CHAPTER V
INTUITIONISTIC FUZZY SUB ALGEBRAS AND IDEALS IN BCH – ALGEBRAS

In this chapter using the notion of Intuitionistic Fuzzy sub Algebras and ideals some results have been studied.

5.1 Intuitionistic fuzzy sub algebras

Definition 5.1.1

An intuitionistic fuzzy set $A$ in a BCH - algebra $X$ is called an intuitionistic fuzzy sub algebra of $X$ if it satisfies the following conditions:

1. $\mu_A(x*y) \geq \min\{ \mu_A(x), \mu_A(y) \}$,
2. $\nu_A(x*y) \leq \max\{ \nu_A(x), \nu_A(y) \}$ for all $x, y \in X$.

Example 5.1.2

Let $X = \{ 0, a, b, c, d \}$ with the following Cayley table be a BCH - algebra.

\[
\begin{array}{c|cccc}
* & 0 & a & b & c & d \\
\hline
0 & 0 & 0 & 0 & 0 & d \\
a & a & 0 & 0 & a & d \\
b & b & b & 0 & 0 & d \\
c & c & c & c & 0 & d \\
d & d & d & d & d & 0 \\
\end{array}
\]

Let $A = \langle \mu_A, \nu_A \rangle$ be an IFS in $X$ defined by

$\mu_A(0) = \mu_A(d) = 0.9, \mu_A(a) = \mu_A(b) = \mu_A(c) = 0.09$ and

$\nu_A(a) = \nu_A(b) = \nu_A(c) = 0.9$ and $\nu_A(0) = \nu_A(d) = 0.09$.

By routine calculations $A$ is an intuitionistic fuzzy subalgebra of $X$.

In what follows, let $X$ denote a BCH-algebra unless otherwise specified.
Definition 5.1.3[4]

Let \( f : X \rightarrow Y \) be a mapping of BCH – algebras and \( A \) be an IFS of \( Y \). The map \( f^{-1}(A) \) is the pre – image of \( A \) under \( f \), if \( \mu_{f^{-1}(A)}(x) = \mu_f(f(x)), \)
\( \nu_{f^{-1}(A)}(x) = \nu_f(f(x)). \)

Theorem 5.1.4

If \( A \) is an intutionistic fuzzy sub algebra of \( X \), then
\( \mu_A(0) \geq \mu_A(x) \) and \( \nu_A(0) \leq \nu_A(x). \)

Proof:

Let \( x, y \in A \). Since \( A \) is an intutionistic fuzzy sub algebra, \( \mu_A(x \cdot y) \geq \min\{ \mu_A(x), \mu_A(y) \} \)
and
\( \nu_A(x \cdot y) \leq \max\{ \nu_A(x), \nu_A(y) \} \).
Since \( x \cdot x = 0 \),
\( \mu_A(x \cdot x) = \mu_A(0) \)
and
\( \nu_A(x \cdot x) = \nu_A(0). \)
Now \( \mu_A(x \cdot x) \geq \min\{ \mu_A(x), \mu_A(x) \} = \mu_A(x) \)
and
\( \nu_A(x \cdot x) \leq \max\{ \nu_A(x), \nu_A(x) \} = \nu_A(x). \)
Hence \( \mu_A(0) \geq \mu_A(x) \)
and
\( \nu_A(0) \leq \nu_A(x). \)
Theorem 5.1.5

If \( A = < \mu_A, \nu_A > \) and \( B = < \mu_B, \nu_B > \) are two intuitionistic fuzzy sub algebra of \( X \), then \( A \cap B \) is an intuitionistic fuzzy sub algebra of \( X \).

Proof:

Suppose \( A \) and \( B \) are intuitionistic fuzzy sub algebras of \( X \) and let \( x, y \in X \). Then

\[
\mu_{A \cap B}(x \ast y) = \min\{\mu_A(x \ast y), \mu_B(x \ast y)\}
\]

\[
\geq \min[ \min\{\mu_A(x), \mu_A(y)\}, \min\{\mu_B(x), \mu_B(y)\}]
\]

\[
= \min[ \min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}]
\]

\[
= \min[\mu_{A \cap B}(x), \mu_{A \cap B}(y)].
\]

\[
\nu_{A \cap B}(x \ast y) = \max\{\nu_A(x \ast y), \nu_B(x \ast y)\}
\]

\[
\leq \max[ \max\{\nu_A(x), \nu_A(y)\}, \max\{\nu_B(x), \nu_B(y)\}]
\]

\[
= \max[ \max\{\nu_A(x), \nu_B(x)\}, \max\{\nu_A(y), \nu_B(y)\}]
\]

\[
= \max[\nu_{A \cap B}(x), \nu_{A \cap B}(y)].
\]

Hence \( A \cap B \) is an intuitionistic fuzzy sub algebra of \( X \).

