INTRODUCTION

It is well known that a distributive complemented lattice is a Boolean Algebra which is equivalent to a Boolean Ring with identity. This relation gives a link between Lattice Theory and Modern Algebra. The algebraic structure connecting lattice and group is called $\ell$-group or lattice ordered group. Many common abstractions, namely Dually residuated lattice ordered semigroups, lattice ordered commutative groups, lattice ordered rings and lattice ordered semirings are presented in [25], [20], [15] and [24] respectively.

Ore, O., in [22] has introduced and developed the concept of distributive elements in a lattice. Gratzer, G., and Schmidt, E.T in [10] have brought forward and characterized the standard elements in a lattice and Birkhoff, G., in [1] has introduced neutral elements in lattices. Later Birkhoff, G., Gratzer, G., Schmidt, E.T., Hashimoto, J., Kinugawa, S., and Iqbalunnisa have developed many equivalent conditions for an element of a lattice to be neutral.

The concept of distributive ideal, standard ideal and neutral ideal are called distributive element, standard element and neutral element in the ideal lattice $I(L)$ of a lattice $L$ which have been introduced and analysed by Hashimoto, J., Gratzer, G., and Schmidt, E.T.,

In the present study of commutative lattice ordered ring or commutative \(\ell\)-ring the following problems are discussed.

1. To “Develop a common abstraction which includes Boolean Algebras (Rings) and \(\ell\) - groups as special cases” posed by Birkhoff, G., (Problem No.115 in [2])

2. The problem posed by Hashimoto, J., Gratzer, G., and Schmidt, E.T., becomes “To introduce Distributive \(\ell\)-ideal, standard \(\ell\)-ideal and Neutral \(\ell\) -ideal in a commutative \(\ell\)-ring”.

3. In the case of commutative \(\ell\)-ring the Gratzer’s problem [III . 1. in [8]] is to “Generalize the concept of distributive \(\ell\)-ideal to convex \(\ell\)-subring”

This thesis mainly deals with commutative lattice ordered ring, properties of commutative lattice ordered ring, examples for commutative lattice ordered ring,
characterization Theorem for commutative lattice ordered ring, fundamental theorem of homomorphism, Isomorphism theorems, lattice of $\ell$-ideals in a commutative lattice ordered ring, distributive $\ell$-ideal, characterization theorem for distributive $\ell$-ideal, standard $\ell$-ideal, characterization theorem for standard $\ell$-ideal, neutral $\ell$-ideal, characterization theorem for neutral $\ell$-ideal distributive convex $\ell$-subring and characterization theorem for distributive convex $\ell$-subring.

In chapter I, “Preliminaries”, many definitions and theorems used in this thesis are given.

In chapter II “Commutative lattice ordered ring”, two definitions for commutative lattice ordered ring are introduced and the following observations are made.

1. Two definitions for commutative lattice ordered ring are equivalent.
2. Any commutative lattice ordered ring is a lattice ordered group.
3. A Boolean ring is a commutative lattice ordered ring.
4. Any Boolean algebra is a commutative lattice ordered ring.
5. Any commutative $\ell$ - ring need not be a Boolean Algebra

Chapter III “Properties of commutative lattice ordered ring”, contains the following aspects.

1. Properties of commutative lattice ordered ring.
2. Any commutative lattice ordered ring is a distributive lattice.

3. Any Browerian algebra B is a commutative lattice ordered ring.


The concept of \( \ell \)-ideal is introduced in chapter IV and the following

Theorems are proved.

1. If \( I_1 \) and \( I_2 \) are \( \ell \)-ideals of a commutative lattice ordered ring \( R \), then

   i) \( I_1 \land I_2 \) is a \( \ell \)-ideal of \( R \).

   ii) \( I_1 \cup I_2 \) need not be an \( \ell \)-ideal of \( R \).

   iii) \( I_1 \lor I_2 \) is an \( \ell \)-ideal of \( R \).

   iv) \( I_1 \lor I_2 \) is the smallest \( \ell \)-ideal containing \( I_1 \cup I_2 \).

2. If \( I(R) \) denote the set of all \( \ell \)-ideals of a commutative \( \ell \)-ring \( R \), then

\( I(R) \) is a distributive lattice.

