Chapter 0

Introduction and Summary

Symmetries are used to study the dynamics of a physical system. In classical mechanics symmetries are usually induced by point transformations, that is, they come exclusively from symmetries of the configuration space. The symmetry based techniques are implemented using integrals of motion which are quantities that are conserved along the flow of that system. This idea can be generalized to many symmetries of the entire phase space of the dynamical system. This is done by associating a map from the phase space to the dual of the Lie algebra of the Lie group which is acting on the phase space encoding the symmetry. This map, whose level sets are preserved by the dynamics of any symmetric system is referred as the Momentum map (Standard Momentum map) of the symmetry. Momentum maps are at the centre of many geometrical facts that are useful in variety of fields of both pure and applied Mathematics. Also these maps are very much useful in Physics and Engineering applications.

This thesis grew out of our study on Momentum maps. In this thesis we present the existence results, elementary properties, convexity properties and certain generalizations of the standard momentum maps. The thesis contains four
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In the first chapter main focus is on the standard momentum map associated to Lie group action on a symplectic manifold. The concept of momentum map and some existence results are given. Also some elementary properties of momentum map are discussed. This chapter contains five sections.

In the first section we have given the basic ideas required. Many of the standard results are recalled. First we give the definitions of Lie group action, proper action, Lie algebra action, symplectic manifold, symplectomorphism, Lagrange submanifold. Also we state some basic theorems on symplectic manifolds. Then the definitions of Hamiltonian vector field, Hamiltonian functions, Poisson manifold, Poisson tensor, canonical mappings, Hamiltonian and Poisson dynamical systems are given. The canonical Lie group and Lie algebra actions, almost Hamiltonian actions and Hamiltonian actions are also discussed.

In section 2 the notion of Noether momentum map on a symplectic manifold is introduced.

**Definition 1.2.1** Let \((M, \{\ldots\})\) be a Poisson manifold and \(G\) (respectively \(G\)) a Lie group (respectively Lie algebra) acting canonically on it. Let \(S\) be a set and \(J : M \to S\) a map. We say that \(J\) is a Noether momentum map for the \(G\)-action (respectively \(G\)-action) on \((M, \{\ldots\})\) when the flow \(F_t\) of any Hamiltonian vector field associated to any \(G\)-invariant (respectively \(G\)-invariant) Hamiltonian function \(h \in C^\infty(M)\) preserves the fibers of \(J\). That is,

\[
J \circ F_t = J |_{\text{Dom}(F_t)}.
\]

Then given Chu momentum map whose definition makes essential use of the symplectic structure and some properties are also proved in section 3.

**Definition 1.3.1** Let \((M, \omega)\) be a symplectic manifold and \(G\) be a Lie algebra
acting canonically on it. The *Chu map* is defined as the map \( \Psi : M \to \mathbb{Z}_2(\mathcal{G}) \) given by

\[
\Psi(m)(\xi, \eta) = \omega(m)(\xi_M(m), \eta_M(m)),
\]

for every \( \xi, \eta \in \mathcal{G} \). The fact that \( \Psi \) maps into \( \mathbb{Z}_2(\mathcal{G}) \) is a consequence of the closedness of the symplectic form \( \omega \) and the canonical character of the \( \mathcal{G} \)-action.

Apart from its intrinsic interest as a Noether momentum map, this construction will be extremely important in the statement and proof of a symplectic version of slice theorem, presented in chapter 3.

In section 4 the standard momentum map, whose values are in the dual of Lie algebra of symmetries, is given.

**Definition 1.4.1** Let \( \mathcal{G} \) be a Lie algebra acting canonically on the symplectic manifold \((M, \omega)\). Suppose that for any \( \xi \in \mathcal{G} \), the vector field \( \xi_M \) (infinitesimal generator) is globally Hamiltonian with Hamiltonian function \( J^\xi \in C^\infty(M) \). The map \( J : M \to \mathcal{G}^* \) defined by the relation

\[
< J(z), \xi >= J^\xi(z),
\]

for all \( \xi \in \mathcal{G} \) and \( z \in M \), is called a *standard momentum map* or simply a momentum map of the \( \mathcal{G} \)-action.