Theorem 5.1.6

If \( A \) is an intuitionistic fuzzy sub algebra of \( X \), then the set \( X_A = \{ x \in X : \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0) \} \) is a sub algebra of \( X \).
**Proof:**

Let \( x, y \in X_A \). Then

\[
\mu_A(x) = \mu_A(y) = \mu_A(0)
\]

and

\[
\nu_A(x) = \nu_A(y) = \nu_A(0).
\]

Since \( A \) is an intuitionistic fuzzy sub algebra of \( X \), it follows that

\[
\mu_A(x*y) \geq \min\{ \mu_A(x), \mu_A(y) \} = \min\{ \mu_A(0), \mu_A(0) \} = \mu_A(0) \quad \text{and}
\]

\[
\nu_A(x*y) \leq \max\{ \nu_A(x), \nu_A(y) \} = \max\{ \nu_A(0), \nu_A(0) \} = \nu_A(0).
\]

Therefore \( \mu_A(x*y) = \mu_A(0) \) and \( \nu_A(x*y) = \nu_A(0) \).

Hence \( x*y \in X_A \) and consequently \( X_A \) is a sub algebra of \( X \).

**Theorem 5.1.7**

An intuitionistic fuzzy set \( A \) of \( X \) is an intuitionistic fuzzy sub algebra if and only if for every pair \( \alpha, \beta \in [0, 1] \), the level set \( A_{\alpha, \beta} \) is empty or a sub algebra.

**Proof:**

Suppose \( A \) is an intuitionistic fuzzy sub algebra and \( A_{\alpha, \beta} \neq \emptyset \). Then for any \( x, y \in A_{\alpha, \beta} \), we have

\[
\mu_A(x*y) \geq \min\{ \mu_A(x), \mu_A(y) \} \geq \alpha
\]

and

\[
\nu_A(x*y) \leq \max\{ \nu_A(x), \nu_A(y) \} \leq \beta.
\]

Therefore \( x*y \in A_{\alpha, \beta} \).

Hence \( A_{\alpha, \beta} \) is a sub algebra.

Conversely,

Take \( \alpha = \min\{ \mu_A(x), \mu_A(y) \} \) and \( \beta = \max\{ \nu_A(x), \nu_A(y) \} \), for any \( x, y \in X \).

Since \( A_{\alpha, \beta} \) is a sub algebra \( x*y \in A_{\alpha, \beta} \), \( \mu_A(x*y) \geq \alpha = \min\{ \mu_A(x), \mu_A(y) \} \) and

\[
\nu_A(x*y) \leq \beta = \max\{ \nu_A(x), \nu_A(y) \}.
\]

Hence \( A \) is an intuitionistic fuzzy sub algebra.
Theorem 5.1.8

Any sub algebra of \( X \) can be realized as a level sub algebra of some intuitionistic fuzzy sub algebra of \( X \).

Proof:

Let \( A \) be a sub algebra of a BCH-algebra \( X \) defined by

\[
\mu_A(x) = \begin{cases} 
\alpha & \text{if } x \in A, \\
0 & \text{otherwise}, 
\end{cases}
\quad \text{and} \quad 
\nu_A(x) = \begin{cases} 
\beta & \text{if } x \in A, \\
1 & \text{otherwise}. 
\end{cases}
\]

Where \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta \leq 1 \). Clearly \( A_{<\alpha, \beta>} = A \). Let \( x, y \in X \). If \( x, y \in A \), then \( x \cdot y \in A \).

So \( \mu_A(x) = \mu_A(y) = \mu_A(x \cdot y) = \alpha \) and \( \nu_A(x) = \nu_A(y) = \nu_A(x \cdot y) = \beta \).

Since \( A \) is an intuitionistic fuzzy sub algebra, \( \mu_A(x \cdot y) \geq \min \{ \mu_A(x), \mu_A(y) \} \) and \( \nu_A(x \cdot y) \leq \max \{ \nu_A(x), \nu_A(y) \} \).

If at most one of \( x, y \in A \), then at least one of \( \mu_A(x) \) and \( \mu_A(y) \) is equal to \( \alpha \) and one of \( \nu_A(x) = \nu_A(y) = \beta \). Therefore, \( \min \{ \mu_A(x), \mu_A(y) \} = 0 \) and \( \mu_A(x \cdot y) \geq 0 \) and \( \max \{ \nu_A(x), \nu_A(y) \} = 1 \) and \( \nu_A(x \cdot y) \leq 1 \) which completes the proof.

Theorem 5.1.9

If \( A \) is an intuitionistic fuzzy sub algebra of \( X \) and \( \alpha \in [0, 1] \), then we have

(i) if \( \alpha = 1 \), then the upper level set \( U(\mu_A; \alpha) \) is either empty or a sub algebra of \( X \).

(ii) if \( \alpha = 0 \), then the lower level set \( L(\nu_A; \alpha) \) is either empty or a sub algebra of \( X \).
Proof:

(i): Suppose that, $\alpha = 1$ and $xy \in U(\mu_A; \alpha)$. Then $\mu_A(x) \geq \alpha = 1$ and $\mu_A(y) \geq \alpha = 1$

It follows that $\mu_A(x \cdot y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min(1, 1) = 1$ so that $x \cdot y \in U(\mu_A; \alpha)$.

Hence $U(\mu_A; \alpha)$ is a sub algebra of $X$ when $\alpha = 1$.

