, 1\}$ contain 6 open sets.  

\[\square\]

**Lemma 3.3.8.** An automorphism of the lattice $LFT(X, L)$ maps $\eta \cup \xi$ onto itself.

**Proof:** Let $\alpha \in \eta \cup \xi$ and let $A$ be an automorphism of $LFT(X, L)$. Clearly $A(\alpha)$ is an atom. To prove $A(\alpha) \in \eta \cup \xi$. Otherwise, there exists an atom $\beta$ such that $A(\alpha) \lor \beta$ consists 6 open sets. Also there exists $\beta' \in LFT(X, L)$ such that $A(\beta') = \beta$. Clearly $\beta'$ is an atom. Now $A(\alpha) \lor \beta = A(\alpha) \lor A(\beta') = A(\alpha \lor \beta')$. Therefore $\alpha \lor \beta'$ consists 6 open sets, a contradiction.
3.3. Automorphisms of $LFT(X, L)$ when $L = \{0, 1/2, 1\}$

Also if $\alpha \in \eta \cup \xi$, consider the automorphism $A^{-1}$ and let $A^{-1}(\alpha) = \beta$. Then $A(\beta) = \alpha$. Also $\beta \in \eta \cup \xi$, for otherwise, we can similarly get a contradiction. Thus $A(\eta \cup \xi) = \eta \cup \xi$.

**Theorem 3.3.9.** Every automorphism of $LFT(X, L)$ maps either ($\eta$ onto $\eta$ and $\xi$ onto $\xi$) or ($\eta$ onto $\xi$ and $\xi$ onto $\eta$).

**Proof:** The join of any two distinct atoms in $\eta$ consists 5 open sets, because $\{0, x_{1/2}, 1\} \vee \{0, y_{1/2}, 1\} = \{0, x_{1/2}, y_{1/2}, x_{1/2} \wedge y_{1/2}, 1\}$ if $x \neq y$. The join of any two distinct atoms in $\xi$ consists 5 open sets, because $\{0, x_{1/2}, 1\} \vee \{0, y_{1/2}, 1\} = \{0, x_{1/2}, y_{1/2}, x_{1/2} \wedge y_{1/2}, 1\}$ if $x \neq y$. The join of an element from $\xi$ with an element from $\eta$ always consists 4 open sets. For, $\{0, x_{1/2}, 1\} \vee \{0, y_{1/2}, 1\} = \{0, x_{1/2}, y_{1/2}, 1\}$ since $x_{1/2} \leq y_{1/2}$ holds for all $x, y \in X$.

Let $A$ be an automorphism of $LFT(X, L)$. If $\alpha, \beta \in \eta$, and $A(\alpha) \in \eta$, then $A(\beta) \in \eta$. Otherwise, $A(\alpha) \in \eta$ and $A(\beta) \in \xi$. Therefore $A(\alpha) \vee A(\beta)$ has 4 open sets. But $A(\alpha) \vee A(\beta) = A(\alpha \vee \beta)$. Therefore $\alpha \vee \beta$ has 4 open sets, a contradiction.

Similarly, if $\alpha, \beta \in \eta$ and $A(\alpha) \in \xi$, then $A(\beta) \in \xi$. Thus either every element of $\eta$ is mapped to $\eta$ or every element of $\eta$ is mapped to $\xi$. Similarly we can prove that either every element of $\xi$ is mapped to $\xi$ or every element of $\xi$ is mapped to $\eta$.

**Theorem 3.3.10.** When $X$ is finite, the set of all automorphisms of the lattice $LFT(X, L)$ is precisely given by $\{H_p^*: p \in S(X)\} \cup \{F_p^*: p \in S(X)\}$.

**Proof:** Let $A$ be an automorphism of $LFT(X, L)$. We want to prove that $A = H_p^*$ or $A = F_p^*$ for some $p \in S(X)$. By the Theorem 3.3.9, $A$ maps either ($\eta$ onto $\eta$ and $\xi$ onto $\xi$) or ($\eta$ onto $\xi$ and $\xi$ onto $\eta$).
3.3. Automorphisms of $LFT(X, L)$ when $L = \{0, 1/2, 1\}$

Case (a) : When $A$ maps $\eta$ onto $\eta$ and $\xi$ onto $\xi$.