3. If \( I \) is an \( \ell \)-ideal of a commutative \( \ell \)-ring \( R \) the \( \frac{R}{I} \) is a commutative \( \ell \)-ring with respect to the following operations

   (i) \( (a + I) + (b + I) = (a + b) + I \)

   (ii) \( (a + I)(b + I) = (a \cdot b) + I \)

   (iii) \( (a + I) \lor (b + I) = (a \lor b) + I \)

   (iv) \( (a + I) \land (b + I) = (a \land b) + I \)
where } a + I, b + I \in \frac{R}{I} \text{. This commutative } \ell \text{-ring is called quotient commutative } \ell \text{-ring. }

4. If } \varphi : R_1 \to R_2 \text{ is an onto homomorphism of a commutative } \ell \text{-ring with Kernel } K \text{ then } \frac{R_1}{K} \text{ is isomorphic to } R_2 \text{. Conversely if } R \text{ is a commutative } \ell \text{-ring and } I \text{ an } \ell \text{-ideal of } R \text{, then there exists a homomorphism of } R \text{ onto } \frac{R}{I} \text{.}

5. Let } \varphi : R \to R' \text{ be an onto homomorphism with Kernel } J \text{. } I' \text{ an } \ell \text{-ideal of } R' \text{ and } I = \{ x \in R; \varphi(x) \in I' \}. \text{ Then } I \text{ is an } \ell \text{-ideal of } R \text{ containing } J \text{ and } \frac{R}{I} \cong \frac{R'}{I'} \text{. Moreover}

\[
\frac{R}{I} \cong \left( \frac{R}{J} \right) / \left( \frac{I}{J} \right)
\]

6. Let } R \text{ be a commutative } \ell \text{-ring and } I, J \text{ be two } \ell \text{-ideals of } R \text{. Then}

\[
\frac{I + J}{J} \cong \frac{I}{I \cap J}
\]

In chapter V “Distributive } \ell \text{-ideal in commutative } \ell \text{-ring”, distributive } \ell \text{-ideal, dually distributive } \ell \text{-ideal, standard } \ell \text{-ideal, dually standard } \ell \text{-ideal and neutral } \ell \text{-ideal are introduced. The characterization theorems and the relation
between the above said \( \ell \)-ideals are proved. The following observations are also made.

1) Every ideal of a Boolean algebra is a distributive \( \ell \)-ideal.

2) If \( D_1 \) and \( D_2 \) are distributive \( \ell \)-ideals of a commutative \( \ell \)-ring \( R \) then

\[ D_1 \lor D_2 \] and \( D_1 \land D_2 \) are also distributive \( \ell \)-ideals.

3) Let \( D \) be an \( \ell \)-ideal of a commutative \( \ell \)-ring \( R \). Then the following conditions are equivalent.

   i) \( D \) is distributive

   ii) The map \( \varphi: X \rightarrow D \lor X \) is a homomorphism of \( I(R) \) onto

   \[ [D] = \{ x \in I(R) / X \geq D \} \]

   iii) The binary relation \( \theta_D \) on \( I(R) \) defined by

   “\( X \equiv y \ (\theta_D) \Leftrightarrow D \lor X = D \lor Y \) where \( X, Y \in I (R) \)” is a congruence relation.

4) If \( D_1 \) and \( D_2 \) are dually distributive \( \ell \)-ideals of a commutative \( \ell \)-ring \( R \), then \( D_1 \lor D_2 \) and \( D_1 \land D_2 \) are dually distributive \( \ell \)-ideals.

5) Let \( D \) be an \( \ell \)-ideal of a commutative \( \ell \)-ring \( R \). Then the following conditions are equivalent.

   i) \( D \) is dually distributive
ii) The map $\varphi: X \rightarrow D \land X$ is a homomorphism of $I(R)$ onto

$[D] = \{X \in I(R) / X \leq D\}$

iii) The binary relation $\theta_D$ on $I(R)$ defined by

"$X \equiv Y (\theta_D) \iff D \land X = D \land Y$ where $X, Y \in I(R)$"

is a congruence relation.