(ii): Suppose that, $\alpha = 0$ and let $x, y \in L(\nu; \alpha)$. Then $\nu_A(x) \leq \alpha = 0$ and $\nu_A(y) \leq \alpha = 0$.

It follows that $\nu_A(x \cdot y) \leq \max\{\nu_A(x), \nu_A(y)\} \leq \min(0, 0) = 0$. Hence $L(\nu_A, \alpha)$ is a sub algebra of $X$ when $\alpha = 0$.

Theorem 5.1.10

Let $f$ be an endomorphism of $X$. If $A$ is an intuitionistic fuzzy sub algebra of $X$, then $B = \{ \mu_{f^{-1}(\alpha)}; \nu_{f^{-1}(\alpha)} \}$ is an intuitionistic fuzzy sub algebra.

Proof:

For any $x, y \in X$ we have

$$\mu_{f^{-1}(\alpha)}(x \cdot y) = \mu_A(f(x \cdot y))$$

$$= \mu_A(f(x) \cdot f(y))$$

$$\geq \min\{\mu_A(f(x)), \mu_A(f(y))\}$$

$$= \min\{\mu_{f^{-1}(\alpha)}(x), \mu_{f^{-1}(\alpha)}(y)\}.$$

Similarly we have, for any $x, y \in X$,

$$\nu_{f^{-1}(\alpha)}(x \cdot y) = \nu_A(f(x \cdot y))$$

$$= \nu_A(f(x) \cdot f(y))$$

$$\leq \max\{\nu_A(f(x)), \nu_A(f(y))\}$$

$$= \max\{\nu_{f^{-1}(\alpha)}(x), \nu_{f^{-1}(\alpha)}(y)\}.$$

This completes the proof.
Theorem 5.1.11

Let $X$ and $Y$ be two BCH-algebras and $f : X \rightarrow Y$ be an epimorphism and let $B = < \mu_B, \nu_B >$ be an intuitionistic fuzzy set of $Y$. If $f^{-1}(B) = < \mu_{f^{-1}(B)}, \nu_{f^{-1}(B)}> is intuitionistic fuzzy sub algebra of $X$, then $B$ is an intuitionistic fuzzy sub algebra of $Y$.

Proof:

Let $x, y \in Y$. Then there exists $a, b \in X$ such that $f(a) = x$, $f(b) = y$. It follows that

$\mu_B(x*y) = \mu_B(f(a)*f(b))$
$= \mu_B(f(a-b))$
$= \mu_{f^{-1}(B)}(a-b)$
$\geq \min\{ \mu_{f^{-1}(B)}(a), \mu_{f^{-1}(B)}(b) \}$
$= \min\{ \mu_B(f(a)), \mu_B(f(b)) \}$
$= \min\{ \mu_B(x), \mu_B(y) \}$

$\nu_B(x*y) = \nu_B(f(a)*f(b))$
$= \nu_B(f(a-b))$
$= \nu_{f^{-1}(B)}(a-b)$
$\leq \max\{ \nu_{f^{-1}(B)}(a), \nu_{f^{-1}(B)}(b) \}$
$= \max\{ \nu_B(f(a)), \nu_B(f(b)) \}$
$= \max\{ \nu_B(x), \nu_B(y) \}$

Hence $B$ is an intuitionistic fuzzy sub algebra of $Y$. 
Theorem 5.1.12

If $A$ is an IFS in $X$ such that the non-empty sets $U(\mu_A ; \alpha)$ and $L(\nu_A ; \alpha)$ are subalgebras of $X$ for all $\alpha \in [0, 1]$, then $A$ is an intuitionistic fuzzy sub algebra of $X$.

Proof:

Suppose that there exist $x_0, y_0 \in X$ such that $\mu_A(x_0 \ast y_0) < \min\{\mu_A(x_0), \mu_A(y_0)\}$

Take $\alpha_0 = 1/2 \left\{ \mu_A(x_0 \ast y_0) + \min\{\mu_A(x_0), \mu_A(y_0)\} \right\}$.

Then $\min\{\mu_A(x_0), \mu_A(y_0)\} > \alpha_0 > \mu_A(x_0 \ast y_0)$. It follows that $x_0, y_0 \in U(\mu_A ; \alpha_0)$ and $x_0 \ast y_0 \notin U(\mu_A ; \alpha_0)$. This is a contradiction and hence $\mu_A$ satisfies the inequality

$\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$. Similarly, suppose that there exists $x_0, y_0 \in X$ such that $\nu_A(x_0 \ast y_0) > \max\{\nu_A(x_0), \nu_A(y_0)\}$.

Take $\beta_0 = 1/2 \left\{ \nu_A(x_0 \ast y_0) + \max\{\nu_A(x_0), \nu_A(y_0)\} \right\}$.

Then $\max\{\nu_A(x_0), \nu_A(y_0)\} \leq \beta_0 < \nu_A(x_0 \ast y_0)$.

It follows that $x_0, y_0 \in L(\nu_A ; \beta_0)$ and $x_0 \ast y_0 \notin L(\nu_A ; \beta_0)$. This is a contradiction and hence $\nu_A$ satisfies the inequality $\nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.

Hence $A$ is an intuitionistic fuzzy sub algebra.

