Chapter 4

Bright soliton on a continuous wave background in Bose-Einstein condensates with time dependent scattering length and trapping potentials

4.1 Introduction

Recently, the dynamics of the optical and matter wave solitons have been reported for dispersion and FR management systems, respectively [200, 201]. The dispersion managed optical solitons are more advantageous in optical fiber medium for the optical communication and storage of the information [202, 203, 204]. The generation and evolution of FR managed matter wave solitons train is important for a number of BEC applications, such as atomic interferometry [205] and different kinds of quantum phase transitions [206]. We recall that the concept of FR management as the variation in the strength and sign of the atomic scattering length (nonlinear coefficient) by tuning the external magnetic field near FR [173]. Recently, the dynamics of a bright soliton [207] has been discussed in the exponentially varying scattering length of BEC. In Ref. [208], the authors have discussed the \(N\)-bright soliton on a CW background propagation in optical fibers. Li et al. have investigated the MI and soliton
on a CW background in an inhomogeneous optical fiber medium with constant non-linearity, frequency chirping, and linear gain/loss effects [209]. Using the Darboux transformation method, the one-soliton on a CW background has been discussed for the integrable Hirota equation describing pulse propagation in optical fibers with higher-order effects [210]. Using auto-Bäcklund transformation, one- and two-soliton solutions with and without the CW background have been constructed in an erbium doped fiber system associated with the higher-order dispersion, self-steepening, and self-induced transparency effects [211]. More recently, the dynamics of bright soliton on a CW background [115, 212] has been discussed in a system of BEC with the exponentially varying atomic scattering length. This chapter is devoted to explore the dynamical evolution of the “cigar shaped” BEC in terms of bright solitons on a CW background under the influence of time varying atomic scattering length, linear and harmonic potentials. The properties of the bright solitons on a CW background such as trajectory, velocity, compression and broadening, and atomic exchange between soliton and CW are discussed in detail for both cases of exponential and periodically varying atomic scattering lengths.

4.2 Theoretical Model

We achieve the “cigar shaped” BEC wherein the radial oscillation frequency ($\omega_\perp$) is very much greater than the axial oscillation frequency ($\omega_0$), i.e., $\omega_\perp \gg \omega_0$. Hence the motion of the condensate along the radial directions is suppressed. The bright soliton has been generated in $^7$Li condensate for the following experimental value of the system parameters: $\omega_0 = 2\pi \times 50 \text{ Hz}$ and $\omega_\perp = 2\pi \times 710 \text{ Hz}$ with the atomic scattering length $a = -0.21 \text{ nm}$ [104]. Under the above physical situation, for the case of $\frac{\omega_0}{\omega_\perp} \ll 1$, the system is effectively 1D. The achievement of the quasi-1D regime has been recently demonstrated in the case of $^7$Li [213] and $^{23}$Na [214] condensates. In quasi-1D geometries, the stable bright soliton and bright matter wave soliton train have been generated in the weakly attractive $^7$Li condensate by means of FR technique [103, 104, 215]. As discussed above, this work is devoted to explore the dynamical
evolution of the “cigar shaped” BEC in terms of bright solitons on a CW background under the influence of time varying atomic scattering length, linear and harmonic potentials. In order to study the dynamical evolution of the BEC, we consider the 1D variable coefficient GP equation with time dependent linear potential is given by [98, 115, 197, 198, 207, 212, 216]

$$i\hbar \frac{\partial \Psi}{\partial \tau} + \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial \bar{z}^2} + \frac{4\pi \hbar^2 a(\tau)}{m} |\Psi|^2 \Psi - f(\tau) \bar{z} \Psi - \frac{1}{2} m \omega_0^2(\tau) \bar{z}^2 \Psi = 0,$$  \hspace{1cm} (4.1)

where $\Psi(\bar{z}, \tau)$ is the macroscopic wave function of the condensate, $a(\tau)$ denotes the time dependent atomic scattering length. Further, $f(\tau)$ and $\omega_0^2(\tau)$ represent arbitrary time dependent coefficients of linear and harmonic potentials, respectively. Finally, $m$ and $\hbar$ are the mass of the atom and Plank’s constant. In order to simplify the Eq. (4.1), we introduce the dimensionless variables

$$t = \omega_\perp \tau, \quad z = \frac{\sqrt{\hbar m}}{\omega_\perp} \bar{z}, \quad \psi = \sqrt{\frac{4\pi a_B \hbar}{\omega_\perp}} \Psi,$$

here $\omega_\perp$ and $a_B$ indicate the transverse trapping potential and Bohr’s atomic radius, respectively. Finally, we obtain the following dimensionless equation:

