6.1 Introduction:

Fuzzy topological vector spaces are introduced by A. K. Katsaras and D. B. Liu [K; L]. In 1981, Katsaras [kat], changed the definition of fuzzy topological vector spaces. After a considerable period of time, J. J. Buckley and Aimien Yan [B; A] opened the way towards the development of fuzzy topological vector space by introducing the notion of fuzzy vector space in a new manner. In this chapter, we establish a relation between fuzzy topological vector spaces and topological group. Also some properties of fuzzy topological vector spaces are discussed.

6.2 Fuzzy topological vector space and topological group

Theorem 6.2.1

Let $\mathcal{X}$ be a fuzzy vector space and let $\tau$ be a topology on $\mathcal{X}$. Then $\tau$ is compatible with the vector space structure of $\mathcal{X}$, that is, addition and scalar multiplication are continuous if and only if
(i) \( \tau \) is compatible with the additive group structure of \( \bar{X} \)

(ii) for each \( a \in R \), the mapping \([u] \to [au] \), (\([u] \in \bar{X} \)) is continuous at

\([u] = \bar{0} \)

(iii) for each \([u] \in \bar{X} \), the mapping \( a \to [au] \), (\( a \in R \)) is continuous at

\( a = 0 \)

(iv) the mapping \((a, [u]) \to [au] \), (\( a \in R, [u] \in \bar{X} \)) is continuous at

\((0, \bar{0}) \).

**Proof:**

Assume that \( \tau \) is compatible with vector space structure. Then conditions (ii)–(iv) follows trivially from the definition of a fuzzy topological vector space over \( R \) and condition (i) results from the continuity of \(([u], [v]) \to [u] + [v] \) and \([u] \to (-1)[u] = [-u] \).

Conversely, let \( a_0 \in R \), \([u_0] \in \bar{X} \), the problem is to show that \((a, [u]) \to [au] \) is continuous at \((a_0, [u_0])\). Let \( U \) be a neighborhood
of $\bar{0}$, then $[a_0 u_0] + U$ is a typical neighborhood of $[a_0 u_0]$. We seek neighborhood $A, W$ of $0, \bar{0}$ such that

$$ (a_0 + A)([u_0] + W) \subseteq [a_0 u_0] + U. $$

Let $V$ be a neighborhood of $\bar{0}$ such that

$$ V + V + V \subseteq U. $$

By (iii), there exists a neighborhood $A_1$ of $0$ such that $A_1[u_0] \subseteq V$. By (ii), there exists a neighborhood $W_1$ of $\bar{0}$ such that $a_0 W_1 \subseteq V$. By (iv), there exists a neighborhood $A_2$ of $0$ and a neighborhood $W_2$ of $\bar{0}$ such that $A_2 W_2 \subseteq V$.

Let $A = A_1 \cap A_2$ and $W = W_1 \cap W_2$, then evidently $A[u_0] \subseteq V$, $a_0 W \subseteq V$ and $AW \subseteq V$. Hence by distributivity,

$$ (a_0 + A)([u_0] + W) \subseteq [a_0 u_0] + A[u_0] + a_0 W + AW $$

$$ \subseteq [a_0 u_0] + V + V + V $$

$$ \subseteq [a_0 u_0] + U. $$

Hence the result.
Definition 6.2.2

If $\mathcal{X}$ is fuzzy vector space, a homothetic mapping is a mapping of the form $[u] \rightarrow [au] + [v]$, $([v] \in \mathcal{X})$. Where $a \in \mathbb{R}$ and $[v] \in \mathcal{X}$ are fixed. If $a \neq 0$, such mapping is bijective with a homothetic inverse mapping namely, $[u] \rightarrow [a^{-1}u] - [a^{-1}v]$.

Note 6.2.3

In a topological group, the translation mapping plays an important role; in a fuzzy topological vector space; the mapping that play the analogous role are homotheties.

Theorem 6.2.4

In a fuzzy topological vector space every non-constant homothety is a homeomorphism.

Proof:

Define the homothety, $f([u]) = [au] + [v]$, where $a \neq 0$. Since the inverse of a homothety is also a homothety, it is enough to show that $f$ is continuous and this is evident from the factorization $[u] \rightarrow [au] \rightarrow [au] + [v]$. 
Note 6.2.5

One consequence of this proposition is that every topology \( \tau \) on a fuzzy topological vector space is translation invariant. A set \( E \subseteq \overline{X} \) is open if and only if each of its translates \([u] + E\) is open. Then \( \tau \) is completely determined by any local base. A local base of a fuzzy topological vector space is thus a collection \( \mathcal{B} \) of neighborhoods of \( \overline{0} \) such that every neighborhood of \( \overline{0} \) contains a member of \( \mathcal{B} \). The open sets of \( \overline{X} \) are then precisely those that are unions of translates of members of \( \mathcal{B} \).

