CHAPTER 4
INDEPENDENT DOM-SATURATION NUMBER OF A GRAPH

4.1 INTRODUCTION

In this chapter, we initiate a study on independent dom-saturation number which was introduced by Arumugam & Subramanian (2007a). Let $is(v)$ denote the minimum cardinality among all maximal independent sets of $G$ containing $v$. Then $is(G) = \max \{is(v) : v \in V(G)\}$ is called the independent dom-saturation number of $G$. An is-set of $v$ (or is-set corresponding to $v$) is a maximal independent set $S$ containing $v$ such that $is(v) = |S|$. A vertex $v \in V$ is called an is-vertex if $is(v) = is(G)$. Let $v \in V$ be such that $is(v) = is(G)$. Any maximal independent set of cardinality $is(G)$ containing $v$ is called an is-set of $G$. Clearly every is-set is a maximal independent set and hence is an independent dominating set. This gives $i(G) \leq is(G) \leq \beta_0(G)$. From Definition 1.3.14, the independent dom-saturation number $is(G)$ can also be denoted by $i^{M,m}(G)$.

We start with an example (see Figure 4.1) to illustrate the concept of independent dom-saturation number of a graph.
Figure 4.1. Graph with \( is(G) = 4 \)

In the graph \( G \), \( i(G) = 2 \), \( is(G) = 4 \) and \( \beta_0(G) = 5 \). Since \( is(v) = is(x) = is(y) = is(z) = 4 \), \( v, x, y \) and \( z \) are all the is-vertices. Moreover \( is(u) = is(w) = 2 \) and \( is(r) = is(s) = 3 \).

It is easy to prove the following observation:

**Observation 4.1.1.**

(i) For the cycle \( C_n \) of length \( n \), \( is(C_n) = \left\lceil \frac{n}{3} \right\rceil \).

(ii) \( is(K_n) = 1 \) and

(iii) \( is(K_{m,n}) = \max \{m, n\} \).

**Observation 4.1.2.** [Sudha (2010)] For any graph \( G \), \( i^{M,m}(G) = 1 \) if and only if \( G \cong K_n \).

**Proposition 4.1.3.** [Sudha (2010)] For any graph \( G \),

\( i^{M,m}(G) \leq n - \delta \).

The following theorem shows that there is no relation between \( IS(G) = \beta_0^{m,M}(G) \) and \( is(G) = i^{M,m}(G) \).

**Theorem 4.1.4.** [Sudha (2010)] Given any two positive integers \( a \) and \( b \), there exists a graph \( G \), with \( i^{M,m}(G) = a \) and \( \beta_0^{m,M}(G) = b \).
4.2 RESULTS ON INDEPENDENT DOM-SATURATION NUMBER

In this section, we study some results related to \( is(G) \).

**Proposition 4.2.1.** For the path \( P_n = (v_1, v_2, ..., v_n) \), \( n \geq 3 \), we have

\[
is(P_n) = \begin{cases} 
  \lfloor n/3 \rfloor + 1 & \text{if } n \equiv 0 \pmod{3} \\
  \lfloor n/3 \rfloor & \text{if } n \equiv 1 \pmod{3} \\
  \lfloor n/3 \rfloor + 2 & \text{if } n \equiv 2 \pmod{3}. \end{cases}
\]

*Proof.* The proof is by induction on \( n \). We consider three cases:

**Case i.** \( n \equiv 0 \pmod{3} \).

Let \( n = 3k \), \( k = 1, 2, ... \). We can easily verify that \( is(P_3) = 2 \) and \( is(P_6) = 3 \). Assume the result is true for \( n \), \( n \geq 9 \) and \( is(P_n) = \lfloor n/3 \rfloor + 1 \). Now we prove that \( is(P_{n+3}) = \lfloor (n + 3)/3 \rfloor + 1 \). Let \( e = v_nv_{n+1} \) be the edge in \( P_{n+3} \) such that \( P_{n+3} - e = P_n \cup P_3 \). Further, any \( is \)-set of \( P_{n+3} \) is the union of \( is \)-set of \( P_n \) and the middle vertex in \( P_3 \). By induction hypothesis, it follows that \( is(P_{n+3}) = \lfloor n/3 \rfloor + 1 + 1 = \lfloor (n + 3)/3 \rfloor + 1 \).

**Case ii.** \( n \equiv 1 \pmod{3} \).

Let \( n = 3k + 1 \), \( k = 1, 2, ... \). It can also be verified that \( is(P_4) = 2 \) and \( is(P_7) = 3 \). Assume the result is true for \( n \), \( n \geq 10 \) and \( is(P_n) = \lfloor n/3 \rfloor \). Now we prove that \( is(P_{n+3}) = \lfloor (n + 3)/3 \rfloor \). Let \( e = v_nv_{n+1} \) be the edge in \( P_{n+3} \) such that \( P_{n+3} - e = P_n \cup P_3 \). By induction, we obtain \( is(P_{n+3}) = \lfloor n/3 \rfloor + 1 = (n + 2)/3 + 1 = \lfloor (n + 3)/3 \rfloor \).

**Case iii.** \( n \equiv 2 \pmod{3} \).
Let \( n = 3k + 2, k = 1, 2, \ldots \). One can easily verify that \( is(P_3) = 2 \) and \( is(P_5) = 4 \). Assume the result is true for \( n, n \geq 11 \) and \( is(P_n) = \lfloor n/3 \rfloor + 2 = (n - 2/3) + 2 \). Now we prove that \( is(P_{n+3}) = \lfloor (n + 3)/3 \rfloor + 2 \). Let \( e = v_n v_{n+1} \) be the edge in \( P_{n+3} \) such that \( P_{n+3} - e = P_n \cup P_3 \). Clearly, any \( is \)-set of \( P_{n+3} \) is union of \( is \)-set of \( P_n \) and middle vertex in \( P_3 \). By induction, \( is(P_{n+3}) = \lfloor n/3 \rfloor + 2 + 1 = (n - 2/3) + 3 = \lfloor (n + 3)/3 \rfloor + 2 \).

It is easy to see that \( is(K_n) = 1 \); the following theorem gives a characterization for graphs \( G \neq K_n \) with \( is(G) = 2 \).

**Theorem 4.2.2.** Let \( G \neq K_n \) be a connected graph. Then \( is(G) = 2 \) if and only if for all vertices \( v \), either one of the following conditions hold:

- (i) \( V(G) - N[v] = \emptyset \),
- (ii) \( V(G) - N[v] \) has a single vertex \( w \),
- (iii) If \( V(G) - N[v] \neq \emptyset \), then there exists \( w \in V(G) - N[v] \) such that every vertex of \( V(G) - N[v] \) is adjacent to \( w \).

**Proof.** Assume that \( is(G) = 2 \). Let \( v \in V(G) \). Since \( is(G) = 2 \), \( is(v) \leq 2 \) for all \( v \in V(G) \). If any \( v \) has \( is(v) = 1 \), then i) holds. Let \( v \in V(G) \) such that \( is(v) = 2 \). Suppose for all \( w \in V(G) - N[v] \), there exists \( u \in V(G) - N[v] \) such that \( w \) is not adjacent to \( v \). Then \( \{u, v, w\} \) is an independent set and so \( is(v) \geq 3 \). It is a contradiction. Hence, (ii) or (iii) holds. The converse is obvious.