After giving examples of such momentum maps, we consider the problem of existence of momentum maps. Its existence is guaranteed when the infinitesimal generators of this action are Hamiltonian vector fields. In other words, if the Lie algebra \( \mathcal{G} \) acts canonically on the Poisson manifold \((M, \{\ldots\})\), then for each \( \xi \in \mathcal{G} \), we require the existence of a globally defined function \( J^\xi \in C^\infty(M) \) such that \( \xi_M = X_{J^\xi} \). In general this is not guaranteed even if there is a canonical Lie
algebra action.

Then various situations on the existence of such momentum maps is given. The following result characterizes the existence of momentum maps in the symplectic case.

**Proposition 1.4.8** Let \((M, \omega)\) be a symplectic manifold and \(\mathcal{G}\) a Lie algebra acting canonically on it. There exists a momentum map associated to this action if and only if the linear map \(\rho : \mathcal{G} \to H^1(M, \mathbb{R})\), by \(\rho([\xi]) = [i_{\xi_\omega} \omega]\) is identically zero.

As a consequence of this proposition we have the following theorem.

**Theorem 1.4.9** Let \((M, \omega)\) be a symplectic manifold and \(\mathcal{G}\) be a Lie algebra acting canonically on it. There exists a momentum map associated to this action if and only if one of the following is true.

1. \(H^1(M, \mathbb{R}) = 0\).
2. \(\mathcal{G} = [\mathcal{G}, \mathcal{G}]\).
3. \(H^1(\mathcal{G}, \mathbb{R}) = 0\).
4. \(\mathcal{G}\) is semisimple.

Then we look at the coadjoint or \(G\)-equivariant momentum maps. Existence results of such momentum maps are given using fixed points of the action, Lie algebra cohomology, \(G\)-invariant 1-form on \(M\) and compact Lie group action. The existence of coadjoint equivariant moment maps for the action of semidirect product \(G_1 \times_{\sigma} G_2\) is also given using conditions on \(G_1\). We have proved two theorems on the existence of coadjoint equivariant momentum maps on the product manifold.

**Theorem 1.4.19** Let \((M_1, \omega_1)\) and \((M_2, \omega_2)\) be two symplectic manifolds and let \(G\) be a Lie group acting canonically on both \(M_1\) and \(M_2\). Suppose the above actions admit coadjoint equivariant momentum maps. Then \(G\) has a coadjoint equivariant momentum map on \((M_1 \times M_2, \pi_1^* \omega_1 - \pi_2^* \omega_2)\) where \(\pi_1\) and \(\pi_2\) are pro-
jections on $M_1$ and $M_2$ respectively.

**Theorem 1.4.20** Let $G_1$ and $G_2$ be Lie groups acting canonically on connected symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ with $G_1$ is connected and $H^1(G_1, \mathbb{R}) = 0$. Suppose the above actions admits coadjoint equivariant momentum maps. If $G = G_1 \times G_2$ then $G$ has a coadjoint equivariant momentum map on $(M_1 \times M_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$ where $\pi_1$ and $\pi_2$ are projections on $M_1$ and $M_2$ respectively.

Infinitesimally equivariant momentum map is defined for the Lie algebra action and the results on existence using the extension of Lie algebra is also given.

In general it is not possible to choose a coadjoint equivariant momentum map, but we could ask whether one can define another action on this space with respect to which we have equivariance.

**Definition** Let $G$ be a Lie group acting canonically on the connected symplectic manifold $(M, \omega)$ with associated momentum map $J : M \to \mathcal{G}^*$. If $\sigma : G \to \mathcal{G}^*$ is the non-equivariance one cocycle of $J$, we define the affine action of $G$ on $\mathcal{G}^*$ with cocycle $\sigma$ by

\[
\Theta : G \times \mathcal{G}^* \to \mathcal{G}^*, \text{ given by } \\
\Theta(g, \mu) = \text{Ad}_g^{-1}\mu + \sigma(g).
\]

**Proposition 1.4.34** : The affine action $\Theta$ of $G$ on $\mathcal{G}^*$ determines a left action. The momentum map $J : M \to \mathcal{G}^*$ is equivariant with respect to the symplectic action $\phi$ on $M$ and the affine action $\Theta$ on $\mathcal{G}^*$.