Theorem 6.2.6

If \( \overline{X} \) is a fuzzy topological vector space over \( \mathbb{R} \) and \( \overline{M} \) fuzzy vector sub space over \( \mathbb{K} \) then \( \overline{M} \), with the relative topology is also a fuzzy topological vector space over \( \mathbb{R} \).

Proof:

The restriction of the continuous mappings \( ([u], [v]) \rightarrow [u] + [v] \) and \( (\alpha, [u]) \rightarrow [\alpha u] \), to \( \overline{M} \times \overline{M} \) and \( \mathbb{R} \times \overline{M} \) respectively are continuous for the relative topology of \( \overline{M} \).
Theorem 6.2.7

In a fuzzy topological vector space, the closure of a fuzzy vector subspace is a fuzzy vector subspace.

Proof:

If \( \overline{M} \) is a fuzzy vector subspace of a fuzzy topological vector space over \( \mathbb{R} \), then the continuous mapping \( ([u], [v]) \rightarrow [u] + [v], \) which maps \( \overline{M} \times \overline{M} \rightarrow \overline{M} \) also maps \( \text{clo}(\overline{M} \times \overline{M}) = \text{clo} \overline{M} \times \text{clo} \overline{M} \rightarrow \text{clo} \overline{M} \), then \( [u] \in \text{clo} \overline{M} \) whenever \( [u], [v] \in \text{clo} \overline{M} \). Similarly the continuous mapping \( (a, [u]) \rightarrow [au], \) which maps \( \mathbb{R} \times \overline{M} \rightarrow \overline{M} \) also maps

\[
\text{clo}(\mathbb{R} \times \overline{M}) = \mathbb{R} \times \text{clo} \overline{M} \rightarrow \text{clo} \overline{M}.
\]

i.e.; \( [au] \in \text{clo} \overline{M}, \forall a \in \mathbb{R} \) and \( [u] \in \text{clo} \overline{M} \).

Theorem 6.2.8

In a fuzzy topological vector space \( \overline{X} \) over \( \mathbb{R} \)

(a) every neighborhood of \( \overline{0} \) contains a balanced neighborhood of \( \overline{0} \) and

(b) every convex neighborhood of \( \overline{0} \) contains a balanced convex neighborhood of \( \overline{0} \).
Proof : (a)

Suppose U is a neighborhood of \( \bar{0} \) in \( \bar{X} \). Since scalar multiplication is continuous, there is a \( \delta > 0 \) and there is a neighborhood \( V \) of \( \bar{0} \) in \( \bar{X} \) such that \( aV \subset U \) whenever \( |a| < \delta \).

Let \( W \) be the union of all these sets \( aV \). Then \( W \) is a neighborhood of \( \bar{0} \), \( W \) is balanced and \( W \subset U \).

Proof : (b)

Suppose U is a convex neighborhood of \( \bar{0} \) in \( \bar{X} \). Let \( A = \bigcap aU \), where \(-1 \leq a \leq 1\). Choose \( W \) as in part (a). Since \( W \) is balanced, \( a^{-1}W = W \), when \( |a| = 1 \). Hence \( W \subset aU \). Thus \( W \subset A \), which implies that the interior of \( A \) is a neighborhood of \( \bar{0} \). Clearly, interior of \( A \) is a subset of \( U \). Being an intersection, with convex sets, \( A \) is convex. Hence, so is intA. To prove that intA is a neighbourhood with desired property, we have to show that intA is balanced; for this it is enough to prove that \( A \) is balanced.

Choose \( c \) such that \( 0 \leq c \leq 1 \), then

\[
    cA = \bigcap_{|a|-1} caU.
\]

Since \( aU \) is a convex set that contains \( \bar{0} \), we have

\[
    caU \subset aU.
\]
Thus \( cA \subset A \). Which completes the proof.

**Theorem 6.2.9**

Suppose \( V \) is a neighborhood of \( \vec{0} \) in a fuzzy topological vector space over \( \mathbb{R} \).