Theorem 5.1.13

If $A$ is a sub algebra of a BCH algebra $X$, then the IFS $\overline{A} = \langle \chi_A, \chi_A \rangle$ is an intuitionistic fuzzy sub algebra of $X$ where $\chi_A$ is the characteristic function of $A$.

Proof:

Let $x, y \in X$. If $x, y \in A$, then $x \ast y \in A$ since $A$ is a sub algebra of $X$. Hence $\chi_A(x \ast y) = 1 \geq \min\{\chi_A(x), \chi_A(y)\}$. 

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Also we have
\[ 0 = 1 - \chi_A (x \ast y) = \chi_A (x \ast y) \leq \max \{ \chi_A (x), \chi_A (y) \}. \]
If \( x \in A \) and \( y \notin A \) (or \( x \notin A \) and \( y \in A \)), then \( \chi_A (x) = 1 \) and \( \chi_A (y) = 0 \).

or \( \chi_A(x) = 0 \) and \( \chi_A(y) = 1 \).

Thus we have
\[ \max \{ \chi_A(x), \chi_A(y) \} = \max \{ 1 - \chi_A(x), 1 - \chi_A(y) \} = \max (0, 1) = 1 \geq \chi_A (x \ast y) \]
or \( \max (1, 0) = 1 \geq \chi_A (x \ast y) \).

This completes the proof.

**Theorem 5.1.14**

Let \( A \) be a non-empty subset of \( X \). If \( \bar{A} = \langle \chi_A, \bar{\chi}_A \rangle \) satisfies

(i) \( \mu_A (x \ast y) \geq \min \{ \mu_A(x), \mu_A(y) \} \)
or

(ii) \( \nu_A (x \ast y) \leq \max \{ \nu_A(x), \nu_A(y) \} \),

then \( A \) is a sub algebra of \( X \).

**Proof:**

Suppose \( \bar{A} = \langle \chi_A, \bar{\chi}_A \rangle \) satisfies
\[ \mu_A (x \ast y) \geq \min \{ \mu_A(x), \mu_A(y) \} \].

Let \( x, y \in A \). Then
\[ \chi_A(x \ast y) \geq \min \{ \chi_A(x), \chi_A(y) \} = \min \{ 1, 1 \} = 1 \]
so that \( \chi_A(x \ast y) = 1 \), that is \( x \ast y \in A \). Now suppose that
\[ \bar{A} = \langle \chi_A, \bar{\chi}_A \rangle \] satisfies \( \nu_A (x \ast y) \leq \max \{ \nu_A(x), \nu_A(y) \} \). Let \( x, y \in A \). Then
\[ \bar{\chi}_A (x \ast y) \leq \max \{ \bar{\chi}_A (x), \bar{\chi}_A (y) \} \]
\[ \leq \max \{ 1 - \chi_A (x), 1 - \chi_A (y) \} \]
\[ = \max \{ 0, 0 \} = 0. \]

Thus \( \bar{\chi}_A (x \ast y) = 1 - \chi_A (x \ast y) = 0 \), that is \( \chi_A (x \ast y) = 1 \).

That is \( x \ast y \in A \).

Hence \( A \) is a sub algebra of \( X \). This completes the proof.
**Theorem 5.1.15**

A is a sub algebra of X if and only if $\overline{A} = \langle \mu_{\overline{A}}, \nu_{\overline{A}} \rangle$ where

$$
\begin{align*}
\mu_{\overline{A}}(x) &= \begin{cases} 
1 & x \in A, \\
0 & \text{Otherwise},
\end{cases} \\
\nu_{\overline{A}}(x) &= \begin{cases} 
0 & x \in A, \\
1 & \text{Otherwise}.
\end{cases}
\end{align*}
$$

is an intuitionistic fuzzy sub algebra of X.

**Proof:**

Let A be a sub algebra of X. Let $x,y \in X$. If $x,y \in A$, then $x \ast y \in A$

$$
\begin{align*}
\mu_{\overline{A}}(x \ast y) &= 1 \geq \min \{ \mu_{\overline{A}}(x), \mu_{\overline{A}}(y) \} \\
\nu_{\overline{A}}(x \ast y) &= 0 \leq \max \{ \nu_{\overline{A}}(x), \nu_{\overline{A}}(y) \}.
\end{align*}
$$

If $x \notin A$ or $y \notin A$, then $\mu_{\overline{A}}(x) = 0$ or $\mu_{\overline{A}}(y) = 0$, $\nu_{\overline{A}}(x) = 1$ or $\nu_{\overline{A}}(y) = 1$.

Thus we have $\mu_{\overline{A}}(x \ast y) \geq \min \{ \mu_{\overline{A}}(x), \mu_{\overline{A}}(y) \}$ and

$$
\nu_{\overline{A}}(x \ast y) \leq \max \{ \nu_{\overline{A}}(x), \nu_{\overline{A}}(y) \}.
$$

Hence $\overline{A}$ is an intuitionistic fuzzy sub algebra of X.

Conversely, let $\overline{A}$ be an intuitionistic sub algebra of X.

