CHAPTER 4

FUZZY HAHN-BANACH THEOREM*

4.1 Introduction:

Hahn-Banach theorem is one of the basic theorems in functional analysis. As such when the concepts of functional analysis are being extended to fuzzy contexts, analogous theorems are found to be essential. So, various such extensions of Hahn-Banach theorem in the fuzzy context were obtained by authors like G. S. Rhie; In Ah Hwang [R; H], T.V.Ramakrishnan [Ram], Xiao, Jian Zhong [Xia]. However, these are in context, different from that of fuzzy vector spaces that we consider. So, our theorems are different from the existing ones. G. S. Rhie; In Ah Hwang [R; H] proved the analytic form of the fuzzy Hahn-Banach theorem. Xiao, Jian Zhong [Xia] established the Hahn-Banach theorem for fuzzy normed spaces. In this chapter, we give the

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new Fuzzy Hahn-Banach Theorem on a fuzzy vector space over the set of fuzzy real numbers. Also we discuss some applications of the theorem.

4.2 Fuzzy Hahn-Banach theorem

Definition 4.2.1.

Let $u$ and $v$ be two fuzzy vectors, then we define $u \sim v$, if

$$(u - v)(0) = 1 \text{ and } (u - v)(x) = (u - v)(-x), \forall x \in X.$$ 

Theorem 4.2.2.

The above relation $\sim$ is an equivalence relation

Proof:

Similar to theorem 3.2.3

Proposition 4.2.3.

Addition is compatible with equivalence $\sim$.

i.e.; if $u_1 \sim v_1$ and $u_2 \sim v_2$, then $u_1 + u_2 \sim v_1 + v_2$.

Proof:

Similar to theorem 3.2.10
Proposition 4.2.4.

Fuzzy scalar multiplication is compatible with equivalence ~ i.e.; if \( u \sim v \) then \( \alpha u \sim \alpha v \), where \( \alpha \) is a non-negative fuzzy number

Proof:

Similar to theorem 3.2.1

Notation 4.2.5.

The set of equivalence classes of fuzzy vectors is denoted by \( \widetilde{X} \).

The equivalence class containing a fuzzy vector \( u \) will be denoted by \([u]\). Thus definition \([\alpha][u] = [\alpha u]\) for \( \alpha \) non-negative and \([u] + [v] = [u + v]\) are well done. Thus we have the following corollary:

Corollary 4.2.6.

\([u] = [v]\) if and only if \([u] - [v] = \tilde{0}\).

Note 4.2.7.

If \( \alpha \sim \beta \) then \( |\alpha| \sim |\beta| \).

Notation 4.2.8.

The class of absolute values \(|\beta|\) for \( \beta \in [\alpha] \) is denoted by \(|[\alpha]|\).
Definition 4.2.9.

Let $\bar{X}$ be the set of all equivalence classes of fuzzy vectors in the Hausdorff topological vector space $X$. A fuzzy linear functional on $\bar{X}$ is

$$f: \bar{X} \rightarrow \bar{R}$$

satisfying

$$f([\alpha][u] + [\beta][v]) = [\alpha]f([u]) + [\beta]f([v]); [u], [v] \in \bar{X}; [\alpha], [\beta] \in \bar{R}.$$

Theorem 4.2.10. (Fuzzy Hahn-Banach theorem)

Let $\bar{X}$ be a fuzzy vector space over $\bar{R}$ and

1. $M$ is a fuzzy linear subspace of $\bar{X}$

$$P: \bar{X} \rightarrow \bar{R}$$

satisfies

$$P([u] + [v]) \leq P([u]) + P([v])$$

and

$$P([\alpha][u]) = [\alpha]P([u]), \text{ if } [u] \in \bar{X}, [v] \in \bar{X}, [\alpha] \in \bar{R}^+.$$

2. $f: M \rightarrow \bar{R}$ is fuzzy linear functional and $f([u]) \leq P([u])$ on $M$,

then there exists a fuzzy linear functional $\Lambda$ defined on $\bar{X}$ such that

$$\Lambda([u]) \leq P([u]), \text{ if } [u] \in \bar{X} \text{ and } \Lambda([u]) = f([u]), \text{ if } [u] \in M.$$