(a) If \( 0 < r_1 < r_2 < ... \) and \( r_n \rightarrow \infty \) as \( n \rightarrow \infty \), then

\[
\overline{X} = \bigcup_{n=1}^{\infty} r_n V.
\]

(b) Every compact subset \( \overline{K} \) of \( \overline{X} \) is bounded.

(c) If \( \delta_1 > \delta_2 > ... \) and \( \delta_n \rightarrow 0 \) as \( n \rightarrow \infty \) and if \( V \) is bounded, then the collection \( \{\delta_n V, n=1,2,3,...\} \) is a local base for \( \overline{X} \).

**Proof:** (a)

Fix \( [u] \in \overline{X} \). Since \( a \rightarrow [au] \) is a continuous mapping of \( \mathbb{R} \rightarrow \overline{X} \), then the set of all \( a \) with \( [au] \in V \) is open, contains \( \vec{0} \), hence contains \( \frac{1}{r_n} \) for all large \( n \). Thus \( \frac{1}{r_n} [u] \in V \) or \( [u] \in r_n V \), for large \( n \).

**Proof:** (b)

Let \( W \) be a balanced neighbourhood of \( \vec{0} \) such that \( W \subset V \).

Then by (a)

\[
\overline{K} \subset \bigcup_{n=1}^{\infty} nW.
\]
Since $\bar{K}$ is compact, there are integers $n_1 < n_2 < \ldots < n_s$ such that

$$\bar{K} \subset n_1W \cup n_2W \cup \ldots \cup n_sW = n_sW.$$  

The equality holds since $W$ is balanced. If $t > n_s$, it follows that

$$\bar{K} \subset tW \subset tV.$$  

Proof: (c)

Let $U$ be a neighborhood of $\bar{0}$ in $\bar{X}$. If $V$ is bounded, there exists $\delta > 0$ such that $V \subset tU$, $\forall \ t > s$. If $n$ is so large that $\delta_n < 1$, it follows that $V \subset \left(\frac{1}{\delta_n}\right) U$. Hence $U$ actually contains all but finitely many of the sets $\delta_nV$.

Note 6.2.10

This property is often described as neighborhoods of $\bar{0}$ are absorbing.

Theorem 6.2.11

Let $\bar{X}$ and $\bar{Y}$ be fuzzy topological vector spaces over $\mathbb{R}$. If $\Lambda: \bar{X} \to \bar{Y}$ is linear and continuous at $\bar{0}$, then $\Lambda$ is continuous. In fact, $\Lambda$ is uniformly continuous in the following sense: To each
neighborhood $W$ of $\bar{o}$ in $\bar{V}$ there corresponds a neighborhood $V$ of $\bar{o}$ in $\bar{X}$ such that $[v] - [u] \in V \Rightarrow \Lambda [v] - \Lambda [u] \in W$.

**Proof:**

Once $W$ is chosen, the continuity of $\Lambda$ at $\bar{o}$ shows that $\Lambda V \subseteq W$ for some neighborhood $V$ of $\bar{o}$. If $[v] - [u] \in V$, the linearity of $\Lambda$ shows that $\Lambda [v] - \Lambda [u] = \Lambda ([v] - [u]) \in W$. Thus $\Lambda$ maps the neighborhood $[u] + V$ of $[u]$ into the neighborhood $\Lambda [u] + W$ of $\Lambda [u]$. Which says that $\Lambda$ is continuous at $[u]$.

**Theorem 6.2.12**

Let $\bar{X}$ be a fuzzy topological vector space over $\mathbb{R}$. Let $\Lambda : \bar{X} \rightarrow \mathbb{R}$ is linear. Assume $\Lambda [u] \neq 0$, for some $[u] \in \bar{X}$. Then each of the following four properties implies the other three.

(a) $\Lambda$ is continuous.

(b) The null space $N(\Lambda)$ is closed.

(c) $N(\Lambda)$ is not dense in $\bar{X}$.

(d) $\Lambda$ is bounded in some neighborhood $V$ of $\bar{o}$.