**Theorem 4.2.3.** Let \( G \) be a connected graph of order \( n \) with \( is(G) = 2 \). Then \( n - 1 \leq |E(G)| \leq (n - 2)(n + 1)/2 \). Also \( |E(G)| = n - 1 \).
if and only if \( G \) is isomorphic to \( P_3 \) or \( P_4 \). Further \(|E(G)| = (n - 2)(n + 1)/2\) if and only if \( G \cong K_n - e \).

**Proof.** Let \( G \) be a connected graph of order \( n \) with \( is(G) = 2 \). The lower bound is obvious since \( G \) is connected. Since \( is(G) = 2 \), \( G \neq K_n \). Hence, there exists \( u, v \in K_n \) such that \( \text{deg } u \leq n - 2 \) and \( \text{deg } v \leq n - 2 \). Further \( \text{deg } w \leq n - 1 \) for all \( w \in G - \{u, v\} \). Therefore \( \sum \text{deg } v \leq 2(n - 2) + (n - 2)(n - 1) = (n - 2)(n + 1) \) and so \(|E(G)| \leq (n - 2)(n + 1)/2\).

Let \( G \) be a graph with \( is(G) = 2 \) and \(|E(G)| = n - 1 \). Then \( G \) is a tree. Let \( v \in V(G) \) such that \( is(v, G) = 2 \). Then the condition (ii) or (iii) of above theorem holds. If (ii) holds, then \( G \cong P_3 \) and if (iii) holds, then \( G \cong P_4 \). The converse is obvious.

Let \( G \) be a connected graph with \( is(G) = 2 \) and \(|E(G)| = (n - 2)(n + 1)/2 \). Since \(|E(G)| = (n - 2)(n + 1)/2\) if and only if \( G \cong K_n - e \); where \( e \in E(K_n) \), the result follows.

**Theorem 4.2.4.** Given any three positive integers \( a, b \) and \( c \) with \( 2 \leq a \leq b \leq c \), there exists a graph \( G \) with \( i(G) = a \), \( is(G) = b \) and \( \beta_0(G) = c \).

**Proof.** Let \( a, b \) and \( c \) be three positive integers with \( 2 \leq a \leq b \leq c \). We construct a graph \( G = \overline{K_a} + (b\text{-partite graph} - \{\cup_{i=1}^c M_i\}) \) where \( M_i \) is the set of edges in the \( b\)-partite graph which are defined as follows:

\[
M_1 = \{v_{11}v_{21}, v_{11}v_{31}, \ldots, v_{11}v_{b1}\},
\]

\[
M_2 = \{v_{12}v_{22}, v_{12}v_{32}, \ldots, v_{12}v_{b2}\},\ldots
\]
\[ M_c = \{v_{1c}v_{2c}, v_{1c}v_{3c}, \ldots, v_{1c}v_{bc}\}. \] Furthermore, vertex sets of the
\( b \)-partite graph are defined as:
\[ V_1 = \{v_{11}, v_{12}, \ldots, v_{1c}\}, \]
\[ V_2 = \{v_{21}, v_{22}, \ldots, v_{2c}\}, \ldots, \]
\[ V_b = \{v_{b1}, v_{b2}, \ldots, v_{bc}\}. \]

Now we prove that \( i(G) = a \), \( is(G) = b \) and \( \beta_0(G) = c \). Since ev-
ery vertex in \( \overline{K_a} \) is adjacent to all the vertices of \( (b\)-partite \ graph
\( \cup_{i=1}^c M_i) \), \( i(G) = a \). Clearly \( V_i \) is a maximum independent set
and hence \( \beta_0(G) = c \). Further, \( \{v_{11}, v_{21}, \ldots, v_{b1}\}, \{v_{12}, v_{22}, \ldots, v_{b2}\}, \ldots, \)
\( \{v_{1c}, v_{2c}, \ldots, v_{bc}\} \) are maximal independent sets of cardinality \( b \). Hence
\( is(v) = b \) for all \( v \in V_i, i = 1 \) to \( b \). This gives \( is(G) = max \{a, b\} = b \).

**Theorem 4.2.5.** For any graph \( G \), \( is(G) = i(G) \) if and only if \( G \)
is \( i \)-excellent.

**Proof.** If \( G \) is \( i \)-excellent, then \( is(v) = i(G) \) for all \( v \in V(G) \). Hence
\( is(G) = max \{is(v) : v \in V(G)\} = i(G) \). The converse is obvious.

Nordhaus & Gaddum (1956) developed sharp bounds on
the sum and product of the chromatic numbers of a graph and its
complement. Jaeger & Payan (1972) obtained similar results for the
domination number. A recent survey on Nordhaus-Gaddum type re-
sults can be found in Mustapha Aouchiche & Pierre Hansen (2013).

The following theorem gives a Nordhaus-Gaddum type re-
sult for \( is(G) \). For any graph \( G \), we denote \( \overline{is(G)} = is(\overline{G}) \), where \( \overline{G} \)
denotes the complement of \( G \).
**Theorem 4.2.6.** Let $G$ be any graph with at least two vertices. Then

(i) $3 \leq is + \overline{is} \leq n + 1 + (\Delta - \delta)$ and $2 \leq is \overline{is} \leq (n - \delta)(\Delta + 1)$.

(ii) The following are equivalent:

(a) $is + \overline{is} = 3$,

(b) $is \overline{is} = 2$,

(c) $G$ is either $K_2$ or $\overline{K_2}$.

**Proof.** Since $is(G) \geq 1$, and $\overline{is}(G) \geq 2$ when $is = 1$, it follows that $is(G) + \overline{is}(G) \geq 3$ and $is \overline{is} \geq 2$. By Proposition 4.1.3, $is(G) \leq n - \delta$ and hence it follows that $is + \overline{is} \leq (n - \delta) + (\Delta + 1) = n + 1 + (\Delta - \delta)$ and $is \overline{is} \leq (n - \delta)(\Delta + 1)$. It follows that (a) and (b) are both equivalent to $is(G) = 1$ and $\overline{is}(G) = 2$ or $is(G) = 2$ and $\overline{is}(G) = 1$.

It is enough to prove that (a) implies (c). Without loss of generality, we assume that $is(G) = 1$ and $\overline{is}(G) = 2$. Since $is(G) = 1$ if and only if $G \cong K_n$, $G$ is either $K_2$ or $\overline{K_2}$. The converse is obvious.

We now proceed to investigate the relation between $is$ and other graph theoretic parameters such as diameter and chromatic number.

**Theorem 4.2.7.** Let $G$ be a connected graph of order $n \geq 3$. Then $is(G) + diam(G) \leq 2(n - 1)$ and equality holds if and only if $G$ is isomorphic to $P_3$.

**Proof.** For any graph $G$, we have $is(G) \leq n - \delta$ and $diam(G) \leq n - 1$. Hence, $is(G) + diam(G) \leq 2n - (\delta + 1) = 2(n - 1)$. Now let $is(G) + diam(G) = 2(n - 1)$. Then $diam(G) = n - 1$ and
\( is(G) = n - 1 \). Since \( diam(G) = n - 1 \), \( G \) is a tree with \( \Delta = 2 \).

Hence \( G \) is a path. Since \( is(G) = n - 1 \), \( G \cong P_3 \).