We will show that $A = H_p^*$ for some $p \in S(X)$. For $I_{x_{1/2}} \in \eta$, let $A(I_{x_{1/2}}) = I_{y_{1/2}}$ for some $y \in X$. This $y$ is unique. Define $p : X \to X$ by $p(x) = y$. We show that $p \in S(X)$. Let $x, x' \in X$ such that $x \neq x'$. This implies $I_{x_{1/2}} \neq I_{x'_{1/2}}$. Let $A(I_{x_{1/2}}) = I_{y_{1/2}}$ and $A(I_{x'_{1/2}}) = I_{y'_{1/2}}$. Then $p(x') = y'$. Now,

$$x \neq x' \implies I_{x_{1/2}} \neq I_{x'_{1/2}}$$

$$\implies A(I_{x_{1/2}}) \neq A(I_{x'_{1/2}})$$

$$\implies I_{y_{1/2}} \neq I_{y'_{1/2}}$$

$$\implies y \neq y'$$

$$\implies p(x) \neq p(x')$$

Therefore $p$ is one to one. Let $y \in X$ and consider $I_{y_{1/2}}$. Since $A$ is onto and $A$ maps $\eta$ onto $\eta$, there exists $x \in X$ such that $A(I_{x_{1/2}}) = I_{y_{1/2}}$. Then $y = p(x)$ and therefore $p$ is onto. Thus $p \in S(X)$. Now we prove that $A = H_p^*$ on $\eta$.

Now for $I_{x_{1/2}} \in \eta$, $A(I_{x_{1/2}}) = I_{y_{1/2}}$

$$= \{0, y_{1/2}, 1\}$$

$$= \{H_p(0), H_p(x_{1/2}), H_p(1)\}$$

$$= H_p^*(\{0, x_{1/2}, 1\})$$

$$= H_p^*(I_{x_{1/2}}).$$

Thus $A = H_p^*$ on $\eta$.

Now we will prove $A = H_p^*$ on $\xi$.

Claim : $A(I_{x_{1/2}}) = I_{y_{1/2}} \implies A(\bar{I}_{x_{1/2}}) = \bar{I}_{y_{1/2}}$. .................................................. (1)

If $0 \neq B \neq 1 \in L^X$ is not comparable with $x_{1/2}$, then $B(x) = 0$. Also if $0 \neq B \neq 1 \in L^X$ is not comparable with $x_{1/2}$, then $B(x) = 1$. Note that there
exists no complement for $x_{1/2}$ or $x^{1/2}$ in $L^X$ when $L = \{0, 1/2, 1\}$. Therefore there exists no $0 \neq B \neq \bar{1} \in L^X$ such that the join of the atom $\{0, B, \bar{1}\}$ with the atom $I_{x_{1/2}}$ and that the join of the atom $\{0, B, \bar{1}\}$ with $I_{x_{1/2}}$ contain more than 4 open sets. But if $x \neq y$, we can find an atom in $LFT(X, L)$ whose join with $I_{x_{1/2}}$ and $\bar{I}_{y_{1/2}}$ contain more than 4 open sets. For a given $I_{x_{1/2}}$ the above two properties characterize $\bar{I}_{x_{1/2}}$ in $\xi$. Also these two properties are preserved by automorphisms. Therefore it follows that, $A(I_{x_{1/2}}) = I_{y_{1/2}} \implies A(\bar{I}_{x_{1/2}}) = \bar{I}_{y_{1/2}}$.

Now,

$$A(\bar{I}_{x_{1/2}}) = \bar{I}_{y_{1/2}}$$

$$= \{0, y^{1/2}, \bar{1}\}$$

$$= \{H_p(0), H_p(x^{1/2}), H_p(1)\}$$

$$= H_p^*(\{0, x^{1/2}, \bar{1}\})$$

$$= H_p^*(\bar{I}_{x_{1/2}}).$$

Thus $A = H_p^*$ on $\xi$ and hence $A = H_p^*$ on $\eta \cup \xi$.