$$i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial z^2} + R(t) |\psi|^2 \psi - 2\alpha(t) z \psi - \frac{\Omega^2(t)}{2} z^2 \psi = 0,$$  \hspace{1cm} (4.2)

where $R(t) = \frac{a(t/\omega_\perp)}{a_B}$, $\alpha(t) = \frac{\sqrt{\hbar m}}{2\omega_\perp^2 \hbar} f(t/\omega_\perp)$, and $\Omega^2(t) = \frac{\omega^2(t/\omega_\perp)}{\omega_\perp^2}$. In general, the Eq. (4.2) can be written as [217]

$$i \frac{\partial \psi}{\partial t} + \frac{D(t)}{2} \frac{\partial^2 \psi}{\partial z^2} + R(t) |\psi|^2 \psi - 2\alpha(t) z \psi - \frac{\Omega^2(t)}{2} z^2 \psi = 0,$$  \hspace{1cm} (4.3)

where time $t$ is measured in units of $1/\omega_\perp$, coordinate $x$ in units of $a_\perp$, and the condensate wave function $\psi$ in units of $1/\sqrt{4\pi a_B a_\perp^2}$. Here, the parameter $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the characteristic extension length of the ground state wave function of harmonic oscillator. In Eq. (4.3), the time dependent coefficient represents the kinetic energy dispersion of the matter wave. If we choose the dispersion coefficient $D(t) = 1$, Eq. (4.3) leads to Eq. (4.2). Equation (4.3) describes the remote controlling the system of BEC by properly choosing the system parameters.
4.3 BEC soliton on a CW background

In this section, we construct the analytical solution for bright soliton on a CW background for the Eq. (4.3). To proceed further, we reduce the variable GP Eq. (4.3) to the standard NLS equation by using the following transformation [218]:

$$\psi = q(t) \phi(Z, T) \exp \left[ i \left( -\frac{q_t}{q(t)D(t)} z^2 + \alpha_1(t)z + \alpha_2(t) \right) \right], \quad (4.4)$$

where $q(t) = q_0^2 \sqrt{\frac{2R(t)}{D(t)}}$, $Z = \Gamma(z, t)$ and $T = T(t)$. Here $q_0$ is an arbitrary real constant. Substituting Eq. (4.4) in Eq. (4.3), we obtain the standard NLS equation as

$$i \frac{\partial \phi}{\partial T} + \frac{1}{2} \frac{\partial^2 \phi}{\partial Z^2} + |\phi|^2 \phi = 0, \quad (4.5)$$

with the following equations for the scaling parameters

$$T_t - R(t)q^2(t) = 0, $$
$$\Gamma_z - \frac{1}{D(t)}T_t = 0, $$
$$\frac{2\alpha_1(t)}{q(t)} q_t - \alpha_{1t} - 2\alpha(t) = 0, $$
$$\alpha_{2t} + \frac{1}{2} D(t) \alpha_1^2(t) = 0, $$
$$q(t)\Gamma_t + \left( D(t)q(t)\alpha_1(t) - 2zq_t \right) \Gamma_z + \frac{D(t)q(t)}{2i} \Gamma_{zz} = 0, $$
$$\frac{q_t}{D(t)} - \frac{3q^2}{2D(t)q(t)} - \frac{q_t D_t}{D(t)^2} - \frac{1}{2} \Omega^2(t)q(t) = 0. \quad (4.6)$$

Equations (4.6) allows us to study the properties of solutions of Eq. (4.3) from those of the NLS Eq. (4.5). Solving all the equations of Eq. (4.6), we have

$$T(t) = 2q_0^4 \int \frac{R^2(t)}{D(t)} dt, $$
$$\Gamma(z, t) = \frac{q^2(t)}{\sqrt{2q_0^2}} z + \Gamma_1(t), $$
$$\alpha_1(t) = -\frac{2R(t)}{D(t)} \int \frac{\alpha(t)D(t)}{R(t)} dt, $$
$\alpha_2(t) = \frac{1}{2} \int D(t) \alpha_1^2(t) dt,$

$\Gamma_1(t) = -\sqrt{2} q_0^2 \int R(t) \alpha_1(t) dt,$

$\Omega^2(t) = -\frac{1}{D(t)} \left[ \frac{d^2}{dt^2} \ln D(t) + \frac{d}{dt} \frac{1}{R(t)} \frac{d^2}{dt^2} \ln D(t) \frac{d}{dt} \ln R(t) \right].$ \hspace{1cm} (4.7)