In the last section we discuss certain properties of momentum maps. First we prove that the momentum map $J$ is a submersion on the open dense subset of principal orbits in $M$. Then the Noether’s theorem, that is, they are constant on the dynamics of any symmetric Hamiltonian vector field is given. An equivalent
condition for the moment map to be constant on the orbits is given. We establishes
a link between the symmetry of a point and the rank of the momentum map at
the point, called bifurcation lemma. Also proved that the zero level set of the
moment map is locally arc wise connected.

In chapter 2 we consider the action of a torus $T^n$ on a symplectic manifold
$(M, \omega)$. Hamiltonian actions of tori of maximal dimension are a special case of
integrable systems. More than that they are the local form of all integrable
systems with compact level sets. Convexity property of momentum map for the
torus action using Morse theory is also discussed. This chapter contains 2 sections.
In the first section we define Hamiltonian torus action and give examples of it.
Then we prove a Hamiltonian circle action on a compact symplectic manifold has
fixed point.

One of the most striking aspects of momentum maps is the convexity properties
of its image. In section 2 we discuss the convexity properties of momentum map.
We discuss the developments in this area starting from the first convexity result
by Atiyah, Guillemin and Sternberg. They have proved the convexity theorem for
compact $M$ on which a torus acts in a Hamiltonian fashion. Then Guillemin and
Sternberg conjectured and partially proved the convexity theorem to actions of
non-abelian compact groups on compact manifolds. This was completely proved
by Kirwan.

First we strengthen the Poincare Lemma to deal with invariant forms. Then
Darboux theorem for momentum maps is given using the G-relative Darboux
theorem.

**Theorem 2.2.9** If $J$ is the momentum map for a Hamiltonian action of a
torus $T$ on the symplectic manifold $(M, \omega)$ and $m$ a $T$-fixed point then there is an
invariant neighborhood $U$ of $m$ in $M$ and a neighborhood $U'$ of $J(m)$ in $T^*$ such
that \( J(U) \) is \( U' \cap (J(m) + C(\alpha_1, \alpha_2, ..., \alpha_n)) \) where \( T \) is the Lie algebra of \( T \) and \( C(\alpha_1, \alpha_2, ..., \alpha_n) \) is the positive cone spanned by the weights \( \alpha_1, \alpha_2, ..., \alpha_n \) of the action of \( T \) on \( M \).

Now to improve it again, that is, for \( M \) compact \( J(M) \) is a compact convex polytope, Morse theory is used. Then Atiyah - Guillemin - Sternberg convexity theorem is proved.

**Theorem 2.2.27** Let \((M, \omega)\) be a compact connected symplectic manifold, and let \( T \) be a torus acts in a Hamiltonian fashion with associated invariant momentum map \( J : M \to T^* \). Here \( T \) denotes the Lie algebra of \( T \) and \( T^* \) its dual. Then the image \( J(M) \) of \( J \) is a compact convex polytope, called the \( T \)-momentum polytope. Moreover, it is equal to the convex hull of the image of the fixed point set of the \( T \)-action. The fibers of \( J \) are connected.

As a corollary of this convexity theorem, if the \( T \)-action is effective, then there must be at least \( m + 1 \) fixed points and \( \dim M \geq 2m \) where \( 2m \) is the dimension of the torus.

Then prove the convexity theorem to actions of non-abelian compact groups on compact manifolds.

**Theorem 2.2.30** Let \( M \) be a compact connected symplectic manifold on which the compact connected Lie group \( G \) acts in a Hamiltonian fashion with associated equivariant momentum map \( J : M \to G^* \). Here \( G \) denotes the Lie algebra of \( G \) and \( G^* \) is its dual. Let \( T \) be a maximal torus of \( G \), \( T \) its Lie algebra, \( T^* \) its dual, and \( T_+^* \) the positive Weyl chamber relative to a fixed ordering of the roots. Then \( J(M) \cap T_+^* \) is a compact convex polytope, called the \( G \)-momentum polytope. The fibers of \( J \) are connected.

