CHAPTER 5

FUZZY INNER PRODUCT SPACE*

5.1 Introduction:

The concept of fuzzy inner product spaces has been introduced in different ways by several authors like Abdel wahad M. El-Abyad; Hassan M. E-Hamonly [A;H], Kohli J.K.; Kumar Rajesh [K; K], Wu Con Xin; Zhou Yu Jiang [W; Z] etc. However, these are in context, different from that of fuzzy vector spaces that we consider. So, our definition is different from the existing ones. In this chapter, we introduce the new version of fuzzy inner product space on a fuzzy vector space over the set of fuzzy real numbers. We prove that a fuzzy inner product generates a fuzzy norm, but every fuzzy normed space can be made into a fuzzy semi inner product space. Also introduce the concept of fuzzy orthogonality. Some immediate results are also proved.

*Some results contained in this chapter were accepted for publication in the Journal of Fuzzy Mathematics [B; S].
5.2 Fuzzy inner product

**Definition 5.2.1**

Let \( \mathcal{X} \) be a fuzzy vector space over the set of fuzzy numbers. A fuzzy semi inner product on \( \mathcal{X} \) is a function which assigns a fuzzy number \( ([u], [v]) \), \( \forall [u], [v] \in \mathcal{X} \) such that the following hold for all \([u], [v], [w] \in \mathcal{X} \) and \([\alpha] \in \mathbb{R}^+\).

1. \( ([u] + [v], [w]) = ([u], [w]) + ([v], [w]) \).
2. \( ([\alpha][u], [v]) = [\alpha]([u], [v]), [\alpha] \in \mathbb{R}^+ \).
3. \( ([u], [u]) > \bar{0}, \text{ if } [u] \neq \bar{0} \).
4. \( |([u], [u])|^2 \leq ([u], [u]) ([v], [v]) \).

Further more if

(i) \( ([u], [v]) = ([v], [u]), \text{ for every } [u], [v] \in \mathcal{X} \).

(ii) \( ([u], [u]) = \bar{0}, \text{ if and only if } [u] = \bar{0} \), then \( (, ) \) is called a fuzzy inner product.

A fuzzy inner product space is fuzzy vector space with a fuzzy inner product defined in it.

**Theorem 5.2.2**

Let \( \mathcal{X} \) be a fuzzy inner product space. The function \( \| \| : \mathcal{X} \rightarrow \mathbb{R}^+ \) defined by \( \| [u] \|^2 = ([u], [u]) \) is a fuzzy norm on \( \mathcal{X} \).
Proof:

\[ ([u], [u]) \geq \bar{0}. \]

So, \[ \|[u]\|^2 \geq \bar{0} \quad \text{and} \]

\[ \|[u]\|^2 = \bar{0}, \quad \text{if and only if} \quad [u] = \bar{0}. \]

\[ \|[\alpha][u]\|^2 = ([\alpha][u], [\alpha][u]) = [\alpha^2] \|[u]\|^2 \]

i.e.; \[ \|[\alpha][u]\|^2 = [\alpha^2]\|[u]\|^2 = [\alpha^2] \|[u]\|^2, \quad \text{since} \quad [\alpha] \in \bar{R}^+ \]

\[ \|[u] + [v]\|^2 = ([u] + [v], [u] + [v]) \]

\[ = ([u], [u]) + ([u], [v]) + ([v], [u]) + ([v], [v]) \]

\[ = ([u], [u]) + 2([u], [v]) + ([v], [v]) \]

\[ = \|[u]\|^2 + 2([u], [v]) + \|[v]\|^2 \]

\[ \leq \|[u]\|^2 + \|[v]\|^2. \]

Theorem 5.2.3

A fuzzy norm on a fuzzy inner product space satisfies the parallelogram equality

\[ \|[u] + [v]\|^2 + \|[u] - [v]\|^2 = 2\|[u]\|^2 + 2\|[v]\|^2. \]

Proof:

\[ \|[u] + [v]\|^2 + \|[u] - [v]\|^2 \]

\[ = ([u] + [v], [u] + [v]) + ([u] - [v], [u] - [v]) \]

\[ = ([u], [u]) + ([u], [v]) + ([v], [u]) + ([v], [v]) + \]
Theorem 5.2.4

The fuzzy inner product \((u, v)\) is a continuous function on \(X_1 \times X_2\).