From the last equation of Eq. (4.7), the harmonic potential coefficient $\Omega^2(t)$ is defined by the time dependent dispersion $(D(t))$ and atomic scattering length $(R(t))$ parameters. The relation $\Omega^2(t)$ represents the integrability condition of Eq. (4.3). The nonzero CW solution of NLS equation is $\phi_c(Z, T) = \eta e^{i(k_c Z - \omega T)}$, where $\omega = -\frac{1}{2}(2\eta^2 - k_c^2)$ and $k_c$ is an arbitrary real constant. The parameter $\eta$ represents the amplitude of the CW solution. The bright soliton on a CW background has been discussed in detail for various forms of the NLS equations [208, 209, 210, 219]. We find the bright soliton on a CW background for Eq. (4.3) as

$$\psi = q(t) \left[ \eta + \sigma \frac{(\kappa \cosh \theta + \cos \varphi) + i(\rho \sinh \theta + \lambda \sin \varphi)}{\cosh \theta + \kappa \cos \varphi} \right] \exp \left[ i \left( -\frac{q_t}{q(t)D(t)} z^2 + \alpha_1(t)z + \alpha_2(t) + \varphi_c \right) \right] \hspace{1cm} (4.8)$$

where

$$\theta = M_R \Gamma(z, t) - q_0^4 \left[(k_c + k_s)M_R - \sigma M_I\right] \int \frac{R^2(t)}{D(t)} dt - \theta_0,$$

$$\varphi = M_I \Gamma(z, t) - q_0^4 \left[(k_c + k_s)M_I + \sigma M_R\right] \int \frac{R^2(t)}{D(t)} dt - \varphi_0,$$

$$\varphi_c = k_c Z - \sigma T = k_c \Gamma(z, t) + q_0^4 (2\eta^2 - k_c^2) \int \frac{R^2(t)}{D(t)} dt,$$

$$\Gamma(z, t) = \frac{q^2(t)}{\sqrt{2q_0^2}} z + 2\sqrt{2q_0^2} \int \frac{R^2(t)}{D(t)} \left( \int \frac{D(t)\alpha(t)}{R(t)} dt \right) dt,$$

$$\rho = \frac{q}{\Delta}(k_s - k_c + M_I), \hspace{0.5cm} \lambda = 1 - \frac{2\eta^2}{\Delta}, \hspace{0.5cm} \kappa = \frac{\eta}{\Delta}(M_R - \sigma), \hspace{0.5cm} \Delta = \eta^2 + \frac{1}{4}(M_R - \sigma)^2 + \frac{1}{4}(k_s - k_c + M_I)^2$$

and the complex parameter $M = \sqrt{(i(k_s - k_c) - \sigma)^2 - 4\eta^2}$, where $\theta_0$ and $\varphi_0$ are the arbitrary real constants. The parameter $\sigma$ describes the amplitude of the soliton. It should be noted that when the amplitude of the CW $\eta = 0$ and $k_c = 0$, the solution (4.8) reduces to well known bright one-soliton solution of Eq. (4.3). Next, we consider the another physical situation wherein the amplitude of the
soliton vanishes i.e., $\sigma = 0$ and $k_s = 0$, we obtain the CW solution of Eq. (4.3) from Eq. (4.8). Therefore, in general, the solution (4.8) describes a bright soliton on a CW background for the system of BEC. For simplicity we consider $k_c = k_s = k$ in the above solution (4.8), and arrive at $M = \sqrt{\sigma^2 - 4\eta^2}$. From this relation, it is clear that the stability of the solution described by the amplitudes/powers of the soliton and CW, respectively. MI can only be possible when the soliton’s peak power is less than four times the power of CW solution i.e., $\sigma^2 < 4\eta^2$ [159, 164, 209, 210]. With the help of this physical process, bright soliton on a CW can eventually be utilized to explain the formation of a train of matter wave solitons which was already discussed in other physical systems [159, 164, 209, 210]. The stable soliton formation on a CW background takes place when $\sigma^2 > 4\eta^2$. Under this physical condition, we obtain the bright soliton on a CW background for Eq. (4.3) from Eq. (4.8) as follows:

$$\psi = q_0^2 \sqrt{\frac{2R(t)}{D(t)}} \left[ -\eta + M_R \frac{M_R \cos \varphi - i\sigma \sin \varphi}{\sigma \cosh \theta - 2\eta \cos \varphi} \right] \exp \left[ i \left( -\frac{q_0}{q(t)D(t)} \right) \frac{z^2}{z^2 + \alpha_1(t)z + \alpha_2(t) + \varphi_c} \right]$$

(4.9)

The above analytical result represents the exact BEC bright soliton on a CW background where the condensate is trapped along the axial direction. In what follows, first we analyze the bright soliton on a CW background in an exponentially varying atomic scattering length for different forms of linear potential. Then, the subsequent section, we discuss the BEC soliton in periodically varying atomic scattering length.