6) If D is an $\ell$-ideal of a commutative $\ell$-ring R, then the following conditions are equivalent.

i) $D$ is a distributive $\ell$-ideal

ii) $D$ is a dually distributive $\ell$-ideal

7) If $S$ is an $\ell$-ideal of a commutative $\ell$-ring R, then the following conditions are equivalent.

i) $S$ is standard $S$

ii) The binary relation $\theta_S$ on $I(R)$ defined by “$X \equiv Y(\theta_S)$ if and only if $(X \land Y) \lor S_1 = X \lor Y$ for some $S_1 \leq S$” is a congruence relation

iii) $S$ is distributive and for all $X, Y \in I(R), S \land X = S \land Y$,

$S \lor X = S \lor Y$ implies $X = Y$
8) If $S_1$ and $S_2$ are standard $\ell$-ideals of a commutative $\ell$-ring $R$, then $S_1 \vee S_2, S_1 \wedge S_2$ are standard $\ell$-ideals of $R$.

9) Let $S$ be an $\ell$-ideal of a commutative $\ell$-ring $R$. Then the following conditions are equivalent.

i) $S$ is dually standard

ii) The binary relation $\theta_S$ on $I(R)$ defined by

\[ X \equiv Y(\theta_S) \text{ if and only if } (X \vee Y) \wedge S_1 = X \wedge Y \text{ for some } S_1 \geq S \text{ where } X, Y \in I(R) \]

is a congruence relation

iii) $S$ is dually distributive and for all $X, Y \in I(R)$,

\[ S \wedge X = S \wedge Y, S \vee X = S \vee Y \implies X = Y \]

10) If $S_1$ and $S_2$ are dually standard $\ell$-ideals of a commutative $\ell$-ring $R$, then $S_1 \vee S_2, S_1 \wedge S_2$ are dually standard $\ell$-ideals

11) In a commutative $\ell$-ring $R$, the following conditions are equivalent.

i) $S$ is a standard $\ell$-ideal

ii) $S$ is a dually standard $\ell$-ideal

12) Let $N$ be an $\ell$-ideal of a commutative $\ell$-ring $R$. Then the following conditions are equivalent.
i) N is Neutral

ii) N is distributive, N is dually distributive and for all

\[ X, Y \in I(R), N \land X = N \land Y, N \lor X = N \lor Y \implies X = Y \]

13) In a commutative ℓ-ring R, neutral ℓ-ideal ⇔ standard ℓ-ideal
    ⇔ distributive ℓ-ideal.

Chapter VI “Distributive convex ℓ-subring” takes up the following
Theorems and they are proved.

1) For each d in a Boolean ring R, \{0, d\} is a distributive ℓ-subring of R.

2) An ℓ-ideal D of a commutative ℓ-ring R is distributive if and only if it
   is a distributive convex ℓ-subring of R.

3) A dual ℓ-ideal D′ of a commutative ℓ-ring R is distributive if and
   only if it is a distributive convex ℓ-subring of R.

4) Let D be a convex ℓ-subring of a commutative ℓ-ring R. If \( x, y \in R \)
   such that \( x \lor t = y \lor t, x \land s = y \land s, x + u = y + u, xv = yv \)
   for some \( s, t, u, v \in D \) then \( \langle D, \{x\} \rangle = \langle D, \{y\} \rangle \).

5) Let R be a commutative ℓ-ring and D a distributive convex ℓ-subring
   of R. If D satisfies property (P), where (P) is

   \[ \langle D, X \lor Y \rangle = \langle D, X \rangle \lor \langle D, Y \rangle \]
\[ \langle D, X \land Y \rangle = \langle D, X \rangle \land \langle D, Y \rangle \]
\[ \langle D, X + Y \rangle = \langle D, X \rangle + \langle D, Y \rangle \]
\[ \langle D, XY \rangle = \langle D, X \rangle \langle D, Y \rangle \]

for all single element convex ℓ-subring X, Y of R, then the binary relations \( \theta_D \) on R defined by “\( x \equiv y (\theta_D) \) if and only if

\[(x \land y) \land s = (x \lor y) \land s\]
\[(x \land y) \lor t = (x \lor y) \lor t\]
\[(x \land y) + u = (x \lor y) + u\]
\[(x \land y) v = (x \lor y) v\]

for suitable \( s, t, u, v \in D \)” is a congruence relation.