In general Morse theory is not sufficient to study convexity properties of the image of the momentum map. The case of compact symplectic manifolds is rich
but quite particular. For noncompact manifolds the results in the previous chapter no longer hold. Convexity results to compact group actions on noncompact manifolds with proper momentum maps were given by Condevaux, Dazord, and Molino and later by Hilgert, Neeb and Plank. The Lokal-global-prinzip is the main tool in these works. Yael Karshon And Christina Marshall gave a generalization of Lokal-global-prinzip for a proper map. But Petre Birtea, Juan-Pablo Ortega and Tudor S.Ratiu gave a generalization of Lokal-global-prinzip for a closed map. Using this, many stronger results in convexity are obtained.

In chapter 3 we discuss the convexity property for a general Lie group action using topological properties. This chapter contains 3 sections. The essential attributes underlying the convexity theorems for momentum maps are the openness of the map onto its image and the local convexity data. The classical convexity theorems given in Chapter 2 are also satisfy these conditions. In this chapter more general theorems on convexity are given using the topological ingredients.

To do convexity results using topological properties we need normal form for the momentum map which we have discussed in section 1. Most of the technical behavior of proper Lie group action is a direct consequence of the existence of slices and tubes; they provide a privileged system of semiglobal coordinates in which the group action takes on a particularly simple form. Proper symplectic Lie group actions turnout to behave similarly: the tubular chart can be constructed in such a way that the expression of the symplectic form is very natural and, moreover, if there is a momentum map associated to this canonical action, this construction provides a normal form for it. We start with the Witt-Artin decomposition of the tangent space. Then the construction of a symplectic tube \((Y_r, \omega_Y)\) at a point \(m\) of a symplectic manifold \((M, \omega)\) is given. Then the symplectic slice theorem is given.

**Theorem 3.1.7** Let \((M, \omega)\) be a symplectic manifold and \(G\) be a Lie group acting
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properly and canonically on it. Let \( m \in M \), and let \( (Y_r, \omega_{Y_r}) \) be the \( G \)-symplectic tube at that point constructed in proposition 3.1.6. Then there is a \( G \)-invariant neighborhood \( U \) of \( m \) in \( M \) and a \( G \)-equivariant symplectomorphism \( \phi : U \rightarrow Y_r \) satisfying \( \phi(m) = [e, 0, 0] \).

Tubewise Hamiltonian action is defined and sufficient conditions for the action to be tubewise Hamiltonian also given. Then the expression of the momentum map in the slice coordinate, which is usually referred to as the Marle-Guillemin-Sternberg normal form is given. 

**Theorem 3.1.12** Let \((M, \omega)\) be a connected symplectic manifold and \( G \) be a Lie group acting properly and canonically on it. Suppose that this action has an associated momentum map \( J : M \rightarrow G^* \) with non equivariance cocycle \( \sigma : G \rightarrow G^* \). Let \( m \in M \), and \((Y_r, \omega_{Y_r})\) be the symplectic tube at \( m \) that models a \( G \)-invariant open neighborhood \( U \) of the orbit \( G.m \) via the \( G \)-equivariant symplectomorphism \( \phi : (U, \omega_U) \rightarrow (Y_r, \omega_{Y_r}) \). Then the canonical left \( G \)-action on \((Y_r, \omega_{Y_r})\) admits a momentum map \( J_{Y_r} : Y_r \rightarrow G^* \) given by the expression

\[
J_{Y_r}([g, \rho, \upsilon]) = Ad_{g^{-1}}^*(J(m) + \rho + J_V(\upsilon)) + \sigma(g).
\]

The map \( J_{Y_r} \times \phi \) is a momentum map for the canonical \( G \)-action on \((U, \omega_U)\). Moreover, if the group \( G \) is connected, this momentum map satisfies \( J \mid_U = J_{Y_r} \times \phi \).

In section 2 we discuss the convexity properties of the image of the momentum map. We give the statement of Lokal-global-prinzip and a generalization of it for a closed map using some topological vector space results.