### 4.4 Exponentially varying atomic scattering length

For the ease of discussion, initially we consider the dispersion coefficient to be $D(t) \equiv 1$ and the exponentially varying atomic scattering length, $R(t) = R_0 e^{\xi t}$ [115, 207, 212], where $R_0$ is an arbitrary constant. The parameter $\xi$ is defined by the ratio between the longitudinal ($\omega_0$) and transverse ($\omega_\perp$) frequencies of the trapping potential. By using the values of $D(t)$ and $R(t)$, we have found the harmonic potential coefficient to be of the form $\Omega^2(t) = -\xi^2$. Under these physical conditions, the variable GP Eq.
(4.3) is reduced to

\[ i\psi_t + \frac{1}{2}\psi_{zz} + R_0 e^{i\xi t} |\psi|^2 \psi - 2\alpha(t) z\psi + \frac{1}{2} \xi^2 z^2 \psi = 0. \]  (4.10)

The above Eq. (4.10) is not only valid for BEC but also applicable to the plasma physics and optical fiber systems. More recently, the dynamics of a bright soliton were discussed in BEC for the case of exponentially varying atomic scattering length when \( \alpha(t) = 0 \) by using the extended tanh-function method [207]. By using the Darboux transformation method, the dynamics of a bright soliton on a CW background has been discussed in BEC when \( \alpha(t) = 0 \) [115, 212]. In order to discuss the role of time independent atomic scattering length (\( \xi = 0 \)), the soliton without background solution has been discussed in a system of plasma with the following three different forms of \( \alpha(t) \) i.e., constant, \( t \) (linearly increase in time), and oscillatory [220]. The soliton on a CW background [116] and cnoidal wave background [221] have been discussed in detail for Eq. (4.10) when \( \xi = 0 \). In this work, we investigate the dynamics of the bright BEC soliton on a CW background when \( \alpha(t) \neq 0 \) and \( \xi \neq 0 \). From this analysis, the linear potential parameter \( \alpha(t) \) is shown to be responsible for the changes in the trajectory and speed of the soliton. Moreover, we prove that the atomic exchange between the soliton and CW, and compression/broadening of the soliton mainly depends on the time varying atomic scattering length \( R(t) = e^{i\xi t} \). Thus, the impact of \( \alpha(t) \) and \( \xi \) over the dynamics of the soliton propagating on a CW background has been completely studied.

We start the analysis from Eq. (9), obtain the BEC soliton on a CW background of Eq. (4.10) is given by

\[ \psi_e = \sqrt{2R_0 q_0^2 e^{i\xi t/2}} \left[ -\eta + M_R \frac{M_R \cos \varphi - i\sigma \sin \varphi}{\sigma \cosh \theta - 2\eta \cos \varphi} \right] \exp \left[ i \left( -\frac{\xi}{2} z^2 + \alpha_1(t) z + \alpha_2(t) + \varphi_c \right) \right], \]  (4.11)

where \( \theta = M_R \Gamma(z, t) + \frac{kM_R R_0^2 q_0^2}{\xi} (1 - e^{2\xi t}) - \theta_0 \), \( \varphi = \sigma M_R R_0^2 q_0^2 (1 - e^{2\xi t}) - \varphi_0 \), and \( \varphi_c = k \Gamma(z, t) + \frac{R_0^2 q_0^2}{2\xi} (2\eta^2 - k^2)(e^{2\xi t} - 1) \). Here, \( \psi_e \) indicates BEC soliton on a CW background in the exponentially varying atomic scattering length. By substituting the values of
$D(t)$ and $R(t)$ in Eq. (4.7), the other physical parameters have been determined as follows: $\alpha_1 = -2e^{\xi t} \int \alpha(t)e^{-\xi t} dt$, $\alpha_2 = -\frac{1}{2} \int \alpha^2(t) dt$, $\Gamma_1(t) = -\sqrt{2} R_0 q_0^2 \int \alpha_1(t)e^{\xi t} dt$, $\Gamma(z,t) = \sqrt{2} R_0 q_0^2 z e^{\xi t} + \Gamma_1(t)$ and $T(t) = \frac{R_0 q_0}{\xi} (e^{2\xi t} - 1)$. We observed from solution (4.11), the term $e^{\xi t/2}$ plays a crucial role in deciding the compression/broadening of the soliton when the value of $\xi$ is positive/negative.