Proof:

Let \(X\) be a fuzzy inner product space and let \(\{u_n\}\) and \(\{v_n\}\) be sequences in \(X\) such that \(\lim u_n = u\) and \(\lim v_n = v\), then

\[
|([u_n] , [v_n]) - ([u], [v])| = |([u_n] , [v_n]) - ([u_n] , [v]) + ([u_n] , [v]) - ([u], [v])|
\]

\[
\leq |([u_n] , [v_n]) - ([u_n] , [v])| + |([u_n] , [v]) - ([u], [v])|
\]

\[
= |([u_n] , [v_n] - [v])| + |([u_n] - [u], [v])|
\]

\[
\leq ||u_n|| \cdot ||v_n - v|| + ||u_n - u|| \cdot ||v||
\]

Since \(\{u_n\}\) and \(\{v_n\}\) converges,

\[
\lim ([u_n] , [v_n]) = ([u], [v]).
\]

This completes the proof.
**Theorem 5.2.5**

Every fuzzy normed space can be made into a fuzzy semi inner product space.

**Proof:**

Let \( \bar{X} \) be a fuzzy normed space. By fuzzy Hahn-Banach theorem corresponding to every \([u_0] \in \bar{X}\), there exists a bounded fuzzy linear map \( f_{u_0} \) on \( \bar{X} \) such that

\[
f_{u_0}([u_0]) = \|[u_0]\|^2 \quad \text{and} \quad \| f_{u_0} \| \leq \|[u_0]\|.
\]

Define \( f_{\lambda}([u]) = ([u], [v]) \), then

(i) \( f_{\lambda}([u_1] + [u_2]) = ([u_1] + [u_2], [v]) = f_{\lambda}([u_1]) + f_{\lambda}([u_2]) = ([u_1], [v]) + ([u_2], [v]). \)

(ii) \( f_{\lambda}([\alpha][u]) = ([\alpha][u], [v]) = [\alpha] f_{\lambda}([u]) = [\alpha]([u], [v]). \)

(iii) \( f_{\lambda}([v]) = \|[v]\|^2 > \bar{0}. \)

i.e.; \( ([v], [v]) > \bar{0}, \) when \([v] \neq \bar{0}\)

(iv) Consider \( \| f_{\lambda}([u]) \| \leq \| f_{\lambda} \| \|[u]\| \leq \|[v]\| \|[u]\| \)
\[
\text{i.e.; } \| f_\ast([u]) \|^2 \leq \| [v] \|^2 \| [u] \|^2 = f_\ast([v]) f_\ast([u]).
\]

### 5.3. Finite Fuzzy Orthogonal Set

**Definition 5.3.1**

A subset \( E \) of a fuzzy inner product space \( \bar{X} \) is said to be fuzzy orthogonal in \( \bar{X} \), if \( ([u], [v]) = 0 \), \( \forall [u] \neq [v] \in E \).

**Definition 5.3.2**

A fuzzy orthogonal set \( E \) in a fuzzy inner product space is said to be complete, if there exists no other fuzzy orthogonal set properly containing \( E \).

**Theorem 5.3.3**

In a fuzzy inner product space \( \bar{X} \), a fuzzy orthogonal set \( E \) is complete if and only if \( [u] \perp E \Rightarrow [u] = 0 \).

**Proof:**

Let \( E \) be complete and \( [u] \neq 0 \) be an element of \( \bar{X} \) such that \( [u] \perp E \). Hence \( \{ [u], E \} \) is a fuzzy orthogonal set which properly contains \( E \). But this is a contradiction to the hypothesis that \( E \) is complete. Hence we must have \( [u] = 0 \).
Conversely let \([u] \perp E \Rightarrow [u] = 0\). We will prove that \(E\) is complete. Let \(E\) be not complete, then by definition there exists a fuzzy orthogonal set \(F\) such that \(F\) properly contains \(E\). In such cases there exists a \([u] \in F - E\), where \([u] \perp E\) and \([u] \neq 0\). This is a contradiction. Hence \(E\) is complete.

**Theorem 5.3.4**

Every finite fuzzy orthogonal set \(\{[e_i]\}\) with \(([e_i], [e_j]) = [1]\) (where \(1(a) = 1\), if \(a=1\) and \(1(a) = 0\), if \(a \neq 1\)) in a fuzzy inner product space \(\bar{X}\) is linearly independent; and hence we can find an expression for a fuzzy vector \([u]\), which is expressible as a linear combination of the elements of the fuzzy orthogonal sets of non-zero fuzzy vectors.