**Theorem 3.2.14** Let \( f : X \rightarrow V \) be a closed map with values in a finite dimensional Euclidean vector space \( V \) and \( X \) a connected, locally connected, first countable, and normal topological space. Assume that \( f \) has local convexity data
and is locally fiber connected. Then

(i) All the fibers of $f$ are connected.

(ii) $f$ is open on to its image.

(iii) The image $f(X)$ is a closed convex set.

Using this we obtained that the convexity is rooted on the map being open onto its image and having local convexity data. Next we look at the convexity for momentum maps. Then a generalization of Atiyah-Guillemin-Sternberg Convexity theorem for non compact manifolds is given.

**Theorem 3.2.21** Let $M$ be a paracompact connected symplectic manifold on which a torus $T$ acts in a Hamiltonian fashion. Let $J_T : M \to T^*$ be an associated momentum map which we suppose is closed. Then the image $J_T(m)$ is a closed convex locally polyhedral subset in $T^*$. The fibers of $J_T$ are connected and $J_T$ is open on to its image.

Further generalization of convexity results are obtained in two cases: when the momentum map has connected fibers and the case when the momentum map has only the locally fiber connectedness property. In the first case we have the theorem :

**Theorem 3.2.28** Let $J_T : M \to T^*$ be the momentum map of a torus action which has connected fibers. Then $J_T$ is open on to its image if and only if $CJ_T(M^{reg}) := J_T(M) \setminus J_T(M^{reg})$ (where $M^{reg}$ denotes the union of all regular orbits) does not disconnect any region in $J_T(M)$. Moreover, the image of the momentum map is locally convex and locally polyhedral.

In the second case suppose that $M_{J_T}$ is a Hausdorff space. Then we have the following theorem.

**Theorem 3.2.34** Let $J_T : M \to T^*$ be the momentum map of a torus action of a connected symplectic manifold $(M, \omega)$. Suppose that $M_{J_T}$ is a Hausdorff space. Then $J_T$ is open on to its image if and only if $J_T(M)$ is locally compact,
$CJ_T(M^{reg})$ does not disconnect any region in $J_T(M)$, and $J_T$ satisfies the connected component fiber condition. Moreover under these hypothesis the image of the momentum map is locally convex and locally polyhedral.

Next a generalization of Kirwan’s convexity result for a paracompact connected symplectic manifold $(M, \omega)$ is given.

**Theorem 3.2.39** Let $M$ be a paracompact connected Hamiltonian $G$-manifold with $G$ a compact connected Lie group. If the momentum map $J_G$ is closed then $J_G(M) \cap T^*_+$ is a closed convex locally polyhedral set. Moreover, $J_G$ is $G$-open on to its image and all its fibers are connected.

Also we prove non-abelian analogues of the Theorems 3.2.28 and 3.2.34.

Pull backs by $J$ of smooth functions on $G^*$ are called collective functions. A collective function is clearly constant on the level sets of the momentum map. The converse need not be true. A momentum map has the division property if any smooth function on $M$ that is locally constant on the level set of $\Phi$ is a collective function. In section 3 we generalize a result on division property of momentum map by replacing the compactness of the Lie group with proper and effective action.

**Theorem 3.3.13** Let $G$ be a connected abelian Lie group acting properly and effectively on a connected symplectic manifold $(M, \omega)$. Let $J : M \rightarrow G^*$ be a proper momentum map associated to this action. Then $J$ has the division property if and only if every smooth function on $M$ that is locally constant on the level set of $\Phi$ is a collective function.

Then we prove that Torus action has division property if $J_T$ is closed and semi-proper.

**Theorem 3.3.17** Let $M$ be a paracompact connected symplectic manifold on which a torus $T$ acts in a Hamiltonian fashion. If the associated momentum map
$J_T$ is closed and semi proper as a map into some open subset of $T^*$, then $J$ has the division property.