**Theorem 4.2.8.** Let \( G \) be a connected graph. Then \( is(G) + \chi(G) \leq n + 1 \).

**Proof.** Since \( is(G) \leq \beta_0 \) and by Theorem 1.4.3, \( \chi(G) \leq n - \beta_0 + 1 \), we get \( is(G) + \chi(G) \leq n + 1 \).

**Theorem 4.2.9.** Let \( T \) be any tree. Then \( is(T) = n - \delta \) if and only if \( T \) is a star.

**Proof.** Assume that \( is(T) = n - \delta \). Then there exists a vertex \( v \) in \( T \) such that \( is(v) = n - \delta \). Therefore, \( is(v) = n - 1 \), since \( T \) is a tree consisting of \( n - 1 \) pendant edges. Hence \( T = K_{1,n} \). The converse is obvious.

**Proposition 4.2.10.** For any graph \( G \) of size \( m \), \( is(T_2(G)) = is(G) + 2m \) where \( T_2(G) \) is a trestled graph of index 2 which is defined in 1.2.32.

**Proof.** Let \( T_2(G) \) be a trestled graph of index 2. For any edge \( uv \) in \( G \), let \( u'v' \) and \( u''v'' \) be the edges in \( T_2(G) \) such that \( u \) is adjacent to \( u',u'' \) and \( v \) is adjacent to \( v',v'' \). Any maximal independent set of \( T_2(G) \) contains exactly two vertices on each pair of edges associated to an edge of \( G \). Hence, \( is \)-sets of \( T_2(G) \) corresponding to a given vertex \( z \) of \( G \) are of cardinality \( is(G) + 2m \). Therefore, \( is(z,T_2(G)) = is(z,G) + 2m \) for all \( z \in V(G) \). Hence the desired result follows.
Theorem 4.2.11. Let $G$ be a connected unicyclic graph. Then $is(G) + \chi(G) = n$ if and only if $G$ is isomorphic to one of the graphs $C_4$, $C_5$, $G_i$, $i = 1$ to $12$ in Figure 4.2.

![Graphs $G_1$ to $G_{12}$](image)

Figure 4.2. Unicyclic graphs with $is(G) + \chi(G) = n$

Proof. Clearly $is(G) + \chi(G) = n$ for all the graphs $C_4$, $C_5$ and graphs $G_i$, $i = 1$ to $12$ shown in Figure 4.2. Conversely, let $G$ be a connected unicyclic graph with $is(G) + \chi(G) = n$ and $C_r$ be the unique cycle in $G$. We distinguish into two cases:

Case i. $r$ is an even.
Here $\chi = 2$, so that $is(G) = n - 2$. Hence, for all vertices $v \in V(G)$, we have $is(v) \leq n - 2$. Since $is(G) \leq is(C_r) + n - r$, $n - 2 \leq [r/3] + n - r$. Hence $r - 2 \leq [r/3]$ implies $r = 4$. Thus the length of $C_r$ is 4. Let $v \in V(G)$ be such that $deg v = 2$ and $is(v) = n - 2$. In $V(G) - N(v)$, there are $n - 2$ vertices. In order to get $is(v) = n - 2$, $V(G) - N(v)$ must be an independent set. Hence $G$ is isomorphic to one of the graphs $C_4, G_1$ and $G_2$. Let $v \in V(G)$ be such that $deg v \geq 3$. In $V(G) - N(v)$, there are at most $n - 3$ vertices. Hence, $is(v) \leq n - 3$. Hence, there exists no unicyclic graph $G$ having $is$-vertex $v$ of $deg v \geq 3$ and $is(G) + \chi(G) = n$.

Case ii. $r$ is an odd.

Here $\chi = 3$, so that $is(G) = n - 3$. Hence, for all vertices $v$ such that $is(v) \leq n - 3$.

Subcase i: $r \geq 5$.

Since $is(G) \leq is(C_r) + n - r$, $n - 3 \leq [r/3] + n - r$. Hence $r - 3 \leq [r/3]$ implies that $r = 5$. Thus the length of $C_r$ is 5. Let $v \in V(G)$ be such that $deg v = 2$ and $is(v) = n - 3$. In $V(G) - N(v)$, there are $n - 2$ vertices. Since $is(v) = n - 3$ and $G$ is connected, the induced subgraph $G - N(v)$ is a union of $n - 4$ isolated vertices and $K_2$. Hence $G$ is isomorphic to one of the graphs $C_5, G_6$, and $G_7$. For any vertex $v \in V(G)$ such that $deg v = 3$, $V(G) - N(v)$ is not an independent set. Hence, there is no $v \in V(G)$ having $deg v = 3$ is an $is$-vertex. Let $v \in V(G)$ be such that $deg v \geq 4$ and $V(G) - N(v)$ has at most $n - 4$ vertices. Hence, $is(v) \leq n - 4$. Hence, there exists no unicyclic graph $G$ having $is$-vertex $v$ of $deg v \geq 4$ and
\( \text{is}(G) + \chi(G) = n. \)

Subcase ii: \( r = 3. \)

Let \( C_r \) be the cycle of length 3. Let \( v \in V(G) \) be such that \( \text{deg} \ v = 2 \) and \( \text{is}(v) = n - 3. \) In \( V(G) - N(v) \), there are \( n - 2 \) vertices. Since \( \text{is}(v) = n - 3 \) and \( G \) is connected, the induced subgraph \( G - N(v) \) is a union of \( n - 4 \) isolated vertices and \( K_2. \) Hence, \( G \) is isomorphic to one of the graphs \( G_3, G_4, G_8 \) and \( G_{12}. \) Let \( v \) be such that \( \text{deg} \ v = 3 \) and \( \text{is}(v) = n - 3. \) In \( V(G) - N(v) \), there are \( n - 3 \) vertices. In order to get \( \text{is}(v) = n - 3, \) \( V(G) - N(v) \) must be an independent set. Hence, \( G \) is isomorphic to one of the graphs \( G_5, G_9, G_{10} \) and \( G_{11}. \) Let \( v \in V(G) \) be such that \( \text{deg} \ v \geq 4 \) and \( V(G) - N(v) \) has at most \( n - 4 \) vertices. Hence, \( \text{is}(v) \leq n - 4. \) Hence, there is no unicyclic graph \( G \) having \( \text{is}-\text{vertex} \ v \) of \( \text{deg} \ v \geq 4 \) and \( \text{is}(G) + \chi(G) = n. \)

Fricke et al. (2002) called a vertex of a graph \( G \) to be \textit{good} if it is contained in some \( \gamma \)-set of \( G. \) \( G \) is called an \textit{excellent} graph if every vertex of \( G \) is good. They characterized trees which are excellent.

Haynes & Henning (2003) characterized total domination excellent trees. In a similar way, we define the concept of excellent graphs with respect to independent dom-saturation.

**Definition 4.2.12.** A vertex \( v \) of a graph \( G \) is called \textit{is-good} if it is contained in some \( \text{is} \)-set of \( G. \) A graph \( G \) is called \textit{is-excellent} if every vertex of \( G \) is \textit{is-good}. 
Example 4.2.13. The complete graph $K_n$ and the complete bipartite graph $K_{n,n}$ are $is$-excellent graphs.

In the following proposition, we determine the independent dom-saturation number of expansion and corona of graphs.

Proposition 4.2.14. For any graph $G$, we have

(i) $is(exp(G,r)) = r.is(G)$.