Let $x,y \in X$. If $x,y \in A$, then

$$
\begin{align*}
\mu_{\overline{A}}(x \ast y) &\geq \min \{ \mu_{\overline{A}}(x), \mu_{\overline{A}}(y) \} = 1 \\
\nu_{\overline{A}}(x \ast y) &\leq \max \{ \nu_{\overline{A}}(x), \nu_{\overline{A}}(y) \} = 0.
\end{align*}
$$

So $x \ast y \in A$. Hence A is a sub algebra of X.

**Theorem 5.1.16**

If IFS $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy sub algebra of X, then

$$
\begin{align*}
\mu_A(0 \ast x) &\geq \mu_A(x) \\
\nu_A(0 \ast x) &\leq \nu_A(x)
\end{align*}
$$

for all $x \in X$. 

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**Proof:**

For any $x \in X$, we have

\[
\mu_A(0 \ast x) \geq \min \{ \mu_A(0), \mu_A(x) \} \\
\geq \min \{ \mu_A(x \ast x), \mu_A(x) \} \\
\geq \min \{ \min \{ \mu_A(x), \mu_A(x) \}, \mu_A(x) \} \\
= \mu_A(x),
\]

and

\[
\nu_A(0 \ast x) \leq \max \{ \nu_A(0), \nu_A(x) \} \\
\leq \max \{ \nu_A(x \ast x), \nu_A(x) \} \\
\leq \max \{ \max \{ \nu_A(x), \nu_A(x) \}, \nu_A(x) \} \\
= \nu_A(x).
\]

This completes the proof.

**Definition 5.1.17**

An IFS $A = <\mu_A, \nu_A>$ in $X$ is called an intuitionistic fuzzy closed ideal if

(i) $\mu_A(0 \ast x) \geq \mu_A(x)$ and $\nu_A(0 \ast x) \leq \nu_A(x)$

(ii) $\mu_A(x) \geq \min \{ \mu_A(x \ast y), \mu_A(y) \}$, and $\nu_A(x) \leq \max \{ \nu_A(x \ast y), \nu_A(y) \}$ for all $x, y \in X$.

**Theorem 5.1.18**

Every intuitionistic fuzzy sub algebra in $X$ satisfying

$\mu_A(x) \geq \min \{ \mu_A(x \ast y), \mu_A(y) \}$ and $\nu_A(x) \leq \max \{ \nu_A(x \ast y), \nu_A(y) \}$ is an intuitionistic fuzzy closed ideal in $X$. 
Proof:

Using Theorem 5.1.15, the proof is straightforward.

Theorem 5.1.19

Every intuitionistic fuzzy closed ideal of $X$ is an intuitionistic fuzzy sub algebra in $X$.

Proof:

Let $A = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy closed ideal of a BCH-algebra $X$ and let $x, y \in X$. Then

\[
\mu_A(x \ast y) \geq \min \left\{ \mu_A((x \ast y) \ast x), \mu_A(x) \right\}
\]

\[
\geq \min \left\{ \mu_A((x \ast x) \ast y), \mu_A(x) \right\}
\]

\[
= \min \left\{ \mu_A(0 \ast y), \mu_A(x) \right\}
\]

\[
\geq \min \left\{ \mu_A(x), \mu_A(y) \right\},
\]

and

\[
\nu_A(x \ast y) \leq \max \left\{ \nu_A((x \ast y) \ast x), \nu_A(x) \right\}
\]

\[
\leq \max \left\{ \nu_A((x \ast x) \ast y), \nu_A(x) \right\}
\]

\[
= \max \left\{ \nu_A(0 \ast y), \nu_A(x) \right\}
\]

\[
\leq \max \left\{ \nu_A(x), \nu_A(y) \right\}.
\]

Hence $A$ is an intuitionistic fuzzy sub algebra of $X$. 

Theorem 5.1.20

Every intuitionistic fuzzy sub algebra of $X$ is not an intuitionistic fuzzy closed ideal of $X$.  

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Proof:

Let $X = \{0, a, b, c, d\}$ with the following Cayley table be a BCH algebra.

<table>
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<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<td>0</td>
<td>0</td>
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<td>0</td>
<td>d</td>
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<tr>
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<td>b</td>
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<td>c</td>
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<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $A = <\mu_A, \nu_A>$ be an IFS in $X$ defined by

$\mu_A(0) = \mu_A(d) = 0.9$, $\mu_A(a) = \mu_A(b) = \mu_A(c) = 0.09$ and

$\nu_A(a) = \nu_A(b) = \nu_A(c) = 0.9$ and $\nu_A(0) = \nu_A(d) = 0.09$.

Then $A$ is an intuitionistic fuzzy sub algebra of $X$.

But $\mu_A(a) = 0.09 < 0.9 = \min \{ \mu_A(a \cdot d), \mu_A(d) \}$.

Hence $A$ is not an intuitionistic fuzzy closed ideal.

This completes the proof.

Theorem 5.1.21

Let $A = <\mu_A, \nu_A>$ be an intuitionistic fuzzy sub algebra of $X$ such that

$\mu_A(x \cdot y) \leq \mu_A(y \cdot x)$, and $\nu_A(x \cdot y) \geq \nu_A(y \cdot x)$ for all $x, y \in X$. Then $A$ is an intuitionistic fuzzy closed ideal of $X$.