6) If D is a convex ℓ-subring of R such that the relation \( \theta_D \) defined by

\[ x \equiv y (\theta_D) \] if and only if

\[(x \land y) \land s = (x \lor y) \land s\]
\[(x \land y) \lor t = (x \lor y) \lor t\]
\[(x \land y) + u = (x \lor y) + u\]
\[(x \land y) v = (x \lor y) v\]

for suitable \( s, t, u, v \in D \) is a congruence relation then D is a distributive convex ℓ-subring of R.
7) Let $f: x \rightarrow x'$ be a homomorphism of $R$ onto $R'$ and let $D$ a distributive convex $\ell$-subring of $R$. Then the homomorphic image $D'$ of $D$ is a distributive $\ell$-subring of $R'$

8) Let $R$ be a commutative $\ell$-ring, $D$ a distributive convex $\ell$-subring and $I$ an $\ell$-ideal of $R$ such that $D \leq I$. Then $I$ is a distributive convex $\ell$-subring of $R$ if and only if $I/D$ is a convex $\ell$-subring in $\frac{R}{I}$ and $\frac{R}{I}$ is isomorphic to $\frac{R/D}{I/D}$.

9) Let $R$ be a commutative $\ell$-ring, $D$ a distributive convex $\ell$-subring and $I$ an $\ell$-ideal of $R$ such that $I \cap D \neq \phi$. Then $I \cap D$ is a distributive convex $\ell$-subring of $I$ and $\langle I, D \rangle / D$ is isomorphic to $I/I \cap D$.
CHAPTER - I

PRELIMINARIES

In this chapter are listed, a number of definitions and results which are made use of throughout the thesis. The symbols $\leq, \not\in, +, \cdot, -, \lor, \land$ and $*$ will denote inclusion, non-inclusion, sum, product, difference, least upper bound, greatest lower bound and symmetric difference in a lattice $L$ or commutative $\ell$-ring $R$ (whenever they are defined), while symbols $\subseteq, \cup, \cap, \in, \notin$ and $\emptyset$ will refer to set inclusion, union, intersection, membership, non-membership and empty set. Small letters $a, b, \ldots$ will denote elements of the lattice $L$ or commutative $\ell$-ring $R$ and Greek letters $\theta, \phi$ will stand for congruence relation on a lattice.

**Definition 1.1:**

Let $L$ be a lattice and $a \in L$. Then $a$ is called a distributive element if and only if

$$a \lor (x \land y) = (a \lor x) \land (a \lor y), \text{ for all } x, y \in L.$$  

$a$ is called a standard element if and only if

$$x \land (a \lor y) = (x \land a) \lor (x \land y), \text{ for all } x, y \in L.$$  

$a$ is called a neutral element if and only if

$$(a \lor x) \land (x \lor y) \land (y \lor a) = (a \lor x) \lor (x \lor y) \lor (y \lor a) \text{ for all } x, y \in L.$$  

a is called a dually distributive element if and only if
\[ a \land (x \lor y) = (a \land x) \lor (a \land y), \text{ for all } x, y \in L. \]

a is called dually standard element if and only if
\[ x \lor (a \land y) = (x \lor a) \land (x \lor y), \text{ for all } x, y \in L. \]

**Definition 1.2:**

Let \( L \) be a lattice and \( \theta \subseteq L \times L \). Then \( \theta \) is said to be a congruence relation if it satisfies the following:

(i) \( \theta \) is reflexive:

\[ x \equiv x (\theta), \text{ for all } x \in L. \]

(ii) \( \theta \) is symmetric:

\[ x \equiv y (\theta) \Rightarrow y \equiv x (\theta), \text{ for all } x, y \in L. \]

(iii) \( \theta \) is transitive:

\[ x \equiv y (\theta) \text{ and } y \equiv z (\theta) \Rightarrow x \equiv z (\theta) \text{ for all } x, y, z \in L. \]

(iv) Substitution property:

\[ x \equiv x_1 (\theta) \text{ and } y \equiv y_1 (\theta) \]
\[ \Rightarrow x \lor y \equiv x_1 \lor y_1(\theta) \quad \text{and} \]
\[ x \land y \equiv x_1 \land y_1(\theta), \quad \text{for all } x, x_1, y, y_1 \in L. \]

**Theorem 1.1:**

A reflexive and symmetric binary relation \( \theta \) on a lattice \( L \) is a congruence relation if and only if the following three properties are satisfied