(ii) $is(cor(G,r)) = IS(G) + (|V(G)| - IS(G))r$.

Proof. (i) Let $D$ be any maximal independent set of $exp(G,r)$. Then each set $I_v$ in $D$ corresponds to a vertex $v$ in $G$. Note that \{v : I_v \subseteq D\} is a maximal independent set of $G$. Let $z$ be any $is$-vertex of $G$ and $S$ be any $is$-set of $G$ containing $z$ in $G$. For every two vertices $v, w$ in $S$, there corresponds two sets $I_v, I_w$ in $exp(G,r)$. Since $v$ and $w$ are non-adjacent, all the vertices of $I_v \cup I_w$ are independent. Hence $\cup I_v, v \in S$ is a maximal independent set containing $w, w \in I_z$. Hence $is(exp(G,r)) = r.is(G)$.

(ii) Let $v \in cor(G,r)$ and $D$ be any maximal independent set of $cor(G,r)$ containing $v$. For every vertex $v$ of $G$, $D$ contains either $v$ or all of $r$ leaves adjacent to $v$. Hence, for $D$ to be as small as possible, $D$ must contain as many vertices of $G$ as possible, namely a maximum independent set $IS(v)$ containing $v$. Hence, $is(v) = IS(v) + (|V(G)| - IS(v))r$ for all $v \in V(G)$. If $v$ is an $IS$-vertex of $G$, then $is(v) = IS(G) + (|V(G)| - IS(G))r$. 

If $v$ is a leaf attached to $w \in V(G)$ and $w$ belongs to some $\beta_0$-set, then $D$ contains $|V(G)| - \beta_0 + 1$ leaves and $\beta_0 - 1$ vertices. Hence, $is(v) = (|V(G)| - \beta_0 + 1)r + \beta_0 - 1$. If $v$ is a leaf attached to $w \in V(G)$ and $w$ does not belong to any $\beta_0$-set, then $is(v) = (|V(G)| - \beta_0)r + \beta_0$. Since $IS(G) \leq IS(v)$ and $IS(G) \leq \beta_0(G)$, we have $is(G) = max \{is(v)\} = IS(G) + (|V(G)| - IS(G))r$.

In the following proposition, we determine the independent dom-saturation number for central graph of cycles.

**Proposition 4.2.15.** For the cycle $C_n = (v_1, v_2, ..., v_n)$, $n \geq 3$, we have $is(C(C_n)) = n - 1$.

**Proof.** For each edge $v_iv_j$ ($i < j$ and $1 \leq i, j \leq n$), let $e_i$ be the subdivided vertex in $C(C_n)$. Now $deg v_i = n - 1$ and $deg e_i = 2$. Since $\{v_i, e_{i+1}, e_{i+2}, ..., e_{i+n-2}\}$ is a maximal independent of cardinality $n - 1$, $is(v_i) = n - 1$ and $is(e_j) = n - 1$. Hence, $is(C(C_n)) = max \{n - 1, n - 1\} = n - 1$.

We now proceed to investigate the independent dom-saturation number for the central graph, middle graph and total graph of the star graphs as well as the double star graphs.

**Proposition 4.2.16.** For any star graph $K_{1,n}$, $n \geq 2$, we have

(i) $is(C(K_{1,n})) = n$,

(ii) $is(M(K_{1,n})) = n + 1$,

(iii) $is(T(K_{1,n})) = n$. 
Proof. (i) For each edge $vv_i$ ($1 \leq i \leq n$) in $K_{1,n}$, let $e_i$ be the subdivided vertex in $C(K_{1,n})$. Now $\{v, v_i\}$, $1 \leq i \leq n$ is a maximal independent set of cardinality 2. Hence $is(v) = 2$ and $is(v_i) = 2$.

Since $\{e_1, e_2, \ldots, e_n\}$ is a maximal independent set of cardinality $n$, $is(e_i) = n$ for all $i = 1$ to $n$. Hence, $is(C(K_{1,n})) = \max \{2, n\} = n$.

(ii) For each edge $vv_i$ ($1 \leq i \leq n$) in $K_{1,n}$, let $e_i$ be the subdivided vertex in $M(K_{1,n})$. Since each $e_i$ ($1 \leq i \leq n$) induces a clique of order $n$, $\{v, v_1, v_2, \ldots, v_n\}$ is a maximal independent set of cardinality $n + 1$. Hence, $is(v) = n + 1$ and $is(v_i) = n + 1$ for all $i = 1$ to $n$.

Clearly $\{e_k\} \cup (\bigcup_{i=1}^n v_i - \{v_k\})$ is the maximal independent set of cardinality $n$. Hence, $is(e_i) = n$ for all $i = 1$ to $n$ and so $is(M(K_{1,n})) = \max \{n + 1, n\} = n + 1$.

(iii) Since $v$ is adjacent to all the vertices of $T(K_{1,n})$, $is(v) = 1$. Now $\{e_k\} \cup (\bigcup_{i=1}^n v_i - \{v_k\})$ is the maximal independent set of cardinality $n$.

Hence, $is(e_i) = n$ for all $i = 1$ to $n$ and so $is(T(K_{1,n})) = \max \{1, n\} = n$.

**Proposition 4.2.17.** For any double star graph $K_{1,n,n}$, $n \geq 2$, we have

(i) $is(C(K_{1,n,n})) = 2n$,

(ii) $is(M(K_{1,n,n})) = n + 2$,

(iii) $is(T(K_{1,n,n})) = n + 1$.

Proof. For each edge $vv_i$ and $v_iu_i$ ($1 \leq i \leq n$), let $e_i$ and $s_i$ be the subdivided vertices respectively in $K_{1,n,n}$.

(i) For every $k = 1$ to $n$, $\{v\} \cup \{v_k\} \cup (\bigcup_{i=1}^n s_i - \{s_k\})$ is a maximal
independent set of cardinality $n + 1$. Hence, $is(v) = n + 1$. Then
\{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$ is a maximal independent set con-
taining $e_i$ (or $s_i$). Hence, $is(e_i) = 2n$ and $is(s_i) = 2n$. Also note that
\{\{v_i\} \cup \{u_i\} \cup \{e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\} \cup \{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\}\}$ is
a maximal independent set containing $v_i$ (or $u_i$). Hence, $is(v_i) = 2n$
and $is(u_i) = 2n$. Hence, $is(C(K_{1,n,n})) = \max\{n + 1, 2n\} = 2n$.
(ii) For any $k = 1$ to $n$, \{u_k\} \cup \{e_k\} \cup (\{\bigcup_{i=1}^n s_i\} - \{s_k\})$ is a max-
imal independent set of cardinality $n + 1$. Hence $is(e_i) = is(u_i) = 
is(s_i) = n + 1$. Moreover, \{v\} \cup \{\bigcup_{i=1}^n s_i\}$ is a maximal independ-
ent set of cardinality $n + 1$. Hence $is(v) = n + 1$. Finally, for any
$k = 1$ to $n$, \{u_k\} \cup \{v_k\} \cup \{v\} \cup (\{\bigcup_{i=1}^n s_i\} - \{s_k\})$ is a maximal
independent set of cardinality $n + 2$. Hence, $is(v_i) = n + 2$ and so
$\text{is}(M(K_{1,n,n})) = \max\{n + 1, n + 2\} = n + 2$.
(iii) Now \{v\} \cup \{\bigcup_{i=1}^n s_i\}$ is a maximal independent set of cardinality
$n + 1$. Hence, $is(v) = n + 1$. For any $k = 1$ to $n$,
\{u_k\} \cup \{e_k\} \cup (\{\bigcup_{i=1}^n s_i\} - \{s_k\})$ is a maximal independent set of car-
dinality $n + 1$. Hence, $is(e_i) = is(u_i) = is(s_i) = n + 1$. Since
\{v_1, v_2, \ldots, v_n\}$ is a maximal independent set of cardinality $n$, we
conclude that $is(v_i) = n$ and so $is(T(K_{1,n,n})) = \max\{n + 1, n\} = 
n + 1$. 
4.3 CHANGING AND UNECHANGING OF INDEPENDENT DOM-SATURATION IN GRAPHS