Proof:

Let $A = <\mu_A, \nu_A>$ be an intuitionistic fuzzy sub algebra of $X$ which satisfies the inequalities...
\[ \mu_A(x \ast y) \leq \mu_A(y \ast x) \text{ and } \nu_A(x \ast y) \geq \nu_A(y \ast x) \text{ for all } x, y \in X. \]

By Theorem 5.1.16, \( \mu_A(0 \ast x) \geq \mu_A(x) \) and \( \nu_A(0 \ast x) \leq \nu_A(x) \) for all \( x \in X \). Then
\[
\mu_A(x) = \mu_A(x \ast 0) \geq \mu_A(0 \ast x) = \mu_A(y \ast y \ast x)
\]
\[
= \mu_A((y \ast x) \ast y) \geq \min\{\mu_A(y \ast x), \mu_A(y)\} \geq \min\{\mu_A(x \ast y), \mu_A(y)\},
\]
and
\[
\nu_A(x) = \nu_A(x \ast 0) \leq \nu_A(0 \ast x) = \nu_A((y \ast y) \ast x)
\]
\[
= \nu_A((y \ast x) \ast y) \leq \max\{\nu_A(y \ast x), \nu_A(y)\} \leq \max\{\nu_A(x \ast y), \nu_A(y)\}.
\]

Hence \( A \) is an intuitionistic fuzzy closed ideal of \( X \).

**Theorem 5.1.22**

If \( A \) is an IFS in \( X \) such that the non-empty sets \( U(\mu_A;\alpha) \) and \( L(\nu_A;\alpha) \) are closed ideals of \( X \) for all \( \alpha \in [0, 1] \), then \( A \) is an intuitionistic fuzzy closed ideal of \( X \).

**Proof:**

Suppose that there exist \( x_0, y_0 \in X \) such that \( \mu_A(x_0) < \min\{\mu_A(x_0 \ast y_0), \mu_A(y_0)\} \).

Take \( \alpha_0 = 1/2 \{ \mu_A(x_0) + \min\{\mu_A(x_0 \ast y_0), \mu_A(y_0)\} \} \). Then
\[
\min\{\mu_A(x_0 \ast y_0), \mu_A(y_0)\} \geq \alpha_0 > \mu_A(x_0)\]

It follows that \( x_0 \ast y_0, y_0 \in U(\mu_A;\alpha_0) \) and \( x_0 \not\in U(\mu_A;\alpha_0) \). This is a contradiction and hence \( \mu_A \) satisfies the inequality
\[
\mu_A(x) \geq \min\{\mu_A(x \ast y), \mu_A(y)\} \text{ for all } x, y \in X.
\]

Similarly, suppose that there exist \( x_0, y_0 \in X \) such that \( \nu_A(x_0) > \max\{\nu_A(x_0 \ast y_0), \nu_A(y_0)\} \).

Take \( \beta_0 = 1/2 \{ \nu_A(x_0) + \max\{\nu_A(x_0 \ast y_0), \nu_A(y_0)\} \} \). Then
\[
\max\{\nu_A(x_0 \ast y_0), \nu_A(y_0)\} \leq \beta_0 < \nu_A(x_0).
\]

It follows that \( x_0 \ast y_0, y_0 \in L(\nu_A;\beta_0) \) and \( x_0 \not\in L(\nu_A;\beta_0) \). This is a contradiction, and hence \( \nu_A \) satisfies the inequality \( \nu_A(x) \leq \max\{\nu_A(x \ast y), \nu_A(y)\} \) for all \( x, y \in X \).

Now assume that there exists \( x_0 \in X \) such that \( \mu_A(0 \ast x_0) < \mu_A(x_0) \).
Take $\alpha_0 = 1/2 \{ \mu_A(0 \ast x_0) + \mu_A(x_0) \}$. Then $\mu_A(0 \ast x_0) \leq \alpha_0$ and $\mu_A(x_0) \geq \alpha_0$. It follows that $x_0 \in U(\mu_A; \alpha_0)$ but $0 \ast x_0 \not\in U(\mu_A; \alpha_0)$. This is a contradiction. Hence $\mu_A(0 \ast x) \geq \mu_A(x)$ for all $x \in X$. Similarly, we get $\nu_A(0 \ast x) \leq \nu_A(x)$ for all $x \in X$.

Hence $A$ is an intuitionistic fuzzy closed ideal.

**Theorem 5.1.23**

If $\{A_i\}_{i \in I}$ is a family of intuitionistic fuzzy sub algebras of $X$, then

$\bigcap_{i \in I} A_i$ is an intuitionistic fuzzy sub algebra of $X$, where $\bigcap_{i \in I} A_i = \langle \bigwedge_{i \in I} \mu_{A_i}, \bigvee_{i \in I} \nu_{A_i} \rangle$.