**Proof:**

Since $N(\Lambda) = \Lambda^{-1}(\{0\})$ and $\{0\}$ is the closed subset of $\mathbb{R}$,
(a) ⇒ (b). By hypothesis, \( N(A) \neq \bar{X} \). Hence (b) ⇒ (c). Assume (c) holds. That is assume that the complement of \( N(A) \) has non-empty interiors.

\[
([u] + V) \cap N(A) = \emptyset,
\]

(16)

for some \([u] \in \bar{X}\) and some balanced neighborhood \( V \) of \( \bar{o} \). Then \( \Lambda V \) is a balanced subset of \( R \). Thus either \( \Lambda V \) is bounded, in which case (d) holds, or \( \Lambda V = R \). In the later case there exists \([v] \in V\) such that \( \Lambda[v] = -\Lambda[u] \) and so \([v] + [u] \in N(A)\) in contradiction to (16). Then (c) ⇒ (d). Finally if (d) holds, then \(|\Lambda[u]| < M\),

\[\forall [u] \in V \text{ and for some } M < \infty. \text{ If } r > 0 \text{ and if } W = \left(\frac{r}{M}\right)V, \text{ then } \]

\[|\Lambda[u]| < r, \quad \forall [u] \in W. \text{ Hence } \Lambda \text{ is continuous at } \bar{o}, \text{ by the theorem 6.2.11, and this implies (a).}\]

**Definition 6.2.13**

A fuzzy topological vector space over \( R \) is said to be metrizable, if it is metrizable as a topological space. In view of (2.3.17), a separated fuzzy topological vector space \( \bar{E} \) is metrizable if and only if there exists a fundamental sequence of neighbourhoods of \( \bar{0} \), in which case there exists a metric \( d \), compatible with the topology of \( \bar{E} \), which is additively invariant.
Lemma 6.2.14

Let $\bar{E}$ be a fuzzy vector space, with a metrizable topology such that

(i) the topology is compatible with the additive group structure

(ii) for each $[u] \in \bar{E}$, the mapping $a \to [au]$ is continuous at $a = 0$ and

(iii) for each $a \in \mathbb{R}$, the mapping $[u] \to [au]$ is continuous at $[u] = 0$;

then the topology is compatible with the vector space structure. That is $\bar{E}$ is a metrizable fuzzy topological vector space over $\mathbb{R}$.

Proof:

The point is that in the presence of metrizability, condition (iv) of [6.2.1] is superfluous. Thus in view of [6.2.1], it will suffice to verify that $(a, [u]) \to [au]$ is continuous at $(0, 0)$.

We observe first that the mapping $(a, [u]) \to [au]$ is separately continuous in $a$ and in $[u]$; this follows from (i) - (iii) and the formulas $[au_0] - [a_0u_0] = (a - a_0)u_0$ and $[a_0u] - [a_0u_0] = a_0(u - u_0)$.

Let $d$ be any metric that generates the topology. Assuming $a_n \to 0$ and $d([u_n], 0) \to 0$, we wish to show that $d([a_nu_n], 0) \to 0$.

Suppose to the contrary that there exists an $\varepsilon > 0$ such that
d([a_n u_n], \bar{0}) \geq \varepsilon, \text{ for infinitely many } n. \text{ We can suppose that } d([a_n u_n], \bar{0}) \geq \varepsilon, \forall n. \text{ Let } U \text{ be a neighbourhood of } \bar{0} \text{ such that } d([u] + [v], \bar{0}) < \varepsilon, \forall [u],[v] \in U \text{ (this is possible by (i)). We can suppose } U \text{ to be closed and symmetric.}

For each } n, \text{ let }
\begin{align*}
C_n &= \{a \in \mathbb{R}: [a u_i] \in U\}; \forall i \geq n \\
&= \bigcap_{i \geq n} \{a: [a u_i] \in U\}.
\end{align*}

Since each of the sets \{a: [a u_i] \in U\} is closed (it is the inverse image of the closed set \(U\) under the continuous mapping \(a \rightarrow [a u_i]\)), it follows that the \(C_n\) are closed. Moreover,
\[
\bigcup_{n=1}^{\infty} C_n = \mathbb{R}.
\]

For, given any \(a \in \mathbb{R}\), one has \([a u_i] \rightarrow \bar{0}\), by (iii); consequently there exists an index \(n\) such that \([a u_i] \in U\), \(\forall i \geq n\). i.e. \(a \in C_n\).

According to the relation (i), \(R\) is the union of a sequence of closed sets \(C_n\). It is a classical property of \(R\) is that at least one of the sets \(C_n\) must have an interior point. Let \(a_0\) be an interior point of \(C_k\).