For any graph theoretic parameter, the study of determining the effect of removal of an edge or a vertex from the graph has important applications such as fault tolerance in networks. The terminology of ‘Changing and Unchanging’ was first suggested by Harary (1982). Carrington et al. (1991) surveyed the problems involved in changing and unchanging domination number of a graph. A detailed study of the effects on $\gamma(G)$ when a vertex or an edge is deleted from $G$ or an edge is added to $G$ is given in chapter 5 of Haynes et al. (1998a). In this section, we study the effect of removal of an edge on the independent dom-saturation number of a graph. We also consider the related problem of determining the minimum number of edges whose removal increases the independent dom-saturation number.

The following are immediate from the definitions:

Example 4.3.1.

(i) For the complete bipartite graph $K_{m,n}$, $m, n \geq 2$ with bipartition $(V_1, V_2)$ and $|V_1| = m \geq n = |V_2|$, we have $\text{is}(K_{m,n}) = m$, $\text{is}(K_{m,n} - v) = m$ for all $v \in V_2$ and $\text{is}(K_{m,n} - v) = m - 1$ for all $v \in V_1$.

(ii) For the complete graph $K_n$, we have $\text{is}(K_n) = 1$, $\text{is}(K_n - v) = 1$ for all $v \in V(K_n)$ and $\text{is}(K_n - e) = 2$ for all $e \in E(K_n)$. 
(iii) Consider the graph given in Figure 4.3.

\[
\begin{array}{c}
\text{Figure 4.3. Graph } G_1 \text{ with } V = V^0 \cup V^- \\
\end{array}
\]

In \( G_1 \), we have \( is(G_1) = 3, \ is(G_1 - u_1) = is(G_1 - u_2) = is(G_1 - u_3) = is(G_1 - u_5) = is(G_1 - u_4) = 3 \) and \( is(G_1 - u_6) = 2 \).

For any \( e_i \in E(G_1) \), we have \( is(G_1 - e_1) = is(G_1 - e_6) = 4 \) and
\( is(G_1 - e_2) = is(G_1 - e_3) = is(G_1 - e_4) = is(G_1 - e_5) = 3 \).

(iv) Consider the graph given in Figure 4.4.

\[
\begin{array}{c}
\text{Figure 4.4 Graph } G_2 \text{ with } V = V^0 \cup V^+ \\
\end{array}
\]

In \( G_2 \), we have \( is(G_2) = 3, \ is(G_2 - u) = is(G_2 - w) = 4 \) and
\( is(G_2 - r) = is(G_2 - s) = is(G_2 - x) = is(G_2 - z) = 3 \).

(v) Consider the graph given in Figure 4.5.

\[
\begin{array}{c}
\text{Figure 4.5 Graph } G_3 \text{ with } V = V^0 \cup V^+ \\
\end{array}
\]

In \( G_3 \), we have \( is(G_3) = 5, \ is(G_3 - e_1) = is(G_3 - e_2) = 4 \) and
\( is(G_3 - e_3) = is(G_3 - e_4) = is(G_3 - e_5) = is(G_3 - e_6) = 5 \).
The above examples show that when a vertex is removed from a graph \( G \), the independent dom-saturation number may increase or decrease or remain unchanged. Following the notation used in the case of domination [Haynes et al. (1998a)], we partition the vertex set \( V \) into subsets \( V^0, V^+, \) and \( V^- \) as follows:

\[
\begin{align*}
V^0 &= \{ v \in V : is(G - v) = is(G) \}; \\
V^+ &= \{ v \in V : is(G - v) > is(G) \}; \\
V^- &= \{ v \in V : is(G - v) < is(G) \}.
\end{align*}
\]

We observe that when an edge is removed from \( G \), the independent dom-saturation may increase or decrease or remain unchanged. Hence, we partition the edge set \( E \) into subsets \( E^0, E^+, \) and \( E^- \) as follows:

\[
\begin{align*}
E^0 &= \{ uv \in E : is(G - uv) = is(G) \}; \\
E^+ &= \{ uv \in E : is(G - uv) > is(G) \}; \\
E^- &= \{ uv \in E : is(G - uv) < is(G) \}.
\end{align*}
\]

**Theorem 4.3.2.** Let \( G \) be a graph with \( \delta = n - 2 \). Let \( v \) be a vertex of degree \( \delta = n - 2 \) and let \( V - N[v] = \{ u \} \). Then \( u \in V^- \).
Proof. Since $\delta = n - 2$, it follows that $is(w) \leq 2$ for all $w \in V(G)$. Since $v$ is of degree $n - 2$, $is(v) = 2$. Hence, $is(G) = 2$. Let $G_1 = G - u$. Since $v$ is adjacent to every vertex of $G_1$, it follows that $is(v) = 1$. Hence, $is(G_1) = 1$ so that $u \in V^-.$

**Corollary 4.3.3.** If $G$ is $n - 2$-regular, then $V = V^-.$

Proof. Let $u \in V$ and $V - N[u] = \{v\}$. Since $G$ is $n - 2$-regular it follows that $V - N[v] = \{u\}$ and hence by Theorem 4.3.2, $u \in V^-.$ Since $u$ is arbitrary, we have $V = V^-.$

**Theorem 4.3.4.** Let $G$ be a graph with $\delta = 1$. Let $uv$ be a pendant edge of $G$. If $v$ is a pendant vertex and $u$ is an is-vertex of $G$, then $v \in V^0$.

Proof. Let $u$ be an is-vertex which is adjacent to $v$. Clearly $v$ is not contained in any is-set of $G$ corresponding to $u$. Hence, when a vertex $v$ is removed from $G$, $is(u)$ cannot increase. Hence $is(G - v) = is(G)$ and so $v \in V^0$.

**Theorem 4.3.5.** Let $G$ be a graph with $\delta = 1$. Let $e = uv$ be a pendant edge of $G$. If $v$ is a pendant vertex and $u$ is an is-vertex of $G$, then $e \in E^+.$

Proof. Let $u$ be the is-vertex which is incident with $e$. Let $G_1 = G - e$. Since $v$ is an isolated vertex of $G_1$, $v$ can be included in any maximal independent set of $G_1$. Hence $is(G_1, u) = is(G, u) + 1 = is(G) + 1$ and so $e \in E^+.$
Example 4.3.6.

(i) For the complete graph $K_n$, $E = E^+$.  

(ii) In $C_5$, we have $is = 2$ and $is(C_5 - e) = is(P_5) = 3$ for any edge $e \in E(C_5)$. Hence $E = E^+$. In general, for the $C_n$ where $n \equiv 0 \pmod{3}$, $E = E^+$.  