**Proof:**

**Step 1**

Let \(S = \{[e_1], [e_2], ..., [e_n]\}\) be a fuzzy orthogonal set of \(\bar{X}\) so that \(([e_i], [e_j]) = \begin{cases} \bar{0}, & \text{if } i \neq j \\ [1], & \text{if } i = j. \end{cases}\)

Now consider a linear combination of these fuzzy vectors say

\[
\sum_{i=0}^{n} [\alpha_i][e_i] = \bar{0}.
\] (14)
If the above relations imply that \( \alpha_i = \tilde{0}, \forall i \), then \( S \) is linearly independent. Now consider the fuzzy inner product
\[
(\Sigma[\alpha_i][e_i], [e_j]) = (\tilde{0}, [e_j]) = \tilde{0}, \quad \forall j, \text{ by (14)}
\]
or
\[
(\alpha_i) ([e_i], [e_j]) + \ldots + (\alpha_j) ([e_j], [e_j]) + \ldots + (\alpha_n) ([e_n], [e_j]) = \tilde{0}
\]
That is; all the \( \alpha_i \) in \( \Sigma \alpha_i [e_i] = \tilde{0} \) are fuzzy zero and hence the set \( S \) is linearly independent.

**Step 2**

Let \([u]\) be any fuzzy vector in \( \mathcal{V} \) which is expressible as a linear combination of the elements of a fuzzy orthogonal set, if
\[
[u] = \Sigma [\alpha_i][e_i], \quad \text{then}
\]
\[
([u], [e_j]) = (\Sigma[\alpha_i][e_i], [e_j]) = \alpha_j ([e_j], [e_j]) = \alpha_j
\]
i.e.;
\[
[\alpha_j] = ([u], [e_j])
\]
So,
\[
[u] = \sum_{i=1}^{n} ([u], [e_i]) [e_i].
\]
Theorem 5.3.5

Let \{[e_1], [e_2], \ldots, [e_n]\} be a finite fuzzy orthogonal set with 

\(([e_i], [e_j]) = 1\) in a fuzzy inner product space \(\mathcal{X}\). If \(u\) is any fuzzy vector in \(\mathcal{X}\), then 

\[ \sum_{i=1}^{n} ([u], [e_i])^2 \leq \|u\|^2. \]

Further, \(u - \sum_{i=1}^{n} ([u], [e_i])[e_i] \perp [e_j]\), for each \(j\).

Proof:

Step 1

Since the set \{[e_1], [e_2], \ldots, [e_n]\} is a finite fuzzy orthogonal set.

We have \(([e_i], [e_j]) = 0\), for \(i \neq j\) and also given that

\(([e_i], [e_j]) = 1\), for \(i = j\). Choose \([v] = [u] - \sum_{i=1}^{n} ([u], [e_i])[e_i]\).

We know that \(\|v\|^2 \geq 0\)

i.e.; \(\|u\|^2 - \sum_{i=1}^{n} ([u], [e_i])[e_i]^2 \geq 0\).

For the sake of simplicity let us call \(([u], [e_i]) = [\alpha_i]\).

i.e.; \(\|u\|^2 - \sum_{i=1}^{n} [\alpha_i][e_i]^2 \geq 0\)

or

\([u] - \sum_{i=1}^{n} [\alpha_i][e_i], [u] - \sum_{j=1}^{n} [\alpha_j][e_j] \) \(\geq 0\)
In the last term performing sum with respect to $j$ first, all the terms will vanish except one term when $j = i$.

So \[ ||[u]||^2 - (\sum_{i=1}^{n} [\alpha_i] [e_i], [u]) - \sum_{j=1}^{n} [\alpha_j] ([u], [e_j]) + \sum_{i=1}^{n} [\alpha_i][\alpha_i] \geq \tilde{0}. \]

Now $[\alpha_i] = ([u], [e_i])$ and $[\alpha_j] = ([u], [e_j])$.

**Step 2**

Consider the fuzzy inner product

\[(u) - (\sum_{i=1}^{n} ([u],[e_i])([e_i]), [e_j]), \text{ for each } j\]

\[= ([u], [e_j]) - (\sum_{i=1}^{n} ([u],[e_i])(e_i), [e_j]).\]
\[ ([u], [e_j]) - \sum_{i=1}^{n} ([u],[e_i]) ([e_i],[e_j]) \]
\[ = ([u], [e_j]) - ([u], [e_j]) ([e_j], [e_j]) \]
\[ = ([u], [e_j]) - ([u], [e_j]) = \bar{0}, \text{ for each } j \]

Now \(([u], [v]) = \bar{0} \Rightarrow [u] \text{ is orthogonal to } [v].\)

ie. \([u] - \sum_{j=1}^{n} ([u],[e_j]) [e_j] \perp [e_j], \text{ for each } j.\)

**Theorem 5.3.6**

Let \(\bar{X}\) be a fuzzy inner product space and let \(\{[e_i]\}\) be a fuzzy orthogonal set with \(([e_i], [e_i]) = 1\) in \(\bar{X}\). Then the following conditions are all equivalent to one another.