**Claim:** $A(I_{x_{1/2}}) = I_{y_{1/2}} \implies A(I_{x_1}) = I_{y_1}$. .........................................................(2)

Let $J_1 = \{0, x_{1/2}, \bar{1}\}$ and $J_2 = \{0, x_1, \bar{1}\}$. Since, $x_{1/2} \leq x_1, J_1 \vee J_2$ contains only 4 open sets. Also for every atom $\{0, B, \bar{1}\}$ such that $0 \neq B \neq \bar{1}$, if $J_2 \vee \{0, B, \bar{1}\}$ contains 6 open sets, then $J_2 \vee \{0, B, \bar{1}\}$ contains $J_1$. For a given $I_{x_{1/2}}$, the above two properties characterize $I_{x_1}$. Also these two properties are preserved by automorphisms. Therefore it follows that $A(I_{x_{1/2}}) = I_{y_{1/2}} \implies A(I_{x_1}) = A(I_{y_1})$.

Now take an atom $\{0, \bar{C}, 1\}$. $\bar{C} \neq C \neq 1$ which is not in $\eta \cup \xi$. To show that $A(\{0, C, 1\}) = H_p^*(\{0, C, 1\})$. Let $A(\{0, C, 1\}) = \{0, C', \bar{1}\}$ where $0 \neq C' \neq \bar{1} \in L^X$. To show that $C' = H_p(C)$ where $H_p(C) = \bigvee \{(p(x))_l : x_l \in C\}$. For this, it suffices to prove that $x_l \in C \iff (p(x))_l \in C'$ for all $x \in X$ and $l \in L, l \neq 0$.

For $x_l \in C$, let $A(I_{x_l}) = I_{(p(x))_l}$. Then $\{0, x_l, \bar{1}\} \vee \{0, C, 1\}$ contains 4 elements.
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Therefore $A(I_{x_1} \cup \{0, C, 1\}) = A(I_{x_1}) \cup A(\{0, C, 1\}) = I_{p(x)_l} \cup \{0, C', 1\}$ contain 4 elements. This implies $(p(x))_l \in C'$. For, when $l = 1/2$, $x_{1/2}$ is not complemented in $L^X$. When $l = 1, x_1 \in C \implies x_{1/2} \in C \implies (p(x))_{1/2} \in C'$. Therefore $C'$ is not the lattice complement of $(p(x))_l$. Thus $(p(x))_l \in C'$.

By a similar argument, we can show that $(p(x))_l \in C' \implies x_1 \in C$. Hence $x_l \in C \iff (p(x))_l \in C'$.

Thus $A = H^*_p$ on all atoms of $LFT(X, L)$ and since $LFT(X, L)$ is atomistic, it follows that $A = H^*_p$ on $LFT(X, L)$.

**Case (b)**: When $A$ maps $\eta$ to $\xi$ and $\xi$ to $\eta$.

We will show that $A = F^*_p$ for some $p \in S(X)$. For $I_{x_{1/2}} \in \eta$, let $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}}$ for some $y \in X$. This $y$ is unique. Define $p : X \to X$ by $p(x) = y$. We show that $p \in S(X)$. Let $x, x' \in X$ such that $x \neq x'$. This implies $I_{x_{1/2}} \neq I_{x'_{1/2}}$. Let $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}}$ and $A(I_{x'_{1/2}}) = \bar{I}_{y'_{1/2}}$. Then $p(x') = y'$. Now,

\[
\begin{align*}
x \neq x' & \implies I_{x_{1/2}} \neq I_{x'_{1/2}} \\
& \implies A(I_{x_{1/2}}) \neq A(I_{x'_{1/2}}) \\
& \implies \bar{I}_{y_{1/2}} \neq \bar{I}_{y'_{1/2}} \\
& \implies y \neq y' \\
& \implies p(x) \neq p(x').
\end{align*}
\]

Therefore $p$ is one to one. Let $y \in X$ and consider $\bar{I}_{y_{1/2}}$. Since $A$ maps $\eta$ onto $\xi$ onto, there exists $x \in X$ such that $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}}$. Then $y = p(x)$ and therefore
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$p$ is onto. Thus $p \in S(X)$.