Then considered the general case. Let $G$ be a compact connected Lie group acting on a compact connected symplectic manifold $M$ in a Hamiltonian fashion with a momentum map $J: M \to G^*$. Put a $G$-invariant metric on $G^*$, and use it to identify $G^*$ with $G$. Let $G_{reg}$ be the elements of $G$ whose stabilizers under the coadjoint action of $G$ are tori, that is, if,

$$G_{reg} = \{ \xi \in G : \text{stabilizer of } \xi \text{ is a torus} \}.$$ 

Then we prove the following theorem.

**Theorem 3.3.21** Let $M$ be a paracompact connected symplectic Hamiltonian $G$-manifold with $G$ a compact connected Lie group. If the associated momentum map $J$ is closed and semi proper as a map into some open subset of $G^*$, then $J$ has the division property if the image $J(M)$ is contained the $G_{reg}^*$.

In chapter 4 we discuss certain generalizations of standard momentum map. This chapter contains 3 sections. The first section is on cylinder valued momentum maps, which has the important property of being always defined, unlike the standard momentum map. To introduce cylinder valued momentum maps, we need connections on a principal fiber bundle. Then we define holonomy bundle and some properties are discussed. The definition of cylinder valued momentum map is given as a generalization of the standard momentum map.

**Definition 4.1.12** For $(z, \mu) \in M \times G^*$, let $M \times G^*(z, \mu)$ be the holonomy bundle through $(z, \mu)$ and let $h(z, \mu)$ be the holonomy group of $\alpha$ with reference point $(z, \mu)$. The reduction theorem guarantees that $(M \times G^*(z, \mu), M, \pi_{M \times G^*(z, \mu)}, h(z, \mu))$ is a reduction of $(M \times G^*, M, \pi, G^*)$ For simplicity we use $(\tilde{M}, M, \tilde{P}, \tilde{h})$ instead of $(M \times G^*(z, \mu), M, \pi_{M \times G^*(z, \mu)}, h(z, \mu))$. Let $\tilde{K} : \tilde{M} \subset M \times G^* \to G^*$ be the projection into the $G^*$-factor.
Consider now the closure $\overline{h}$ of $h$ in $G^*$. Since $\overline{h}$ is a closed subgroup of $(G^*, +)$, the quotient $D := \frac{G^*}{\overline{h}}$ is a cylinder, that is, it is isomorphic to the abelian Lie group $\mathbb{R}^a \times T^b$ for some $a, b \in \mathbb{R}$. Let $\pi_D : G^* \to \frac{G^*}{\overline{h}}$ be the projection. Define $K : M \to \frac{G^*}{\overline{h}}$ to be the map that makes the following diagram commutative:

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{K} & G^* \\
\downarrow \tilde{P} & & \downarrow \pi_D \\
M & \xrightarrow{K} & \frac{G^*}{\overline{h}}
\end{array}
$$

In other words, $K$ is defined by $K(m) = \pi_D(v)$, where $v \in G^*$ is any element such that $(m, v) \in C$. This is well defined because if we have two points $(m, v), (m, v') \in \tilde{M}$, then $(m, v), (m, v') \in \tilde{P}^{-1}(m)$, that is, there exists $\rho \in \overline{h}$ such that $v' = v + \rho$. So $\pi_D(v) = \pi_D(v')$.

Then the map $K : M \to \frac{G^*}{\overline{h}}$ is referred as a cylinder valued momentum map associated to the canonical $G$ action on $(M, \omega)$. The definition of $K$ depends on the choice of the holonomy bundle, that is, if we let $\tilde{M}_1$ and $\tilde{M}_2$ are two holonomy bundles of $(M \times G^*, M, \pi, G^*)$. Then

$$
K_{M_1} = K_{M_2} + \pi_D(\tau)
$$

where $\tau \in G^*$.

We look at certain properties of Cylinder valued momentum maps. Cylinder valued momentum maps are genuine generalizations of the standard ones in the sense that whenever a Lie algebra action admits a standard momentum map, there is a cylinder valued momentum map that coincides with it.

**Proposition 4.1.16** Let $(M, \omega)$ be a connected paracompact symplectic manifold and $G$ a Lie algebra acting canonically on it. Let $K : M \to \frac{G^*}{\overline{h}}$ be a cylinder valued momentum map. Then there exists a standard momentum map if and only if
\[ h = \{0\} \]. In this case \( K \) is a standard momentum map.