Proof:

Step 1:

$M \neq \bar{X}$. Choose a fuzzy vector $[u_1] \in \bar{X}$ and $[u_1] \notin M$ and define

$$M_1 = \{ [u] + [\alpha][u_1] : [u] \in M, [\alpha] \in \bar{R}^+ \}.$$

Then $M_1 \subset \bar{X}$ also for
\[ [v^1] = [u^1] + [\alpha^1][u_1] \in M_1, \quad [v^2] = [u^2] + [\alpha^2][u_1] \in M_1, \]
\[ [v^1] + [v^2] = ([u^1] + [u^2]) + ([\alpha^1] + [\alpha^2])[u_1] \in M_1. \]
\[ [\beta] ([u] + [\alpha][u_1]) = [\beta][u] + ([\beta][\alpha])[u_1] \in M_1, \]

since \([\beta][u] \in M\) and \([\beta][\alpha] \in \mathbb{R}^+\). i.e. \(M_1\) is closed under addition and fuzzy scalar multiplication. So \(M_1\) is a fuzzy vector subspace of \(\overline{X}\).

Since \(f([u]) + f([v]) = f([u] + [v]) \leq P([u] + [v])\), but
\[ P([u] + [v]) = P([u] - [u_1] + [u_1] + [v]) \]
\[ \leq P([u] - [u_1]) + P([u_1] + [v]) \]
\[ f([u]) + f([v]) \leq P([u] - [u_1]) + P([u_1] + [v]) \]
\[ f([u]) - P([u] - [u_1]) \leq P([u_1] + [v]) - f([v]); \quad [u], [v] \in M \quad (8) \]

Let \([\gamma]\) be the least upper bound of the left side of (8) as \([u]\)
ranges over \(M\), then
\[ f([u]) - P([u] - [u_1]) \leq [\gamma] \leq P([v] + [u_1]) - f([v]) \]
\[ f([u]) - P([u] - [u_1]) \leq [\gamma] \Rightarrow f([u]) - [\gamma] \leq P([u] - [u_1]) \quad \text{and} \quad (9) \]
\[ f([v]) + [\gamma] \leq P([v] + [u_1]) \quad (10) \]

Define \(f_1\) on \(M_1\) by
\[ f_1([u] + [\alpha][u_1]) = f([u]) + [\alpha][\gamma] \quad (11) \]

To prove that \(f_1 = f\) on \(M\)
\[ [u] \in M \Rightarrow [u] = [u] + 0 [u_1] \]
\[ f_1([u]) = f_1([u] + 0 [u_1]) = f([u]) + 0 [\gamma] = f([u]), \quad \text{thus} \]
\[ f_i([u]) = f([u]), \] for every \([u] \in M.\]

Hence \( f_i \) is an extension of \( f \) over \( M. \) Also to prove that \( f_i \) defined above is fuzzy linear on \( M. \) \([v^1], [v^2] \in M_1, \) so that

\[ [v^1] = [u^1] + [\alpha^1][u_1], \quad [v^2] = [u^2] + [\alpha^2][u_1], \quad [\beta] \in \bar{R}^+, \text{then} \]

\[ [\beta][v^1] + [v^2] = ([\beta][u^1] + [u^2]) + ([\beta][\alpha^1] + [\alpha^2])[u_1] \]

\[ f_i([v^1]) = f([u^1]) + [\alpha^1][\gamma] \]

\[ f_i([v^2]) = f([u^2]) + [\alpha^2][\gamma] \]

\[ f_i([\beta][v^1] + [v^2]) = f([\beta][u^1] + [u^2]) + ([\beta][\alpha^1] + [\alpha^2])[\gamma] \]

\[ = [\beta]f([u^1]) + f([u^2]) + [\beta][\alpha^1][\gamma] + [\alpha^2][\gamma] \]

\[ = [\beta](f([u^1]) + [\alpha^1][\gamma]) + f([u^2]) + [\alpha^2][\gamma] \]

\[ = [\beta]f_i([u^1]) + [\alpha^1][u_1]) + f_i([u^2] + [\alpha^2][u_1]) \]

\[ = [\beta]f_i([v^1]) + f_i([v^2]). \]