In order to find the trajectory of the soliton, we assume that the amplitude of the CW is equal to zero. The bright BEC soliton when the amplitude of the CW is zero, is given by

$$\psi_e^\text{sol} = \sqrt{2R_0 \sigma q_0^2 e^{\xi t/2}} \text{sech}{\theta e^{i(-\xi z^2 + \alpha_1(t)z + \alpha_2(t) + \varphi + \varphi_c)}}$$  \hspace{1cm} (4.12)

From the above relation, we find the trajectory of the soliton as follows

$$z = -2e^{-\xi t} \int_0^t e^{\xi t} \left( \int_0^t \alpha(t)e^{-\xi t} dt \right) dt + \frac{\sqrt{2}}{\xi} k R_0 q_0^2 \sinh(\xi t), \hspace{1cm} (4.13)$$

with the arbitrary constant $\theta_0 = 0$. From this relation, it is clear that the trajectory of the soliton depends on the linear potential parameter $\alpha(t)$. Fig. 4.1(a) represents the soliton’s trajectory for different forms of the linear potential parameter $\alpha(t)$. In Fig. 4.1(a), dotted, dashed, and solid lines correspond to parabolic, inverse S-type, and oscillatory type trajectories, respectively. The velocity of the soliton $v_s = \frac{dz}{dt}$. Fig. 4.1(b) depicts the velocity of the soliton for the same forms of $\alpha(t)$.

Now, we discuss the different forms of BEC soliton on a CW background for different forms of $\alpha(t)$ as follows. Fig. 4.2(a) shows the dynamics of the BEC soliton on a CW background in harmonic potential when $\alpha(t) = 0$. At this juncture, we point out that the soliton on a CW background has been discussed for the system of BEC trapped in harmonic potential with exponentially varying atomic scattering length in the absence of linear potential coefficient ($\alpha(t) = 0$) [115, 212]. Now, we turn to explore the dynamics of a BEC soliton on a CW background for $\alpha(t) \neq 0$. Fig. 4.2(b-d) depict the BEC soliton on a CW background for three different forms of the $\alpha(t)$. First, we consider the value of $\alpha(t) = 0.075$ wherein the trajectory of the soliton is parabola as seen in Fig. 4.2(b). In the second case, $\alpha(t) = 0.03t$, the trajectory of the soliton is an inverse S-type, which is depicted in Fig. 4.2(c). Next
Figure 4.1: (a) The trajectory of the bright BEC soliton given by Eq. (4.13) for different forms of $\alpha(t)$: $\alpha(t) = 0.075$ (dotted line), $\alpha(t) = 0.03t$ (dashed line), $\alpha(t) = 2\cos(2.5t)$ (solid line). Other system parameter values are $q_0 = 1.5$, $R_0 = 0.5$, $\xi = 0.05$, and $k = 0.05$. (b) The velocity of the soliton for same forms of $\alpha(t)$. 
we consider $\alpha(t) = 2 \cos(\beta t)$, the trajectory of the soliton becomes oscillatory with the period of $2\pi/\beta$ as shown in Fig. 4.2(d) for $\beta = 2.5$. The two dips at the wings of the bright BEC soliton imitates the characteristics of the breather soliton. The main characteristic of BEC soliton on a CW background is the amplitude of the BEC soliton oscillates periodically without splitting of the soliton along the propagation direction in a system of BEC i.e., the breathers like soliton can be generated in BEC. This is clearly seen in Fig. 4.2. It is to be noted that the atomic scattering length parameter $\xi$ can be positive/negative. As seen in Fig. (4.2), the atomic density of the BEC soliton monotonously grows (compression) owing to the increase in the absolute value of the atomic scattering length. It is intrestiong to point put that the width of the BEC soliton becomes narrow in the harmonic potential both in an exponential way during the propagation when the parameter $\xi$ is positive. Thus based on the compression process, one can generate and achieve the BEC soliton with the desired atomic peak density wherein we fix the absolute value of the atomic scattering length. Further, the soliton undergoes broadening when the parameter $\xi$ is negative.