Now for, $I_{x_{1/2}} \in \eta$, \[ A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}} \]
\[ = \{0, y^{1/2}, 1\} \]
\[ = \{F_p(1), F_p(x_{1/2}), F_p(0)\} \]
\[ = F_p^*(\{1, x_{1/2}, 0\}) \]
\[ = F_p^*(I_{x_{1/2}}). \]

Thus $A = F_p^*$ on $\eta$.

Now we will prove $A = F_p^*$ on $\xi$. Let $x \in X$. Then,

\[ F_p^*(\bar{I}_{x_{1/2}}) = F_p^*(\{0, x^{1/2}, 1\}) \]
\[ = \{F_p(0), F_p(x^{1/2}), F_p(0)\} \]
\[ = \{0, y_{1/2}, 1\} \]
\[ = I_{y_{1/2}} \text{ where } p(x) = y. \]

Therefore it suffices to show that $A(\bar{I}_{x_{1/2}}) = I_{y_{1/2}}$.

Claim : $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}} \implies A(\bar{I}_{x_{1/2}}) = I_{y_{1/2}}$. .............................................. (3)

This can be proved by a similar argument used in the proof of the claim(1).

Thus we get $A = F_p^*$ on $\xi$ and hence $A = F_p^*$ on $\eta \cup \xi$.

Claim : $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}} \implies A(I_{x_{1/2}}) = \bar{I}_{y_{1}}$. ................................................ (4)

$I_{x_{1/2}} \lor I_{x_{1}}$ contains 4 open sets and for every atom $\{0, B, 1\}, 0 \neq B \neq 1$, if the join $\{0, B, 1\} \lor I_{x_{1}}$ contains 6 open sets, then $\{0, B, 1\} \lor I_{x_{1}}$ contains $I_{x_{1/2}}$. For a given $I_{x_{1/2}}, I_{x_{1}}$ is characterized by these two properties. These two properties are preserved by automorphisms. Also the join $\bar{I}_{y_{1/2}} \lor \bar{I}_{y_{1}} = \{0, y^{1/2}, 1\} \lor \{0, y^0, 1\}$ contains only 4 open sets, since $y^0 < y^{1/2}$. Also for every atom $\{0, B, 1\}$;
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If the join $\{0, B, 1\} \cup \overline{I}_{y_1}$ contains 6 open sets, then $\{0, B, 1\} \cup \overline{I}_{y_1}$ contains $\overline{I}_{y_1/2}$. For, we have $\{0, B, 1\} \cup \{0, y^0, 1\} = \{0, 1, B, y^0, B \vee y^0, B \wedge y^0\}$. If $B(y) = 1, B \vee y^0 = 1$ and if $B(y) = 0, B \leq y^0$ and hence $B \wedge y^0 = B$. Therefore if $\{0, B, 1\} \cup \overline{I}_{y_1}$ contain 6 open sets, then, $B(y) = 1/2$. But then, $B \vee y^0 = y^{1/2}$.

Therefore $\overline{I}_{y_1/2} = \{0, y^{1/2}, 1\}$ is contained in $\{0, B, 1\} \cup \overline{I}_{y_1}$. Also for a given $\overline{I}_{y_1/2}$, $\overline{I}_{y_1}$ is characterized by these two properties. Thus we obtain the claim.

Now take an atom $\{0, D, 1\} \not\in \eta \cup \xi$. To show that $A(\{0, D, 1\}) = F_p^*(\{0, D, 1\})$ for some $p \in S(X)$. Let $A(\{0, D, 1\}) = \{0, D', 1\}$. To show that $D' = \text{comp}(p(D))$.