In section 2 we discuss Lie group valued momentum maps. We define Lie group valued momentum maps and then show that it is a Noether Momentum Map.

**Definition 4.2.1** Let \( G \) be an Abelian Lie Group whose Lie algebra \( G \) acts canonically on a symplectic manifold \((M, \omega)\). Let \((.,.)\) be some bilinear symmetric non-degenerate form on the Lie algebra \( G \). The map \( J : M \to G \) is called a \( G \)-valued momentum map for the \( G \)-action on \( M \) whenever

\[ i_{\xi_m} \omega(m).v_m = (T_m(L_{J(m)}^{-1} \circ J)(v_m), \xi) \]

for any \( \xi \in G \), \( m \in M \), and \( v_m \in T_m M \).

For abelian symmetries, cylinder valued momentum maps are closely related to the so-called Lie group valued momentum maps. This relationship is discussed in detail.

**Theorem 4.2.4** Let \((M, \omega)\) be a connected paracompact symplectic manifold and \( G \) Abelian Lie algebra acting canonically on it. Let \( h \subset G^* \) be the holonomy group associated to the connection \( \alpha \) associated to the \( G \)-action and let \((.,.) : G \times G \to \mathbb{R} \) be a bilinear symmetric non degenerate form on \( G \). Let \( f : G \to G^* \), \( \tilde{f} : \mathbb{G} \to \mathbb{G}^* \) and let \( T := f^{-1}(h) \) be as in the statement of above proposition. Let \( G \) be a connected Abelian Lie group whose Lie algebra is \( G \) and suppose that there exists a \( G \)-valued momentum map \( A : M \to G \) associated to the \( G \)-action whose definition uses the form \((.,.)\)

(i) If \( \exp : G \to G \) is the exponential map, then \( h \subset f(\text{Ker } \exp) \).

(ii) \( h \) is closed in \( G^* \).

Let \( J := \tilde{f}^{-1} \circ K : M \to \mathbb{G} \), where \( K \) is a cylinder valued momentum map for the \( G \)-action on \((M, \omega)\). If \( f(\text{Ker } \exp) \subset h \), then \( J : M \to \mathbb{G} = \mathbb{G}_{\text{Ker } \exp} \simeq G \) is a \( G \)-valued momentum map that differs from \( A \) by a constant in \( G \). Conversely, if
\[ h = f(Ker \exp), \text{ then } J : M \rightarrow \frac{\mathcal{g}}{Ker \exp} \simeq G \] is a \( G \)-valued momentum map.

In section 3 we discuss a generalization of the standard momentum map not involving the group action. The classical notion of momentum map from Weinstein’s point of view is given first. To do this we recall some ideas related to the symplectic category. Then we look at the standard momentum map in a more general set up as a map \( \tilde{J} : M \times G \rightarrow G^* \). In this case we have shown that \( \tilde{J} \) is a momentum map. Then introduce the notion of generalization of the momentum map, where the group action is replaced by a family of symplectomorphisms.

Let \((M, \omega)\) be a symplectic manifold, \(S\) an arbitrary manifold and \(f_s, s \in S\), a family of symplectomorphisms of \(M\) depending smoothly on \(s\). For \(p \in M\) and \(s_o \in S\), let \(g_{s_o,p} : S \rightarrow M\) be the map, \(g_{s_o,p}(s) = f_s \circ f_s^{-1}(p)\). Then the derivative at \(s_o\) is given by

\[
(dg_{s_o,p})_{s_o} : T_{s_o}S \rightarrow T_pM.
\]

From this we get the linear map

\[
(dg_{s_o,p})_{s_o} : T_{s_o}S \rightarrow T_p^*M.
\]

Now, let \(J\) be the map of \(M \times S\) into \(T^*S\) which is compatible with the projection, \(M \times S \rightarrow S\) in the sense

\[
M \times S \xrightarrow{J} T^*S \xrightarrow{\downarrow} S
\]

commutes; and for \(s_o \in S\) let \(J_{s_o} : M \rightarrow T_{s_o}^*S\) be the restriction of \(J\) to \(M \times \{s_o\}\).
Definition 4.3.14 \( J \) is a \textit{momentum map} if, for all \( s \) and \( p \),

\[
(dJ_{s_0})_p : T_pM \to T^*_sS
\]

is the transpose of the map \((dg_{s_0,p})_{s_0}\).