Hence \( f_i \) is a fuzzy linear on \( M_1 \) from (9), (10) and (11) it follows that

\( f_i \leq p \) on \( M_1. \)

**Step 2:**

Let \( \varphi \) be the collection of all ordered pairs \((M^1, f^1)\) where \( M^1 \) is a fuzzy subspace of \( \bar{X} \) that contains \( M \) and \( f^1 \) is a fuzzy linear functional on \( M^1 \) that extends \( f \) and satisfies \( f^1 \leq p \) on \( M^1. \) \( \varphi \) is partially ordered by the order \( \leq \) defined by \((M^1, f^1) \leq (M^2, f^2)\) if \( M^1 \subseteq M^2 \) and \( f^1 = f^2 \) on \( M^1. \) By Hausdorff maximality theorem there exists a
maximal totally ordered sub collection $\Omega$ of $\emptyset$. Let $K$ be the 
collection of all $M^i$ such that $(M^i, f^i) \in \Omega$, then $K$ is totally ordered by 
set inclusion and $M_0$, the union of all members of $K$ in the fuzzy 
vector subspace of $\bar{X}$. If $u \in M_0$ then $u \in M^i$ for some $M^i \in K$. Define 
$\Lambda u = f^i(u)$, where $f^i$ is the function which occur in the pair $(M^i, f^i) \in \Omega$.
Hence $\Lambda$ is fuzzy linear and $\Lambda \leq P$. If $M_0$ were a proper subspace of $\bar{X}$, the 
first part of the proof would give a further extension of $\Lambda$ and this would 
contradict the maximality of $\Omega$. Then $M_0 = \bar{X}$. This completes the proof.

**Theorem 4.2.11.**

Let $\bar{X}$ be a fuzzy topological vector space over $\bar{R}$ and let $p$ be a fuzzy semi norm on $\bar{X}$. Let $M$ be a fuzzy linear subspace of $\bar{X}$ and 
f a fuzzy linear functional defined on $M$ such that $|f([u])| \leq p([u])$, if 
$[u] \in M$. Then, there exists a fuzzy linear functional $\Lambda$ defined on $\bar{X}$ 
such that

$$-p([u]) \leq \Lambda([u]) \leq p([u]), \ \forall [u] \in \bar{X}$$

**Proof:**

$$|f([u])| \leq p([u]), \ \forall [u] \in M \Rightarrow f([u]) \leq p([u]).$$ Since $p$ is a fuzzy 
semi norm,
\[ p([u] + [v]) \leq p([u]) + p([v]) \quad \text{and} \quad p([\alpha][u]) = ||\alpha|| p([u]) \quad \text{where} \quad [u], [v] \in \overline{X} \quad \text{and} \quad [\alpha] \in \overline{R}. \]

Then by Fuzzy Hahn-Banach theorem, there exists a fuzzy linear functional \( \Lambda \) defined on \( \overline{X} \) such that

\[ \Lambda([u]) \leq p([u]), \quad \forall [u] \in \overline{X} \quad \text{and} \quad -p([u]) \leq -\Lambda([u]), \quad -p([-u]) \leq -\Lambda([-u]), \]

\[ -p([-u]) \leq \Lambda([-u]) \leq p([-u]). \]

Since \( p \) is a fuzzy semi norm \( p([-u]) = p([u]) \), we get

\[ -p([u]) \leq \Lambda([u]) \leq p([u]), \quad \forall [u] \in \overline{X} \]

**Corollary 4.2.12.**

\( \overline{X} \) is a fuzzy normed space and \([u_0] \in \overline{X} \), there exists a fuzzy linear functional \( \Lambda \) is defined on \( \overline{X} \) such that

\[ \Lambda([u_0]) = ||[u_0]|| \quad \text{and} \quad |\Lambda([u])| \leq ||[u]||, \quad \forall [u] \in \overline{X}. \]

**Proof:**

If \([u_0] = \overline{0} \), take \( \Lambda = \overline{0} \). If \([u_0] \neq \overline{0} \), apply the above theorem with \( p([u]) = ||[u]|| \). \( M \), the one dimensional space generated by \([u_0] \) and \( f([\alpha][u_0]) = ||[\alpha]| ||[u_0]|| \) on \( M \).
**Definition 4.2.13.**