**Proof:**

Let $x, y \in M$. Then

$$(\bigcap_{i \in I} \mu_{A_i})(x \ast y) = \bigwedge_{i \in I} \mu_{A_i}(x \ast y)$$

$\geq \bigwedge_{i \in I} \mu_{A_i}(x) \land \mu_{A_i}(y)$$

$= \{ \bigwedge_{i \in I} \mu_{A_i}(x) \} \land \{ \bigwedge_{i \in I} \mu_{A_i}(y) \}$

$= (\bigcap_{i \in I} \mu_{A_i})(x) \land (\bigcap_{i \in I} \mu_{A_i})(y)$.

$$(\bigcap_{i \in I} \nu_{A_i})(x \ast y) = \bigvee_{i \in I} \nu_{A_i}(x \ast y)$$

$\leq \bigvee_{i \in I} \{ \nu_{A_i}(x) \lor \nu_{A_i}(y) \}$

$= \{ \bigvee_{i \in I} \nu_{A_i}(x) \} \lor \{ \bigvee_{i \in I} \nu_{A_i}(y) \}$

$= (\bigcap_{i \in I} \nu_{A_i})(x) \lor (\bigcap_{i \in I} \nu_{A_i})(y)$.

Hence $\bigcap_{i \in I} A_i$ is an intuitionistic fuzzy sub algebra of $X$. 

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Theorem 5.1.24

Let I be a sub algebra of X. If the intuitionistic fuzzy set A = (μ_A, ν_A) in X is defined by

\[
\mu_A(x) = \begin{cases} 
  p & \text{if } x \in I, \\
  s & \text{otherwise}, 
\end{cases} \\
\nu_A(x) = \begin{cases} 
  u & \text{if } x \in I, \\
  v & \text{otherwise}. 
\end{cases}
\]

for all x ∈ X where 0 ≤ s < p, 0 ≤ v < u and p + u ≤ 1, s + v ≤ 1, then A is an intuitionistic fuzzy sub algebra of X and \( U(\mu_A; p) = I = L(\nu_A; u) \).

Proof:

Let x, y ∈ X.

If at least one of x and y does not belong to I, then

\[
\mu_A(x * y) \geq s = \min \{\mu_A(x), \mu_A(y)\}, \\
\nu_A(x * y) \leq v = \max \{\nu_A(x), \nu_A(y)\}.
\]

If x, y ∈ I, then

x * y ∈ I and so \( \mu_A(x * y) = p = \min \{\mu_A(x), \mu_A(y)\} \) and

\( \nu_A(x * y) = u = \max \{\nu_A(x), \nu_A(y)\} \).

Therefore A is an intuitionistic fuzzy sub algebra of X.

Theorem 5.1.25

An IFS A = (μ_A, ν_A) in a BCH algebra X is an intuitionistic fuzzy sub algebra if and only if \( A_{<t,s>} = \{x \in X : \mu_A(x) \geq t, \nu_A(x) \leq s\} \) is a sub algebra of X for \( \mu_A(0) \geq t, \nu_A(0) \leq s \).
Proof:

Suppose that $A$ is an intuitionistic fuzzy sub algebra of $X$ and let
\[ \mu_A(0) \geq t, \nu_A(0) \leq s. \]
Let $x, y \in A_{<t,s>}$.

Then $\mu_A(x) \geq t, \nu_A(x) \leq s$ and $\mu_A(y) \geq t, \nu_A(y) \leq s$.

Therefore $x \cdot y \in A_{<t,s>}$ for all $x, y \in A_{<t,s>}$.

So $A_{<t,s>}$ is a sub algebra.

Conversely, suppose that $A_{<t,s>}$ is a sub algebra of $X$ for $\mu_A(0) \geq t, \nu_A(0) \leq s$.

Let $x, y \in X$ be such that $\mu_A(x) = t_1, \nu_A(x) = s_1, \mu_A(y) = t_2$ and $\nu_A(y) = s_2$.

We may assume that $t_2 \leq t_1$ and $s_2 \geq s_1$ without loss of generality. It follows that

$A_{<t_2,s_2>} \subseteq A_{<t_1,s_1>}$ so that $x, y \in A_{<t_1,s_1>}$.

Since $A_{<t_1,s_1>}$ is a sub algebra of $X$, we have $x \cdot y \in A_{<t_1,s_1>}$.

\[ \mu_A(x \cdot y) \geq t_1 \geq t_2 = \min \{ \mu_A(x), \mu_A(y) \} \quad \text{and} \quad \nu_A(x \cdot y) \leq s_1 \leq s_2 = \max \{ \nu_A(x), \nu_A(y) \}. \]

Hence $A$ is an intuitionistic fuzzy sub algebra of $X$.

5.2 Intuitionistic fuzzy H-ideals

Theorem 5.2.1

An intuitionistic fuzzy subset $A$ of $X$ is an intuitionistic fuzzy H-ideal if and only if for any pair $t, s \in [0,1]$ with $t + s \leq 1$, $A_{<t,s>} = \{ x : \mu_A(x) \geq t \text{ and } \nu_A(x) \leq s \}$ is a H-ideal of $X$ where $A_{<t,s>} \neq \emptyset$.

Proof:

Suppose $A$ is an intuitionistic fuzzy H-ideal of $X$ and $A_{<t,s>} \neq \emptyset$, for any pair
Let \( t, s \in [0,1] \) with \( t + s \leq 1 \). Let \( x \in A_{<t, s>} \), then \( \mu_A(x) \geq t \) and \( \nu_A(x) \leq s \). So \( 0 \in A_{<t, s>} \).