For a suitable \(\delta > 0\), the open set
\[
B = \{a: |a - a_0| < \delta\} \subset C_k.
\]
Let \(A = \{b: |b| < \delta\}\); then \(A\) is a symmetric neighbourhood of \(0\), and
Choose an index $n$ so large that $a_i \in A$, $\forall i \geq n$ (possible, since $a_i \to 0$), $[a_0 u_i] \in U$, $\forall i \geq n$ (possible by (iii)). Let $m = \max \{k, n\}$. Since $m \geq n$, we have $a_m \in A$ (by the definition of $n$), and therefore $a_0 + a_m \in C_k$, by (17). Since $m \geq k$, it follows from the definition of $C_k$ that $(a_0 + a_m)[u_m] \in U$. Also $m \geq n \Rightarrow a_0[u_m] \in U$ (by the definition of $n$) and therefore $[-a_0 u_m] \in -U = U$. Taking $[u] = [-a_0 u_m]$ and $[v] = (a_0 + a_m)[u_m]$; we have $[u], [v] \in U$ and $[u] + [v] = [a_m u_m]$; then $d([a_m u_m], \bar{0}) = d([u] + [v], \bar{0}) < \varepsilon$, by the definition of $U$, where as $d([a_m u_m], \bar{0}) \geq \varepsilon$, by the choice of $\varepsilon$. Which is a contradiction.

Hence the proof.

**Definition 6.2.15**

A sub set $\overline{M}$ of a fuzzy topological vector space over $R$ is said to be bounded, if to every neighborhood $V$ of $\bar{0}$ in $\overline{X}$ corresponds a number $\delta > 0$ such that set $\overline{M} \subset tV$, $\forall t > \delta$.

**Theorem 6.2.16.**

(a) If $d$ is a translation invariant metric on a fuzzy vector space $\overline{X}$ over $R$, then
\[ d([u], 0) \leq nd([u], 0), \quad \forall [u] \in \overline{X} \quad \text{and for } n = 1, 2, 3, \ldots \]

(b) If \{[u_n]\} is a sequence in a metrizable fuzzy topological vector space \(\overline{X}\) over \(\mathbb{R}\) and if \([u_n] \to \overline{0}\) as \(n \to \infty\), then there are positive numbers \(r_n\) such that \(r_n \to \infty\) and \([r_n u_n] \to \overline{0}\).

**Proof:**

Statement (a) follows from

\[ d([u], 0) \leq \sum_{k=1}^{n} d(k[u], (k-1)[u]), \quad \text{by triangle inequality} \]

\[ = n \, d([u], 0), \quad \text{by translation invariant}. \]

To prove (b), let \(d\) be a metric as in (a), compatible with the topology of \(\overline{X}\). Since \(d([u_n], \overline{0}) \to 0\), there is an increasing sequence of positive integers \(n_k\), such that

\[ d([u_n], \overline{0}) < k^{-2}, \quad \text{if } n \geq n_k. \]

Put

\[ r_n = 1, \quad \text{if } n < n_1 \quad \text{and} \]

\[ r_n = k; \quad \text{if } n_k \leq n \leq n_{k+1}. \]

For such \(n\),

\[ d([r_n u_n], \overline{0}) = d([k u_n], \overline{0}) \leq k d([u_n], \overline{0}) < k^{-1}. \]

Hence

\[ [r_n u_n] \to \overline{0} \quad \text{as } n \to \infty. \]

**Theorem 6.2.17.**

Let \(\overline{X}\) be a fuzzy topological vector space over \(\mathbb{R}\) and \(\overline{E} \subset \overline{X}\).

Then the following two properties are equivalent.
(a) $E$ is bounded

(b) If $\{u_n\}$ is a sequence in $E$ and $\{r_n\}$ is a sequence of real numbers such that $r_n \to 0$ as $n \to \infty$, then $[r_nu_n] \to \bar{o}$ as $n \to \infty$.

**Proof:**

Suppose $E$ is bounded. Let $V$ be a balanced neighborhood of $\bar{o}$ in $X$. Then $E \subset tV$, for some $t$. If $[u_n] \in E$ and $r_n \to 0$, there exists $N$ such that $r_n < 1$, if $n > N$. Since $t^{-1}E \subset V$ and $V$ is balanced,

$$[r_nu_n] \in V, \forall n > N. \text{ Thus } [r_nu_n] \to \bar{o}.$$

Conversely, if $E$ is not bounded, there is a neighborhood $V$ of $\bar{o}$ and a sequence $r_n \to \infty$ such that no $r_nV$ contains $E$. Choose $[u_n] \in E$ such that $[u_n] \in r_nV$. Then no $[r^{-1}_nu_n]$ is in $V$, so that $\{[r^{-1}_nu_n]\}$ does not converge to $\bar{o}$.

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