Let $X$ be a fuzzy normed space. A fuzzy linear functional $f$ on $X$ is said to be bounded if there exist an $[\alpha] \in \bar{R}^+$ such that

$$|f([u])| \leq [\alpha] \| [u] \|, \quad \forall [u] \in X.$$

**Definition 4.2.14.**

Let $f$ be a bounded fuzzy linear functional on a fuzzy normed space $X$, then $\| f \|$ is defined by

$$\| f \| = \inf \{ [\alpha] \in \bar{R}^+ : |f([u])| \leq [\alpha] \| [u] \|, \forall [u] \in X \}.$$

**Proposition 4.2.15.**

The norm $\| \|$ defined above is a fuzzy norm.

**Proof:**

1. If $f = 0$, the zero functional then $\| f \| = 0$.

2. If $f \neq 0$, then $\| f \| > 0$.

i.e., $\| f \|$ is non-negative

$$\| f \| = \inf \{ [\alpha] \in \bar{R}^+ : |f([u])| \leq [\alpha] \| [u] \| \}$$

$$= \inf \{ [\alpha] \in \bar{R}^+ : |\| f([u])| \leq [\alpha] \| [u] \| \}$$

$$= \inf \{ [\| f([u])| \| [\alpha] \| [u] \| \}$$

(By replacing $\alpha$ by $[\| f([u])| \| [\alpha] \| [u] \|$, $\| f([u])| \| [\alpha] \| [u] \| \}$)
\[ \|f + g\| = \inf \{ [\alpha] \in \bar{R}^+ : |f([u]) + g([u])| \leq [\alpha] \| [u] \| \} \]

\[ \leq \inf \{ [\alpha_1] + [\alpha_2] \in \bar{R}^+ : |f([u])| + |g([u])| \leq ([\alpha_1] + [\alpha_2]) \| [u] \| \} \]

\[ \leq \inf \{ [\alpha_1] \in \bar{R}^+ : |f([u])| \leq [\alpha_1] \| [u] \| \} + \inf \{ [\alpha_2] \in \bar{R}^+ : |g([u])| \leq [\alpha_2] \| [u] \| \} \]

\[ \leq \| f \| + \| g \| \]

Hence the norm \( \| \| \) defined above is a fuzzy norm.

**Remark 4.2.16.**

If \( f \) is a bounded fuzzy linear map then, \( |f([u])| \leq \| f \| \| [u] \| \).

**Corollary 4.2.17.**

Let \( \bar{X} \) be a fuzzy normed space. Then corresponding to every \( [u_0] \in \bar{X} \), there exists a bounded fuzzy linear map \( f_{u_0} \) on \( \bar{X} \) such that

\[ f_{u_0} ([u_0]) = \| [u_0] \|^2 \quad \text{and} \quad \| f_{u_0} \| \leq \| [u_0] \|. \]
Proof:

Take $\bar{Y} = \text{linear span of } [u_o]$, then $\bar{Y}$ will be a subspace of $\bar{X}$.

Define $f: \bar{Y} \to \mathbb{R}$ by $f([\alpha][u_o]) = [\alpha] \| [u_o] \|^2$.

Then $f$ is a fuzzy linear map on $\bar{Y}$.

ie; $f ([\alpha][u_o] + [\beta][u_o]) = f (([\alpha]+[\beta])[u_o])$

$= ([\alpha]+[\beta]) \| [u_o] \|^2$

$= [\alpha] \| [u_o] \|^2 + [\beta] \| [u_o] \|^2$

$= f([\alpha][u_o]) + f([\beta][u_o])$

$f([\gamma]([\alpha][u_o])) = f(([\gamma][\alpha])[u_o]) = [\gamma][\alpha] \| [u_o] \|^2$

$= [\gamma] \{[\alpha] \| [u_o] \|^2\} = [\gamma] f([\alpha][u_o])$.

Define $p: \bar{X} \to \mathbb{R}$, by $p([u]) = \| [u_o] \| \| [u] \|$.

Then $f([u]) \leq p([u])$ on $\bar{Y}$.