(i) \( x \equiv y (\theta) \iff x \land y \equiv x \lor y (\theta) \)

(ii) \( x \leq y \leq z, x \equiv y (\theta) \) and \( y \equiv z (\theta) \Rightarrow x \equiv z (\theta) \)

(iii) \( x \leq y \) and \( x \equiv y (\theta) \Rightarrow x \land z \equiv y \land z (\theta) \) and \( x \lor z \equiv y \lor z (\theta) \)

for all \( x, y, z \in L. \)

**Definition 1.3:**

A lattice \( L \) is said to be distributive if and only if

\[ a \lor (b \land c) = (a \lor b) \land (a \lor c), \quad \text{for all } a, b, c \in L. \]

**Theorem 1.2:**

In any lattice \( L \), the following are equivalent.

(i) \( a \lor (x \land y) = (a \lor x) \land (a \lor y) \)

(ii) \( a \lor x = a \lor y, \quad a \land x = a \land y \Rightarrow x = y, \quad \text{for all } a, x, y \in L. \)
**Theorem 1.3:**

In a distributive lattice, every element is distributive, standard and neutral.

**Definition 1.4:**

Let $L$ be a lattice and $H \subseteq L \times L$. Then the smallest congruence relation such that $a \equiv b \ (\theta_H)$ for all $(a, b) \in H$ is denoted by $\theta_H$.

**Definition 1.5:**

A non-empty subset $I$ of a lattice $L$ is said to be an ideal if and only if

(i) $x, y \in I \Rightarrow x \lor y \in I$

(ii) $x \in I, t \in L$ and $t \leq x \Rightarrow t \in I$

Let “$a$” be an element of a lattice $L$. Then the set $\{x \in L / x \leq a\}$ form an ideal of $L$ is called principal ideal generated by “$a$” and is denoted by $(a]$

**Theorem 1.4:**

If $L$ is a lattice and $I (L)$, set of all ideals of $L$ then $I (L)$ is a lattice with respect to the following.

$I_1 \leq I_2 \iff I_1 \subseteq I_2$.

$I_1 \lor I_2 = \{x \in L / x \leq x_1 \lor x_2 \text{ for some } x_1 \in I_1, x_2 \in I_2\}$

$I_1 \land I_2 = I_1 \cap I_2$ where $I_1, I_2 \in I (L)$. 
Definition 1.6:

Dual of an ideal is called a dual ideal or a filter. It is defined as follows.

A non-empty subset F of a lattice L is called a filter if and only if

(i) \( x \in F \) and \( y \in F \implies x \land y \in F \).

(ii) \( x \in F \), \( y \in L \) \( y \geq x \implies y \in F \).

Let “a” be an element of a lattice L. Then the set \( \{ x \in L / x \geq a \} \) form a filter of L is called the principal filter generated by “a” and is denoted by \( [a] \).

Definition 1.7:

An ideal of I of a lattice L is called a distributive ideal if and only if

\( I \lor (X \land Y) = (I \lor X) \land (I \lor Y) \) for all \( X, Y \in I (L) \).

That is I is a distributive element of I (L).

Definition 1.8:

An ideal I of a lattice L is called a dually distributive ideal if and only if

\( I \land (X \lor Y) = (I \land X) \lor (I \land Y) \) for all \( X, Y \in I (L) \).

That is I is a dually distributive element of I (L).
Definition 1.9:

An ideal $I$ of a lattice $L$ is called a standard ideal if and only if

$$X \land (I \lor Y) = (X \land I) \lor (X \land Y)$$

for all $X, Y \in I(L)$.

That is $I$ is a standard element of $I(L)$.

Definition 1.10:

An ideal $I$ of a lattice $L$ is called a dually standard ideal if and only if

$$X \lor (I \land Y) = (X \lor I) \land (X \lor Y)$$

for all $X, Y \in I(L)$.

That is $I$ is a dually standard element of $I(L)$.

Definition 1.11:

An ideal $I$ of a lattice $L$ is called a neutral ideal if and only if

$$(I \lor X) \land (X \lor Y) \land (Y \lor I) = (I \land X) \lor (X \land Y) \lor (Y \land I)$$

for all $X, Y \in I(L)$.

That is $I$ is a neutral element of $I(L)$.