From the literature review, we have found that there are papers which deal with the experimental realization of the continuous wave background in BEC. The main characteristic of BEC soliton on a CW background is the amplitude of the BEC soliton oscillates periodically without splitting of the soliton along the propagation direction. The bright matter wave soliton has been experimentally realized with the oscillation of the amplitude of the atomic soliton is about $370 \, \mu m$ in $Li$ condensate which is clearly shown in Fig. (3) of Ref. [103]. Based on these experimental results, our analytical results can also be realized in the real experiment. For that purpose, we need to have a special requirement i.e., the background should be larger than that of the scale of the soliton. For instance, in the recent experiment [103, 104], the length of the background of BEC was about $2L = 370 \, \mu m$. The width of the bright soliton is about $2l = 2 \times 1.4 \, \mu m = 2.8 \, \mu m$, where $l = a_{\perp} = \sqrt{\hbar/m\omega_{\perp}}$. When the above mentioned conditions are satisfied, i.e., especially we need to have $l \ll L$ which is considered to be a necessary condition, then there is a possibility to realize our analytical results experimentally in $Li$ condensates.
Bright soliton on a continuous wave background...

Figure 4.2: Dynamics of the bright BEC soliton compression on a CW background of the solution (4.11) for different forms of $\alpha(t)$. The physical parameter values are $q_0 = 1.5$, $\sigma = 2.5$, $\eta = 1$, $R_0 = 0.5$, $\xi = 0.05$, $k = 0.05$, $\theta_0 = \varphi_0 = 0$. 
Further, we calculate the total number of atoms in BEC soliton under the influence of the exponentially varying atomic scattering length. The total number of atoms in the soliton is calculated as

\[ N_e = \int_{-\infty}^{\infty} (|\psi(z, t)|^2 - |\psi(\pm \infty, t)|^2) \, dz = 4R_0q_0^4 \sqrt{\sigma^2 - 4\eta^2} \, e^{\xi t}. \tag{4.14} \]

From the above relation, the total number of atoms in the soliton on a CW background remains invariant in the presence of linear and harmonic potentials. It is also noted that the total number of atoms in the BEC soliton is a conserved quantity of Eq. (4.10). Then, the integrability of Eq. (4.10) is not affected by the linear potential term \( \alpha(t)x \). Moreover, the exchange of atoms between the soliton and the CW background in the exponentially varying scattering length is

\[ (\Lambda_e)_{s-CW} = \int_{-\infty}^{\infty} |\psi(z, t) - \psi(\pm \infty, t)|^2 \, dz \]

\[ = 2R_0q_0^4 \sqrt{\sigma^2 - 4\eta^2} \left[ 2 + \eta W \cos \varphi \right] e^{\xi t}, \tag{4.15} \]

where

\[ W = \frac{4 \arctan \left( \frac{\sqrt{\sigma^2 + \eta \cos \varphi}}{\sigma^2 - \eta \cos \varphi} \right)}{\sigma^2 - \eta^2 \cos^2 \varphi} \]

with \( \varphi = \frac{\sigma M R R_0^2 q_0^4}{2\xi} (1 - e^{2\xi t}) - \varphi_0. \)

From Eqs. (4.14) and (4.15), the total number of atoms and the atomic exchange are independent of the linear potential \((\alpha(t))\) but certainly depend on the time varying atomic scattering length. Fig. 4.3 depicts the atomic exchange between the soliton and CW background for different values of \( \eta \). If the amplitude of the CW \( \eta = 0 \), there is no exchange of atoms between the soliton and CW, i.e., all the atoms are available only in the soliton. In Fig. 4.3, the thick dashed line clearly indicates that the atoms does not undergo exchange from the soliton to CW in the exponentially varying atomic scattering length when \( \xi = 0.05 \). In other words, we can also say that the soliton only appears and CW disappears. However in the case of nonzero background, the physical situation is entirely different as follows. The thick dotted \((\eta = 1)\) and thick solid \((\eta = 2)\) lines represent the atomic exchange between the soliton and CW background in the exponentially varying atomic scattering length when \( \xi = 0.05 \). The
Figure 4.3: The atomic exchange between the soliton and CW given by Eq. (4.15). Thick dashed line ($\eta = 0$, $\sigma = 1$), thick dotted ($\eta = 1$, $\sigma = \sqrt{5}$) and thick solid ($\eta = 2$, $\sigma = \sqrt{17}$) lines represent the atomic exchange between the soliton and CW when $\xi = 0.05$. The thin lines represent the same for the same physical parameters except $\xi = -0.05$. Thin lines represent the same for the identical physical parameters except $\xi = -0.05$. From Fig. 4.3, we observe that the exchange of atoms between the soliton and CW is faster when we increase the absolute value of atomic scattering length. Based on this physical process, one can infer that the number of atoms in the soliton remains constant against the variation of the scattering length. It is to be noted that when the time dependent atomic scattering length parameter $\xi = 0$, the exchange of atoms between the soliton and the CW background (4.15) reduces to the system of BEC trapped in time dependent linear potential [116].