Also $p([u] + [v]) \leq p([u]) + p([v])$

$p([\alpha][u]) = \|[\alpha]\| p([u])$.

Hence by fuzzy Hahn-Banach theorem, there exists a fuzzy linear functional $f_{u_o}: \bar{X} \to \mathbb{R}$ such that $f_{u_o}([u_o]) = f([u])$ on $\bar{Y}$ and $f_{u_o}([u]) \leq p([u]), \forall [u] \in \bar{X}$.

Also $f_{u_o}([u_o]) = f([u_o]) = \| [u_o] \|$ and

$| f_{u_o}([u]) | \leq p([u]) = \| [u_o] \| \| u \|$

ie; $| f_{u_o}([u]) | \leq \| [u_o] \| \| [u] \|$
Theorem 4.2.18.

Suppose \( f \) be a bounded fuzzy linear functional on a fuzzy normed subspace \( \bar{Y} \) of a fuzzy normed space \( \bar{X} \). Then there exists a bounded fuzzy linear functional \( F \) defined on the whole space having the same fuzzy norm as \( f \).

Proof:

We have \( |f([u])| \leq \| f \| \| [u] \| \).

Define \( p: \bar{X} \to \bar{R} \), by \( p([u]) = f([u]) \) \( \forall [u] \in \bar{X} \).

Then \( p([u] + [v]) \leq p([u]) + p([v]) \).

Also \( |f([u])| \leq p([u]), \forall [u] \in \bar{X} \).

By fuzzy Hanh-Banach theorem we can extend \( f \) to a new fuzzy linear functional \( F \), defined on the whole space \( \bar{X} \) such that

\[ |F([u])| \leq p([u]) = \| f \| \| [u] \|. \]

In view of this result, it is clear that \( F \) is a bound fuzzy linear functional and also that \( \| F \| \leq \| f \| \) \hspace{1cm} (12)

Also we have \( \| F \| = \inf \{ [\alpha] \in \bar{R}^+ : |F([u])| \leq [\alpha] \| [u] \| \} \)

When \( [u] \in \bar{Y} \), \( |f([u])| = |F([u])| \leq \| F \| \| [u] \| \).
i.e.; $|f| \leq \|F\|$  \hspace{1cm} (13)

From (12) & (13) $||f|| = ||F||$.

**Theorem 4.2.19.**

Suppose $p$ is a fuzzy semi norm of a fuzzy vector space $\vec{x}$ over $\vec{R}$. Then

(i) $p(0) = 0$

(ii) $|p([u]) - p([v])| \leq p([u] - [v])$

(iii) $\{[u] : p([u]) = 0\}$ is a fuzzy subspace of $\vec{x}$.

**Proof:**

(i) Since $p([\alpha][u]) = |[\alpha]| p([u])$

Put $[\alpha] = 0$, then $p(0[u]) = |0| p([u]) = 0$.

(ii) $p([u]) = p([u] - [v] + [v]) \leq p([u] - [v]) + p([v])$

i.e.; $p([u]) - p([v]) \leq p([u] - [v])$

Also $p([v]) - p([u]) \leq p([v] - [u])$. But $p([v] - [u]) = p([u] - [v])$.

Hence $|p([u]) - p([v])| \leq p([u] - [v])$.

(iii) If $p([u]) = p([v]) = 0$ and $[\alpha]$ and $[\beta]$ are fuzzy numbers, then

$0 \leq p([\alpha][u] + [\beta][v]) \leq |[\alpha]| p([u]) + |[\beta]| p([v]) = 0$.

This proves (iii).
Theorem 4.2.20.

Let $p_1$ and $p_2$ be fuzzy semi norms on $X_1, X_2$ respectively and define $p$ on $X = X_1 \times X_2$, by $p([u], [v]) = \min (p_1([u]), p_2([v]))$. Then $p$ is a fuzzy semi norm on $X$.