For this, it is enough to show that

- $D(x) = 1 \iff D'(p(x)) = 0$,
- $D(x) = 0 \iff D'(p(x)) = 1$ and
- $D(x) = 1/2 \iff D'(p(x)) = 1/2$.

By a similar kind of argument, $D(x) = 0 \iff D \leq x^0 \iff y_1 \in D' \iff D'(y) = 1 \iff D'(p(x)) = 1$.

Thus we have proved $A = F_p^*$ on all atoms of $LFT(X, L)$. Since $LFT(X, L)$ is atomistic, it follows that $A = F_p^*$ on $LFT(X, L)$.

**Remark 3.3.11.** If $X$ is infinite, there is no automorphism of $LFT(X, L)$ which maps $\eta$ to $\xi$ and $\xi$ to $\eta$. Suppose $A$ maps $\eta$ to $\xi$ and $\xi$ to $\eta$. Since automorphisms
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preserves order, we have, $A(\bigvee \{I_{x/1} : x \in X\}) = \bigvee \{A(I_{x/1}) : x \in X\}$. That is, $A(\bigvee_{x \in X} \{0, x^{1/2}, 1\}) = \bigvee_{x \in X} \{0, x^{1/2}, 1\}$. But this is not possible, because when $X$ is infinite, the cardinality of L.H.S. is $2^{|X|}$ while the cardinality of R.H.S. is $|X|$ where $|X|$ denote the cardinality of the set $X$. Thus we obtain, the following result.

**Theorem 3.3.12.** When $X$ is infinite and $L$ is the lattice $\{0, 1/2, 1\}$, the set of all automorphisms of the lattice $LFT(X, L)$ is precisely $\{H_p^* : p \in S(X)\}$.

**Remark 3.3.13.** In view of the Theorem 3.3.10 and Theorem 3.3.12 we shall make the following conjectures.

**Conjecture 3.3.14.** Let $X$ be a finite set and $L$ be a finite $F$-lattice. Let $S(X)$ denote the set of all bijections on $X$ and $A(L)$ denote the set of all automorphisms of $L$. For $p \in S(X)$ and $g \in A(L)$, define the mappings $H_{p,g}, F_{p,g} : L^X \rightarrow L^X$ by $H_{p,g}(C)(x) = g(C(p^{-1}(x)))$ and $F_{p,g}(C) =$ pseudo-complement of $H_{p,g}(C)$ in $L^X$ where $C \in L^X$ and $x \in X$. Also define the mappings $H_{p,g}^*$ and $F_{p,g}^*$ : $LFT(X, L) \rightarrow LFT(X, L)$ by $H_{p,g}^*(\delta) = \{H_{p,g}(A) : A \in \delta\}$ and $F_{p,g}^*(\delta) = \{F_{p,g}(A) : A \in \delta\}$ for $\delta \in LFT(X, L)$. Then the set of all automorphisms of the lattice $LFT(X, L)$ is precisely given by the set $\{H_{p,g}^* : p \in S(X), g \in A(L)\} \cup \{F_{p,g}^* : p \in S(X), g \in A(L)\}$.

**Conjecture 3.3.15.** Let $X$ be an infinite set and $L$ be a finite $F$-lattice. Let $S(X)$ denote the set of all bijections on $X$ and $A(L)$ denote the set of all automorphisms of $L$. For $p \in S(X)$ and $g \in A(L)$, define the mapping $H_{p,g} : L^X \rightarrow L^X$ by $H_{p,g}(C)(x) = g(C(p^{-1}(x)))$. Also define the mapping $H_{p,g}^* : LFT(X, L) \rightarrow LFT(X, L)$ by $H_{p,g}^*(\delta) = \{H_{p,g}(A) : A \in \delta\}$ for $\delta \in LFT(X, L)$. Then the set of all automorphisms of $LFT(X, L)$ is precisely given by the set $\{H_{p,g}^* : p \in S(X), g \in A(L)\}$. 