4.5 Periodically varying atomic scattering length

Now, we consider the case of periodically varying atomic scattering length defined by $R(t) = m_0 + m \sin(\chi t)$ with $0 < m < 1$ [95, 145, 179]. Here $m_0$ and $\chi$ are the arbitrary constants. Recently, controlling the dynamics of the condensate has been discussed when the time dependent dispersion and nonlinearity coefficients are equal [222] i.e.,
\( D(t) = R(t) \). In this context, we analyze the BEC soliton on a CW background when \( D(t) \) and \( R(t) \) are periodic with same period for all \( t \). From the last equation of Eq. (4.7), the harmonic potential becomes zero (\( \Omega^2(t) = 0 \)) when we substitute the values of dispersion and nonlinearity coefficients. In other words, we can also say that the BEC is trapped only in the time dependent linear potential. According to these physical conditions, the BEC soliton embedded on a CW background in the periodically varying atomic scattering length is given by

\[
\psi = \psi_p = \sqrt{2q_0^2} \left[ -\eta + M_R \frac{M_R \cos \varphi - i\sigma \sin \varphi}{\sigma \cosh \theta - 2\eta \cos \varphi} \right] \exp \left[ i \left( \alpha_1(t) z + \alpha_2(t) + \varphi_c \right) \right],
\]

where \( \theta = M_R \Gamma(z, t) + \frac{2kM_R \delta_0}{\chi} (m - m \cos(\chi t) + m_0 \chi t) - \theta_0, \varphi = \frac{\sigma M_R \delta_0}{\chi} (m \cos(\chi t) - m - m_0 \chi t) - \varphi_0, \) and \( \varphi_c = k \Gamma(z, t) + \frac{m_0 \chi}{2} (2\eta^2 + k^2)(m - m \cos(\chi t) + m_0 \chi t) \). Here, \( \psi_p \) represents the BEC soliton on a CW background in the periodically varying scattering length. Other physical parameters are as follows: \( \alpha_1 = -2 \int_0^t \alpha(t) \, dt, \alpha_2 = -\frac{1}{2} \int_0^t \left[ m_0 + m \sin(\chi t) \right] \alpha_2(t) \, dt, T = \frac{2m \chi}{\chi} \left[ m - m \cos(\chi t) + m_0 \chi t \right], \) and \( \Gamma(z, t) = \sqrt{2q_0^2} (z - \int_0^t \left[ m_0 + m \sin(\chi t) \right] \alpha_1(t) \, dt) \). From Eq. (4.16), the trajectory of the soliton is defined by

\[
z = -2 \int_0^t \left[ m + m_0 \sin(\chi t) \right] \left( \int_0^t \alpha(t') \, dt' \right) \, dt
+ \frac{\sqrt{2kq_0^2} \chi}{\chi} (m - m \cos(\chi t) + m_0 \chi t),
\]

for the arbitrary constant \( \theta_0 = 0 \). From this relation, in the periodically varying atomic scattering length too, the trajectory of the soliton on a CW background strongly depends on the \( \alpha(t) \). Fig. 4.4(a) represents the soliton’s trajectory for three different forms of \( \alpha(t) \) in the periodically varying atomic scattering length. Fig. 4.4(a), dotted, dashed, and solid, lines indicate the snake, snake type with large oscillation, and periodic oscillation type of soliton trajectories for three different forms of \( \alpha(t) \) are as follows: 0.125, 0.025t, 0.25\cos(0.075t). Fig. 4.4(b) depicts the velocity of the soliton for the same forms of \( \alpha(t) \).

Now, we discuss the bright BEC soliton on a CW background under the manipulation of the periodically varying atomic scattering length for the different forms of
Figure 4.4: The trajectory of the bright BEC soliton given by Eq. (4.17) for different forms of $\alpha(t)$: $\alpha(t) = 0.125$ (dotted line), $\alpha(t) = 0.025t$ (dashed line), $\alpha(t) = 0.25\cos(0.075t)$ (solid line). Other system parameter values are $q_0 = 0.75$, $m = 0.95$, $\chi = 0.75$, $m_0 = 0$ and $k = 0.25$. (b) The velocity of the soliton for same forms of $\alpha(t)$. 
\( \alpha(t) \). Fig. 4.5(a) shows the dynamics of the BEC soliton on a CW background in the absence of the linear potential parameter \( \alpha(t) = 0 \). Fig. 4.5(b-d) depict the BEC soliton on a CW background for three different forms of \( \alpha(t) \). First we consider the value of \( \alpha(t) = 0.125 \) which implies the condensate being trapped in time independent linear potential. In this case, the trajectory of the soliton will be a ‘snake’ type as seen in Fig. 4.5(b). In the second case, \( \alpha(t) = 0.025t \) describes the condensate trapped in time dependent linear potential and the trajectory of the soliton is similar snake type with large oscillation as shown in Fig. 4.5(c). Next, we consider \( \alpha(t) = 0.25 \cos(\beta_p t) \) with \( \beta_p = 0.075 \), which represents the condensate trapped in oscillatory time dependent linear potential. In this case, the trajectory of the soliton has periodic oscillation with period \( 2\pi/\beta_p \) as portrayed in Fig. 4.5(d). The main feature of the BEC soliton on a CW background is the periodic peaking property on the soliton, which can be very strong without splitting of the soliton. The peaking property of the atomic density depends on the linear potential parameter \( \alpha(t) \). If we change the form of \( \alpha(t) \), the periodic peaking property of the BEC soliton also changes, which is clearly shown in Fig. 4.5. The total number of atoms in the soliton and the CW background in the periodically varying atomic scattering length is given by