(iii) For the graphs $P_n$, $K_{1,n}$, we have $E = E^0$.  

Definition 4.3.7. Let $G$ be a graph and let $v \in V(G)$. If $v$ lies in every is-set of $G$, then $v$ is called is-fixed. $v$ is called is-free if there exist is-sets $A$ and $B$ such that $v \in A$ and $v \notin B$. $v$ is called is-totally free if $v$ does not lie in any is-set of $G$.  

Remark 4.3.8.  

One can easily verify the following:  

(i) Every vertex of a graph $G$ is ‘is-fixed’ if and only if $G = K_n^c$.  

(ii) There is no graph $G$ such that every vertex of $G$ is ‘is-totally free’.  

(iii) A graph $G$ is ‘is-excellent’ if and only if $G$ does not contain a vertex $v$, which is ‘is-totally free’.  

(iv) In the complete graph $K_n$, every vertex is ‘is-free’.  

(v) Consider the bi-star $G = B(n_1, n_2)$ where $n_1, n_2 \geq 2$. Let $v_1, v_2 \in V(G)$ be such that $deg v_1 = n_1$ and $deg v_2 = n_2$. If $n_1 = n_2$, then every vertex of $G$ is is-free. If $n_1 < n_2$, then $v_1$
and all the pendant vertices adjacent to \( v_2 \) are \textit{is}-fixed and all
the remaining vertices are \textit{is}-totally free.

(vi) Consider the complete bipartite graph \( G = K_{m,n} \). Let \( (X, Y) \)
be the bipartition of \( G \) with \(|X| = m\) and \(|Y| = n\). If \( m = n \),
then every vertex of \( G \) is ‘\textit{is}-free’. If \( m < n \), then every vertex
in \( Y \) is ‘\textit{is}-fixed’ and every vertex in \( X \) is ‘\textit{is}-totally free’.

**Theorem 4.3.9.** If \( v \) is an \textit{is}-fixed vertex of a graph \( G \), then \( v \in V^- \)
(The symbol \( V^- \) is defined in the page 88).

**Proof.** Let \( S \) be any \textit{is}-set of \( G \). Since \( v \) is an \textit{is}-fixed vertex, \( v \)
lies on \( S \). We now prove that \( is(G - v) < is(G) \). If \( is(G - v) =
\textit{is}(G) \), then \( v \) does not lie in an \textit{is}-set of \( G \) which is a contradiction.
If \( is(G - v) > is(G) \), then any \textit{is}-set of \( G - v \) contains at least
two vertices \( r, s \) of \( N(v) \). Hence \( S - \{v\} \cup \{r, s\} \) is a maximal
independent set of cardinality \( is(G) + 1 \) which is a contradiction.
Hence \( v \in V^- \).

Bauer et al. (1983) introduced the concept of \textit{bondage number}, which they called \textit{edge stability number}, to be the minimum
number of edges whose removal increase the domination number.
Fink et al. (1990) studied the same concept and were the first to use
the term \textit{bondage number}. We investigate the analogous problem for
independent dom-saturation number of a graph.

**Definition 4.3.10.** The \textit{is-bondage number} of \( G \) is defined to be
the minimum number of edges whose removal increase the value of \( is(G) \).

**Proposition 4.3.11.** For the path \( P_n = (v_1, v_2, ..., v_n) \), \( n \geq 2 \), we have

\[
ib(P_n) = \begin{cases} 
2 & \text{if } n \equiv 0 \pmod{3} \text{ or } n \equiv 2 \pmod{3} \\
1 & \text{if } n \equiv 1 \pmod{3}.
\end{cases}
\]

**Proof.** We consider three cases:

Case (i) \( n \equiv 1 \pmod{3} \)

Let \( n = 3k+1, \ k = 1, 2, ... \) From Proposition 4.2.1, we have \( is(P_n) = \lceil n/3 \rceil = \lceil (3k+1)/3 \rceil \). Let \( e \) be a pendant edge of \( P_n \). Then \( P_n - e = P_1 \cup P_{3k} \). Hence \( is(P_n - e) = 1 + \lceil 3k/3 \rceil + 1 = \lceil (3k+1)/3 \rceil + 1 > is(P_n) \). Hence \( ib(P_n) = 1 \).

Case (ii) \( n \equiv 0 \pmod{3} \)

Let \( n = 3k, \ k = 1, 2, ... \) From Proposition 4.2.1, we have \( is(P_n) = \lceil n/3 \rceil + 1 \). Let \( e \) be any edge of \( P_n \). Then \( P_n - e = P_{n_1} \cup P_{n_2} \); \( n_1 + n_2 = n \), where \( n_1, n_2 \equiv 0 \pmod{3} \) or \( n_1 \equiv 1 \pmod{3} \) and \( n_2 \equiv 2 \pmod{3} \). Since \( is(v) \) is the minimum cardinality among all maximal independent sets of \( G \) containing \( v \), any \( is \)-set of \( P_n \) is union of \( is \)-set of \( P_{n_1} \) and \( i \)-set of \( P_{n_2} \). Hence, if \( n_1 \equiv 1 \pmod{3} \) and \( n_2 \equiv 2 \pmod{3} \), then \( is(P_{n_1} \cup P_{n_2}) = is(P_{n_1}) + i(P_{n_2}) = \lceil n_1/3 \rceil + \lceil n_2/3 \rceil = (n_1+2)/3 + (n_2+1)/3 = \lceil n/3 \rceil + 1 \). If \( n_1, n_2 \equiv 0 \pmod{3} \), then \( is(P_{n_1} \cup P_{n_2}) = is(P_{n_1}) + i(P_{n_2}) = \lceil n_1/3 \rceil + 1 + \lceil n_2/3 \rceil = \lceil n/3 \rceil + 1 \). Hence \( ib(P_n) \geq 2 \). Let \( e, f \) be two pendant edges of \( P_n \).
Then \( P_n - \{e, f\} = P_1 \cup P_{n-2} \cup P_1 \), where \( n - 2 \equiv 1 \ (mod \ 3) \). Hence 
\( is(P_n - \{e, f\}) = 1 + \lceil n/3 \rceil + 1 > is(P_n) \). Thus \( ib(P_n) = 2 \).

Case (iii) \( n \equiv 2 \ (mod \ 3) \)

Similar arguments as in Case (ii), we can prove that \( ib(P_n) = 2 \).

**Theorem 4.3.12.** For the cycle \( C_n = (v_1, v_2, ..., v_n) \), \( n \geq 3 \), we have

\[
ib(C_n) = \begin{cases} 
1 & \text{if } n \equiv 0 \ (mod \ 3) \text{ or } n \equiv 2 \ (mod \ 3) \\
2 & \text{if } n \equiv 1 \ (mod \ 3).
\end{cases}
\]

**Proof.** From Observation (i) of 4.1.1, we have \( is(C_n) = \lceil n/3 \rceil \) for \( n = 3, 4, ... \) Now we consider three cases:

Case i. \( n \equiv 0 \ (mod \ 3) \)

Let \( e \) be any edge of \( C_n \). Then \( C_n - e = P_n \). Hence \( is(C_n - e) = is(P_n) = \lceil n/3 \rceil + 1 > \lceil n/3 \rceil \). Hence, \( ib(C_n) = 1 \).