It is observed that $L$ is a distributive lattice if and only if every ideal is distributive, standard and neutral ideal.

Theorem 1.5:

Let $L$ be a lattice and $I$ an ideal of $L$. Then the following conditions on $I$ are equivalent.
(i) I is a distributive ideal.

(ii) The binary relation \( \theta_I \) on \( L \) is defined by “\( x \equiv y (\theta_I) \) if and only if \( x \lor i = y \lor i \) for some \( i \in I \)” is a congruence relation.

Theorem 1.6:

Let \( I \) be an ideal of a lattice \( L \). Then the following conditions on \( I \) are equivalent.

(i) \( I \) is a standard ideal.

(ii) The equality \( (a \land (I \lor b)) = ((a \land I) \lor (a \land b)) \)
holds for all \( a, b \in L \)

(iii) For any ideal \( J \) of \( L \), \( I \lor J = \{ i \lor j / i \in I, j \in J \} \)

(iv) The binary relation \( \theta_I \) on \( L \) defined by

“\( x \equiv y (\theta_I) \) if and only if \( (x \land y) \lor i = x \lor y \) for some \( i \in I \)”
is a congruence relation.

(v) \( I \) is a distributive ideal and for all \( J, K \in I \) such that
\( I \lor J = I \lor K \) and \( I \land J = I \land K \) \( \Rightarrow \) \( J = K \).
**Theorem 1.7:**

Let I be an ideal of a lattice L. Then the following conditions on I are equivalent.

(i) I is a neutral ideal.

(ii) For all $j, k \in L$, $(I \lor (j)) \land ((j \lor (k])) \land ((k \lor I) =

\quad (I \land (j)) \lor ((J \land (k)) \lor ((k) \land I)

(iii) For all $J, K \in I(L)$, I, J and K generate a distributive sublattice of I(L).

(iv) I is distributive, I is dually distributive and for all $J, K \in I(L)$ such that

$J \lor K = I \lor J$,

$I \land J = I \land K \Rightarrow J = K$.

**Theorem 1.8:**

In a lattice L

(i) Every standard ideal is a distributive ideal.

(ii) Every neutral ideal is a standard ideal.

(iii) Every standard and dually standard ideal is a neutral ideal.

(iv) Every standard and dually distributive ideal is a neutral ideal.
Definition 1.12:

A ring \((R, +, \cdot)\) is called a Boolean ring if and only if \(a \cdot a = a\), for all \(a \in R\).

A Boolean ring \(R\) is called Boolean ring with identity if there exists \(1 \in R\) such that \(1 \cdot a = a\), for all \(a \in R\).

Theorem 1.9:

If \(R\) is a Boolean ring then

(i) \(a + a = 0\)

(ii) \(ab = ba\)

for all \(a, b \in R\)

Definition 1.13:

A Boolean algebra \(B\) is a distributive complemented lattice.

Theorem 1.10:

The following two systems are equivalent.

(i) Boolean algebra

(ii) Boolean ring with identity.
Definition 1.14

A non-empty set $G$ is called lattice ordered group or $\ell$-group if

(i) $(G, +)$ is a group

(ii) $(G, \lor, \land)$ is a lattice

(iii) $a + x \lor y + b = (a + x + b) \lor (a + y + b)$

$a + x \land y + b = (a + x + b) \land (a + y + b)$ for all $a, b, x, y \in G$.

Definition 1.15

A non-empty set $R$ is called lattice ordered ring or $\ell$-ring if

(i) $(R, +, \cdot)$ is a ring

(ii) $(R, \lor, \land)$ is a lattice

(iii) $a + x \lor y + b = (a + x + b) \lor (a + y + b)$

$a + x \land y + b = (a + x + b) \land (a + y + b)$

for all $a, b, x, y \in R$

(iv) $a (x \lor y) b = (a x b) \lor (a y b)$

$a (x \land y) b = (a x b) \land (a y b)$

for all $a, b, x, y \in R$ and $a \geq 0, b \geq 0$
Definition 1.16

A non-empty set $B$ is called a Browerian Algebra if and only if

i) $(B, \leq)$ is a lattice

ii) $B$ has a least element

iii) To each $a, b \in B$, there exists $x = a - b \in B$ such that $b \lor x \geq a$