\[
N_p = \int_{-\infty}^{\infty} \left( |\psi(z, t)|^2 - |\psi(\pm \infty, t)|^2 \right) dz = 4R_0 q_0^4 \sqrt{\rho^2 - 4\eta^2}.
\]  
(4.18)

Moreover, the exchange of atoms between the soliton and the CW background in the periodically varying scattering length is calculated as

\[
(\Lambda_p)_{s-CW} = \int_{-\infty}^{\infty} |\psi(z, t) - \psi(\pm \infty, t)|^2 dz
= 2R_0 q_0^4 \sqrt{\rho^2 - 4\eta^2} (2 + \eta W \cos(\beta T)).
\]  
(4.19)

As discussed in the previous section, if the CW background parameter \( \eta = 0 \) in the above relation, there is no exchange of atoms between the soliton and CW. Fig. 4.6 depicts the atomic exchange between the soliton and CW background. In Fig. 4.6, the solid \( (\eta = 0) \), dashed \( (\eta = 1.5) \) and dotted \( (\eta = 2) \) lines represent the atomic exchange between the soliton and CW in the periodically varying atomic scattering length for different values of \( \sigma \).
Bright soliton on a continuous wave background...

Figure 4.5: Dynamics of Bright BEC soliton on a CW background given by Eq. (4.16) when $R(t) = m_0 + m \sin (\chi t)$ for different forms of $\alpha(t)$. The physical parameter values are $q_0 = 1.5, \sigma = 2.5, \eta = 1, m = 0.95, \chi = 0.75, k = 0.25, m_0 = 0, \theta_0 = \varphi_0 = 0$. 
Bright soliton on a continuous wave background...

Figure 4.6: The atomic exchange between the soliton and CW given by Eq. (4.19). Solid ($\eta = 0, \sigma = 1$), dashed ($\eta = 1.5, \sigma = \sqrt{10}$) and dotted ($\eta = 2, \sigma = \sqrt{17}$) lines represent the atomic exchange between the soliton and CW. Other parameters are $m_0 = 0$, $m = 0.75$ and $\chi = 1$.

4.6 Summary

- In this chapter, we have investigated the soliton on a CW background for a system of BEC described by the variable coefficients GP equation.

- We have observed the two different scenarios based on the power of soliton as well as CW. One is the occurrence of MI when the power of the CW exceeds a quarter of the peak power of the soliton ($\sigma^2 < 4\eta^2$). The second possibility arises when the power of the CW is smaller than a quarter of the peak power of the soliton $\sigma^2 > 4\eta^2$. Under this condition, the resulting solution describes the bright matter wave soliton on CW background.

- The main characteristic of matter wave bright soliton on a CW background is the amplitude of the BEC soliton oscillates periodically without splitting of the soliton propagating in a system of BEC i.e., the breather like solitons can be generated in BEC.
• The compression and broadening of the soliton have been shown to get controlled by the FR managed atomic scattering length.

• In both the case of exponential and periodically varying atomic scattering lengths, the total number of atoms in the soliton and the atomic exchange between the soliton and CW are independent of the linear potential coefficient ($\alpha(t)$) but strongly depends on the FR managed atomic scattering length.

• The trajectory and velocity of the soliton mainly depends on the linear potential coefficient $\alpha(t)$.

• The dynamics of the soliton on a cnoidal wave background which depends on complex parameter $u$, it was clearly indicated in chapter-3. In this chapter, deals with the dynamics of the soliton on CW background depends on the amplituds/powers of the soliton and CW. The properties of the solutions in Chapters 3 and 4 depends on the different physical situations. That is why, we couldn’t combine the general form of results for both chapters 3 and 4.