Case ii. \( n \equiv 1 \ (mod \ 3) \)

Since \( is(C_n) = is(P_n) \) if \( n \equiv 1 \ (mod \ 3) \), \( ib(C_n) \geq 2 \). We now prove that \( ib(C_n) \leq 2 \). Let \( e = v_1v_2 \) and \( f = v_1v_n \) be two edges of \( C_n \) and 
\( C_n - \{e, f\} = P_{n-1} \cup P_1 \). Hence, \( is(C_n - \{e, f\}) = is(P_{n-1}) + 1 = \lceil (n-1)/3 \rceil + 1 + 1 = \lceil n/3 \rceil + 1 > \lceil n/3 \rceil \). Thus \( ib(C_n) = 2 \).

Case iii. \( n \equiv 2 \ (mod \ 3) \)

Let \( e \) be any edge of \( C_n \). Then \( C_n - e = P_n \). Hence, \( is(C_n - e) = is(P_n) = \lceil n/3 \rceil + 2 > \lceil n/3 \rceil \). Thus \( ib(C_n) = 1 \).
Theorem 4.3.13.

(i) For the complete graph $K_n$, $n \geq 1$, $ib(K_n) = 1$.

(ii) For the complete bipartite graph $K_{m,n}$, $m, n \geq 1$, $ib(K_{m,n}) = \max \{m, n\}$.

Proof. (i) From Observation (ii) of 4.1.1, $is(K_n) = 1$. Let $V(K_n) = \{v_1, v_2, ..., v_n\}$. Let $e = v_1 v_2$ be the edge of $K_n$. We notice that $is(v_1, K_n - e) = 2$ and $is(v_2, K_n - e) = 2$. Hence $is(K_n - \{e\}) = \max \{1, 2\} = 2$. Hence $ib(K_n) = 1$.

(ii) From Observation (iii) of 4.1.1, $is(K_{m,n}) = \max \{m, n\}$. Let $(X, Y)$ be the bipartition of $K_{m,n}$ with $|X| = m$ and $|Y| = n$, where $m \leq n$. We have $is(v) = \begin{cases} m & \text{if } v \in X \\ n & \text{if } v \in Y. \end{cases}$

Hence $is(K_{m,n}) = n$. Let $H$ be a spanning subgraph of $K_{m,n}$ such that $is(H) = n + 1$. Then there exists a vertex $v \in X$ such that $v$ is non-adjacent to every vertex of $Y$ in $H$. Hence the number of edges removed from $K_{m,n}$ to obtain $H$ is at least $n$. Then $ib(K_{m,n}) \geq n$.

We obtain $ib(K_{m,n}) \leq n$ by removing all the edges incident with a vertex $v \in X$. Hence the result.
4.4 EDGE INDEPENDENT DOM-SATURATION NUMBER IN GRAPHS

In this section, we discuss edge independent dom-saturation number $eis(G)$ of a graph $G$. Analogous to the domination chain $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$, the six parameters relating to edge independence, edge domination and edge irredundance give rise to the following chain of inequalities:

\[ ir'(G) \leq \gamma'(G) \leq i'(G) \leq \beta_1(G) \leq \Gamma'(G) \leq IR'(G). \]

The two parameters $\gamma'$ and $i'$ are always equal. The edge independent dom-saturation number $eis(G)$ extends the above chain of inequalities and give rise to $ir'(G) \leq \gamma'(G) \leq eis(G) \leq \beta_1(G) \leq \Gamma'(G) \leq IR'(G)$.

**Definition 4.4.1.** Let $G = (V, E)$ be a graph. Let $eis(v, G)$ denote the minimum cardinality among all maximal edge independent sets of $G$ containing $v$. Then $eis(G) = \max \{eis(v) : v \in V(G)\}$ is called the edge independent dom-saturation number of $G$. Thus $eis(G)$ is the least positive integer $k$ with the property that every edge of $G$ lies in a maximal independent set of cardinality $k$.

**Remark 4.4.2.** The parameters edge independent dom-saturation number $eis(G)$ and edge independence saturation number $EIS(G)$ are not comparable. For example,
(i) For the star graph $K_{1,n}$, $eis = EIS$.
(ii) For $C_8$, $eis = 3$ and $EIS = 4$.
(iii) For $P_6$, $eis = 3$ and $EIS = 2$.

**Example 4.4.3.**

(i) For the complete graph $K_n$, $eis(K_n) = \lfloor n/2 \rfloor$.
(ii) For the star graph $K_{1,n}$, $eis(K_{1,n}) = 1$.
(iii) For the cycle $C_n$, $eis(C_n) = \lfloor n/3 \rfloor$.
(iv) For any complete bipartite graph $K_{m,n}$, $eis(K_{m,n}) = \min \{m, n\}$.

**Theorem 4.4.4.** [Sudha (2010)] For any graph $G$,

\[ eis(G) \leq m - \delta' \]

**Theorem 4.4.5.** For the path $(v_1, v_2, \ldots, v_n); n \geq 4$, we have

\[
\begin{align*}
\text{eis}(P_n) &= \begin{cases} 
\lfloor n/3 \rfloor + 1 & \text{if } n \equiv 0 \pmod{3} \\
\lfloor n/3 \rfloor & \text{otherwise}
\end{cases}
\end{align*}
\]

**Proof.** We prove this result by induction on $n$. We consider three cases:

Case i. $n \equiv 0 \pmod{3}$

Let $n = 3k$, $k = 1, 2, \ldots$. We can easily verify that $eis(P_6) = 3$ and $eis(P_9) = 4$. Assume the result is true for $n$, $n \geq 12$. Hence $eis(P_n) = \lfloor n/3 \rfloor + 1$. Now we prove that $eis(P_{n+3}) = \lfloor (n+3)/3 \rfloor + 1$.

Let $e = v_nv_{n+1}$ be the edge in $P_{n+3}$ such that $P_{n+3} - e = P_n \cup P_3$.

Further, any $eis$-set of $P_{n+3}$ is union of $eis$-set of $P_n$ and an edge in $P_3$. By the induction hypothesis, $eis(P_{n+3}) = \lfloor n/3 \rfloor + 1 + 1 = \lfloor (n+3)/3 \rfloor + 1$. 


Case ii. $n \equiv 1 \pmod{3}$
Let $n = 3k + 1, \ k = 1, 2, \ldots$. It can also be verified that $eis(P_4) = 2$ and $is(P_7) = 3$. Assume the result is true for $n, \ n \geq 10$. Hence $eis(P_n) = \lceil n/3 \rceil$. Now we prove that $eis(P_{n+3}) = \lceil (n+3)/3 \rceil$. Let $e = v_nv_{n+1}$ be the edge in $P_{n+3}$ such that $P_{n+3} - e = P_n \cup P_3$. Clearly, any $eis$-set of $P_{n+3}$ is union of $eis$-set of $P_n$ and an edge in $P_3$. By the induction hypothesis, $eis(P_{n+3}) = \lceil n/3 \rceil + 1 = (n+2)/3 + 1 = \lceil (n+3)/3 \rceil$.