(i) \(\{[e_i]\}\) is complete.

(ii) \([u] \perp \{[e_i]\} \Rightarrow [u] = \bar{0}.\)

(iii) If \([u]\) is an arbitrary vector in \(\bar{X}\) then
\[ [u] = \Sigma([u],[e_i]) [e_i]. \]

(iv) If \([u]\) is an arbitrary vector in \(\bar{X}\) then \(\|[u]\|^2 = \Sigma([u],[e_i])^2.\)

**Proof:**

(i) \(\Rightarrow\) (ii).

Let \(\{[e_i]\}\) be complete and \([u] \neq \bar{0}\) be an element of \(\bar{X}\) such that \([u] \perp \{[e_i]\}\). Hence \([u], \{[e_i]\}\) is a fuzzy orthogonal set, which
properly contains \{[e_i]\}. But this is a contradiction to the hypothesis that \{[e_i]\} is complete. Hence \([u] = \bar{0}\).

(ii) \Rightarrow (iii).

Given \([u] \perp \{[e_i]\} \Rightarrow [u] = \bar{0}\), we have to show that

\([u] = \Sigma([u], [e_i]) [e_i].\)

Consider \([u] - \Sigma([u],[e_i]) [e_i]\) and we shall show that it is \( \perp \{[e_i]\}\).

Choose any \([e_j] \in \{[e_i]\}\), then

\(([u] - \sum_{i=1}^{n} ([u],[e_i]) [e_i], [e_j]) = ([u], [e_j]) - (\sum_{i} ([u], [e_i]) [e_i], [e_j])\)

\(= ([u], [e_j]) - \sum_{i} ([u], [e_i]) ([e_i], [e_j]).\)

Sum with respect to \(i\), all terms vanish except one

\(= ([u], [e_j]) - ([u], [e_j]) ([e_j], [e_j])\)

\(= ([u], [e_j]) - ([u], [e_j]) = \bar{0}\)

\(\Rightarrow [u] - \sum_{i=1}^{n} ([u],[e_i]) [e_i] \perp [e_j], \text{ for each } j\)

\(\Rightarrow [u] - \sum_{i=1}^{n} ([u],[e_i]) [e_i] \perp \{[e_i]\}\)

\(\Rightarrow [u] - \sum_{i=1}^{n} ([u],[e_i])[e_i] = \bar{0}, \text{ by the given condition}\)

\(\Rightarrow [u] = \sum_{i=1}^{n} ([u],[e_i]) [e_i].\)
(iii) ⇒ (iv).

Given \([u] = \sum ([u], [e_i]) [e_i]\), to prove \(||[u]||^2 = \sum ([u], [e_i])^2\)

\(||[u]||^2 = ([u], [u]) = (\sum_i ([u], [e_i]) [e_i], \sum_j ([u], [e_j]) [e_j]) = \sum_i \sum_j ([u], [e_i]) ([u], [e_j]) ([e_i], [e_j])\)

Summing with respect to \(j\) first,

\[= \sum_i (\sum_j ([u], [e_i]) ([u], [e_j]) ([e_i], [e_j])) = \sum ([u], [e_i])^2.\]

(iv) ⇒ (i).

Given \(||[u]||^2 = \sum ([u], [e_i])^2\). To prove that \([e_i]\) is complete, let \([e_i]\) be not complete. Then there exists \([v], \{[e_i]\}\), where \([v] \neq \tilde{0}\) is complete.

\[\Rightarrow [v] \perp \{[e_i]\} \Rightarrow ([v], [e_i]) = \tilde{0}, \forall i. \quad (15)\]

Now \(||[u]||^2 = \sum ([u], [e_i])^2\), by (iv).

\(||[v]||^2 = \sum ([v], [e_i])^2 = \tilde{0}, \text{ by (15)}\)

\[\Rightarrow ||[v]||^2 = \tilde{0} \Rightarrow [v] = \tilde{0},\]

which is a contradiction. Hence \([e_i]\) is complete.

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