Proof:

$p([u], [v]) \geq 0$

$p\{[\alpha]([u], [v])\} = p([\alpha][u], [\alpha][v]) = \min \{p_1([\alpha][u]), p_2([\alpha][v])\}$

$= \min \{||\alpha||p_1([u]), ||\alpha||p_2([v])\}$

$= ||\alpha||\min \{p_1([u]), p_2([v])\}$

$= ||\alpha||p([u], [v])$

$p([u_1], [v_1]) + ([u_2], [v_2]) = p([u_1] + [u_2]), ([v_1] + [v_2])$

$= \min \{p_1([u_1] + [u_2]), p_2([v_1] + [v_2])\}$

$\leq \min \{p_1([u_1]) + p_1([u_2]), p_2([v_1]) + p_2([v_2])\}$

$= \min \{p_1([u_1]), p_2([v_1])\} + \min \{p_1([u_2]), p_2([v_2])\}$

$\leq \min \{p_1([u_1]), p_2([v_1])\} + \min \{p_1([u_2]), p_2([v_2])\}$

$= p([u_1], [v_1]) + p([u_2], [v_2]).$
Theorem 4.2.21.

Suppose $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ and $\Lambda$ are fuzzy linear functional on a fuzzy vector space $\bar{X}$. Let $N = \{[u] : \Lambda_1 [u] = \Lambda_2 [u] = \ldots = \Lambda_n [u] = 0\}$. Then the following three properties are equivalent.

1. There are fuzzy numbers $[\alpha_1], [\alpha_2], \ldots, [\alpha_n]$ such that

$$\Lambda = [\alpha_1] \Lambda_1 + \ldots + [\alpha_n] \Lambda_n$$

2. There exists $[\beta] \in \bar{R}$ such that $|\Lambda([u])| \leq [\beta] \sup_{1 \leq i \leq n} |\Lambda_i([u])|$, $[u] \in \bar{X}$

3. $\Lambda [u] = 0$, $\forall [u] \in N$.

Proof:

$(1) \Rightarrow (2)$

$$|\Lambda [u]| \leq |[\alpha_1]| |\Lambda_1 [u]| + \ldots + |[\alpha_n]| |\Lambda_n [u]|$$

$$\leq [\beta] \sup_{1 \leq i \leq n} |\Lambda_i [u]|$$

where $[\beta] = \sup_{1 \leq i \leq n} |[\alpha_i]|$

ie; $|\Lambda [u]| \leq [\beta] \sup_{1 \leq i \leq n} |\Lambda_i ([u])|$.

$(2) \Rightarrow (3)$

For every $[u] \in N$, $\Lambda_i [u] = 0$.

$$|\Lambda [u]| \leq [\beta] \sup_{1 \leq i \leq n} |\Lambda_i [u]|$$

$$= [\beta] \sup_{1 \leq i \leq n} |0|$$
But absolute value of a fuzzy number is non-negative.

Hence  \(|\Lambda[u]| = \bar{0} \implies \Lambda([u]) = \bar{0}, \ \forall [u] \in \mathbb{N}.

(3) \implies (1)

Define \( \Pi: \tilde{\mathbb{R}} \to \tilde{\mathbb{R}}^n \), by \( \Pi([u]) = (\Lambda_1[u], \Lambda_2[u], \ldots, \Lambda_n[u]). \)

Also define a fuzzy linear functional \( F \) on \( \tilde{\mathbb{R}}^n \) (i.e.; \( F: \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}} \)), by

\[
F([u_1], [u_2], \ldots, [u_n]) = \alpha_1[u_1] + \alpha_2[u_2] + \ldots + \alpha_n[u_n],
\]

where \([u_1], [u_2], \ldots, [u_n] \in \tilde{\mathbb{R}}. \)

Hence \( \Lambda: \tilde{\mathbb{R}} \to \tilde{\mathbb{R}} \). i.e.; \( \Lambda = F \circ \Pi \), for some functions \( F \) on \( \tilde{\mathbb{R}}^n \).

Thus,

\[
\Lambda[u] = F(\Pi([u])) = F(\Lambda_1[u_1], \Lambda_2[u_2], \ldots, \Lambda_n[u_n])
\]

\[
= [\alpha_1]\Lambda_1[u_1] + [\alpha_2]\Lambda_2[u_2] + \ldots + [\alpha_n]\Lambda_n[u_n]
\]

\[
= \sum_{i=1}^{n} a_i \Lambda_1[u_i]. \quad \text{Which is (1)}.
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