Then a sufficient condition for the existence of momentum map can be done in a more general set up. We do it in a more general set up which does not involve the group action. After giving a sufficient condition for the existence of momentum map, we have recaptured a generalization of standard momentum map by family of symplectomorphisms and the momentum map associated to Hamiltonian group action.

Let \((M,\omega)\) be a symplectic manifold. Let \(Z, X \) and \( S \) be manifolds and suppose that \( \pi : Z \to S \) is a fibration with fibers diffeomorphic to \( X \). Let \( G : Z \to M \) be a smooth map and let \( g_s : Z_s \to M, Z_s := \pi^{-1} s \)

denote the restriction of \( G \) to \( Z_s \). We assume that \( g_s \) is a Lagrangian embedding and let \( \Lambda_s := g_s(Z_s) \) denote the image of \( g_s \). Thus, for each \( s \in S \), \( G \) imbeds the fiber, \( Z_s = \pi^{-1} s \), into \( M \) as the Lagrangian submanifold, \( \Lambda_s \). Let \( s \in S \) and \( \xi \in T_sS \). For \( z \in Z_s \) and \( w \in T_zZ_s \) tangent to the fiber \( Z_s \),

\[
dG_z w = (dg_s)_z w \in T_{G(z)} \Lambda_s.
\]

So, \( dG_z \) induces a map, which by abuse of language, we will continue to denote by \( dG_z \)

\[
dG_z : \frac{T_zZ}{T_zZ_s} \to \frac{T_mM}{T_m\Lambda_s}, \quad m = G(z).
\]
But \( d\pi_z \) induces an identification \( \frac{T_s Z}{T_z Z_s} = T_s S. \)

Furthermore, we have an identification \( \frac{T_m M}{T_m \Lambda_s} = T_m \Lambda_s. \)

Using the identifications, we have \( dG_z : T_s Z \rightarrow T_s^* Z_s. \) Now, let \( J : Z \rightarrow T^* S \) be a lifting of \( \pi : Z \rightarrow S, \) so that

\[
\begin{array}{ccc}
Z & \xrightarrow{J} & T^* S \\
\pi & \downarrow & \downarrow \\
S & & 
\end{array}
\]

commutes, and for \( s \in S, \) let \( J_s : Z_s \rightarrow T_s^* S \) be the restriction of \( J \) to \( Z_s. \)

**Definition 4.3.16** \( J \) is a momentum map if, for all \( s \) and all \( z \in Z_s, \)

\[
(dJ_s)_z : T_z Z_s \rightarrow T_s^* S
\]

is the transpose of \( dG_z. \)

We have an embedding \((G, J) : Z \rightarrow M \times T^* S.\) from the momentum map \( J : Z \rightarrow T^* S. \) Then we prove a theorem on the existence of momentum maps.

**Theorem 4.3.17** Let \((M, \omega)\) be a symplectic manifold. Let \( Z, X \) and \( S \) be manifolds and suppose that \( \pi : Z \rightarrow S \) is a fibration with fibers diffeomorphic to \( X. \) Let \( G : Z \rightarrow M \) be a smooth map and \( J \) is a momentum map. The pull back by \((G, J)\) of the symplectic form on \( M \times T^* S \) is the pull back by \( \pi \) of a closed two form \( \rho \) on \( S. \) If \([\rho] = 0, \) there exists a momentum map, \( J, \) for which the imbedding \((G, J)\) is Lagrangian.

**Theorem 4.3.18** Let \( J \) be a map of \( Z \) into \( T^* S \) lifting the map, \( \pi, \) of \( Z \) into \( S. \) Then, if the imbedding \((G, J)\) is Lagrangian, \( J \) is a momentum map.