Suppose \( x^* (y * z) \in A_{<t, s>} \), then

\[
\mu_A(x^*(y * z)) \geq t, \mu_A(y) \geq t \text{ and } \nu_A(x^*(y * z)) \leq s, \nu_A(y) \leq s.
\]

By Definition 4.3.18, \( \mu_A(x^* z) \geq \min \{ \mu_A(x^*(y * z)), \mu_A(y) \} \geq t \) and \( \nu_A(x^* z) \leq \max \{ \nu_A(x^*(y * z)), \nu_A(y) \} \leq s \).

So \( x * z \in A_{<t, s>} \). Hence \( A_{<t, s>} \) is a \( H \)-ideal of \( X \).

Conversely, suppose that for each pair \( t, s \in [0,1] \) with \( t + s \leq 1 \), \( A_{<t, s>} \) is either empty or an ideal of \( X \). For any \( x \in X \), setting \( \mu_A(x) = t \) and \( \nu_A(x) = s \), then \( x \in A_{<t, s>} \). Since \( A_{<t, s>} (\neq \emptyset) \) is an ideal of \( X \), we have \( 0 \in A_{<t, s>} \) and hence \( \mu_A(0) \geq t = \mu_A(x) \) and \( \nu_A(0) \leq s = \nu_A(x) \). Thus \( \mu_A(0) \geq \mu_A(x) \) and \( \nu_A(0) \leq \nu_A(x) \) for all \( x \in X \).

If \( \mu_A(x^* z) \geq \min \{ \mu_A(x^*(y * z)), \mu_A(y) \} \) and \( \nu_A(x^* z) \leq \max \{ \nu_A(x^*(y * z)), \nu_A(y) \} \) is not true, then there exist \( x_0, y_0, z_0 \in X \), such that \( \mu_A(x_0 * z_0) < \min \{ \mu_A(x_0 * (y_0 * z_0)), \mu_A(y_0) \} \) and \( \nu_A(x_0 * z_0) > \max \{ v_A(x_0 * (y_0 * z_0)), v_A(y_0) \} \).

Putting \( \mu_A(x_0 * z_0) < t_0 < \min \{ \mu_A(x_0 * (y_0 * z_0)), \mu_A(y_0) \} \) and \( v_A(x_0 * z_0) > s_0 > \max \{ v_A(x_0 * (y_0 * z_0)), v_A(y_0) \} \), then \( x_0 * (y_0 * z_0), y_0 \in A_{<t} \), \( s_0 > \emptyset \). But \( A_{<t} \), \( s_0 > \emptyset \) is an ideal of \( X \), so \( x_0 * z_0 \in A_{<t_0}, s_0 > \). That is, \( \mu_A(x_0 * z_0) \geq t \) and \( v_A(x_0 * z_0) \leq s \) which is contradictory.

Hence \( \mu_A(x^* z) \geq \min \{ \mu_A(x^*(y * z)), \mu_A(y) \} \) and \( \nu_A(x^* z) \leq \max \{ v_A(x^*(y * z)), v_A(y) \} \).

This completes the proof.

**Theorem 5.2.2**

An intuitionistic fuzzy set \( A \) of \( X \) is an intuitionistic fuzzy \( H \)-ideal if and only if for any \( x_0 \in X \), \( X_A < x_0 > = \{ x \in X : \mu_A(x) \geq \mu_A(x_0) \text{ and } v_A(x) \leq v_A(x_0) \} \) is a \( H \)-ideal of \( X \).
Proof:

Putting $\mu_A(x_0) = t$ and $\nu_A(x_0) = s$, by Theorem 5.2.1, $x_A < x_0>$ is a H-ideal.

Conversely, let $A$ be an intuitionistic fuzzy set of a BCH-algebra $X$ and for any $x_0 \in X,$

$X_A < x_0> = \{ x \in X : \mu_A(x) \geq \mu_A(x_0) \text{ and } \nu_A(x) \leq \nu_A(x_0) \}$ is a H-ideal of $X$.

It is clear that $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X.$ For all $x, y \in X,$ putting $\min \{ \mu_A(x*(y*z)) , \mu_A(y) \} = \mu_A(x_0)$ and $\max \{ \nu_A(x*(y*z)) , \nu_A(y) \} = \nu_A(x_0)$

we have $\mu_A(x*(y*z)) \geq \mu_A(x_0)$ and $\nu_A(x*(y*z)) \leq \nu_A(x_0)$

That is, $x*(y*z) \in X_A < x_0>, y \in X_A < x_0>.$ Thus $x*z \in X_A < x_0>$, since $X_A < x_0>$ is a H-ideal.

So $\mu_A(x*z) \geq \mu_A(x_0) = \min \{ \mu_A(x*(y*z)) , \mu_A(y) \}$ and

$\nu_A(x*z) \leq \nu_A(x_0) = \max \{ \nu_A(x*(y*z)) , \nu_A(y) \}$.

Therefore $A$ is an intuitionistic fuzzy H-ideal of $X$. This completes the proof.