Case iii. $n \equiv 2 \pmod{3}$
Let $n = 3k + 2, \ k = 1, 2, \ldots$. It can also be verified that $eis(P_5) = 2$ and $eis(P_8) = 3$. Assume that the result is true for $n, \ n \geq 11$. Hence $eis(P_n) = \lceil n/3 \rceil$. Now we prove that $eis(P_{n+3}) = \lceil (n+3)/3 \rceil$. Let $e = v_nv_{n+1}$ be the edge in $P_{n+3}$ such that $P_{n+3} - e = P_n \cup P_3$. Furthermore, any $eis$-set of $P_{n+3}$ is union of $eis$-set of $P_n$ and an edge in $P_3$. By induction hypothesis, $eis(P_{n+3}) = \lceil n/3 \rceil + 1 = (n+1)/3 + 1 = \lceil (n+3)/3 \rceil$.

**Theorem 4.4.6.** For any graph $G$, $\gamma' = eis$ or $eis - 1$.

**Proof.** Let $e$ be an edge of $G$ with $eis(e) = eis(G)$ and let $A$ be a set of maximal independent edges such that $e \in A$ and $|A| = eis(G)$. Let $B$ be a set of maximal independent edges of minimum cardinality, so that $|B| = \gamma'$. If $e \in \gamma'$, then $eis = \gamma'$. Suppose we assume that $e \notin B$. Let $H = G[B\Delta A]$ be the edge induced subgraph of $G$, induced by the set of edges of $A\Delta B$, where $A\Delta B$ denotes the symmetric difference of $A$ and $B$. Since $e \in A$ and $e \notin B$, we have
$e \in A \Delta B$. Now every vertex of $H$ is incident with at most one edge in $A$ and at most one edge in $B$ and hence has degree 1 or 2. Hence, each component of $H$ is either an even cycle with edges alternately in $A$ and $B$ or a path with edges alternately in $A$ and $B$.

Now, suppose any odd component $C$ has $k$ edges from $A$ and $k + 1$ edges from $B$, then $(B - (E(C) \cap B)) \cup (E(C) \cap A)$ is a maximal independent set of edges of cardinality $\gamma' - 1$. It is a contradiction. Hence any odd component $C$ has $k + 1$ edges from $A$ and $k$ edges from $B$ and in this case $|A| = |B| + 1$, so that $eis = \gamma' + 1$. Hence $eis = \gamma'$ or $\gamma' + 1$.

4.5 INDEPENDENT DOM-SATURATION NUMBER OF SUBDIVISION GRAPHS

Arumugam & Paulraj Joseph (1996) obtained several results concerning domination parameters in subdivision graphs. It is important to study the effect of modifications in a graph $G$ such as deletion or addition of an edge, contraction of an edge, deletion of a vertex etc. Arumugam (1998) considered another type of graph modification and defined dominance subdivision number $sd_\gamma$ of a graph $G$ to be the minimum number of edges that must be subdivided, where each edge can be subdivided at most once, in order to increase the domination number.

In this section, we compute similar problems for the inde-
dependent dom-saturation number of a graph. We compute \( is(S(G)) \) for some classes of graphs and obtain bounds for \( is(S(G)) \). We also introduce and study the concept of \( is \)-subdivision number of a graph. We first determine \( is(S(G)) \) for some standard graphs.

**Example 4.5.1.** For the cycle \( C_n, n \geq 3 \), we have,
\[
is(S(C_n)) = is(C_{2n}) = \lfloor 2n/3 \rfloor.
\]

**Example 4.5.2.** For the path \( P_n, n \geq 2 \), we have
\[
is(S(P_n)) = is(P_{2n-1}) = is(P_m), \text{ where } m = 2n - 1.
\]
Then \( is(P_m) = \begin{cases} 
\lfloor m/3 \rfloor + 1 & \text{if } m \equiv 0 \pmod{3} \\
\lfloor m/3 \rfloor & \text{if } m \equiv 1 \pmod{3} \\
\lfloor m/3 \rfloor + 2 & \text{if } m \equiv 2 \pmod{3}.
\end{cases} \]

**Proposition 4.5.3.** For the complete graph \( K_n, n \geq 3 \), we have
\[
is(S(K_n))) = n - 1.
\]

*Proof.* Let \( V(K_n) = \{u_1, u_2, ..., u_n\} \). Let \( w_{ij} \) be the vertex of \( S(K_n) \) which subdivides the edge \( u_iu_j, 1 \leq i < j \leq n \) and \( i < j \).
Then \( \{w_{ij}\} \cup (V(K_n) - \{u_i, u_j\}) \) is a maximal independent set of cardinality \( n - 2 + 1 = n - 1 \). Hence, \( is(w_{ij}) = n - 1 \) and \( is(u_i) = n - 1 \) for all \( u_i \in V(K_n) \). Therefore, \( is(S(K_n))) = n - 1 \).

**Proposition 4.5.4.** For the star graph \( K_{1,n}, n \geq 1 \), \( is(S(K_{1,n})) = n + 1 \).

*Proof.* Let \( V(K_{1,n}) = \{u, u_1, u_2, ..., u_n\} \) where \( u \) is the centre of the star \( K_{1,n} \). Let \( V(S(K_{1,n})) = V(K_{1,n}) \cup \{v_1, v_2, ..., v_n\} \) where \( v_i \) is
the vertex subdivides the edge $uu_i$. Now $A = \{u, u_1, u_2, ..., u_n\}$ and $B = \{v_1, v_2, ..., v_n\}$ are maximal independent sets in $S(K_{1,n})$. Hence, $is(u) = is(u_i) = n + 1$ and $is(v_i) = n$ for all $i = 1$ to $n$. Therefore, $is(S(K_{1,n})) = n + 1$.

**Proposition 4.5.5.** For the complete bipartite graph $K_{m,n}$, $is(S(K_{m,n})) = m + n - 1$ where $2 \leq m \leq n$.

**Proof.** Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_m\}$ be the bipartition of $K_{m,n}$. Let $w_{ij}$ be the vertex subdividing the edge $x_iy_j$, $1 \leq i \leq m$, $1 \leq j \leq n$. It is clear that $\{y_1\} \cup (\bigcup_{j=1}^{n} w_{2j} - \{w_{21}\}) \cup \{x_1, x_3, ..., x_m\}$ is a maximal independent set containing $y_1$ of cardinality $1 + n - 1 + m - 1 = n + m - 1$. Hence, $is(y_1) = n + m - 1$ and $is(x_1) = is(x_3) = ... = is(x_m) = n + m - 1$. Similarly, $\{y_2\} \cup (\bigcup_{j=1}^{n} w_{2j} - \{w_{22}\}) \cup \{x_1, x_3, ..., x_m\}$ is a maximal independent set containing $y_2$. Hence, $is(y_2) = n + m - 1$ and $is(y_3) = ... = is(y_m) = n + m - 1$. Moreover, $\{x_2\} \cup (\bigcup_{j=1}^{n} w_{1j} - \{w_{11}\}) \cup \{x_3, ..., x_m\}$ is a maximal independent set containing $x_2$ of cardinality $1 + n + m - 2 = n + m - 1$. Hence, $is(x_2) = n + m - 1$ and $is(w_{ij}) = n + m - 1$ for all $i = 1$ to $n$ and $j = 1$ to $m$ and so $is(S(K_{m,n})) = m + n - 1$.

**Proposition 4.5.6.** For any connected graph $G$, we have $is(G) < is(S(G)) \leq \max \{n, m\}$.

**Proof.** Let $V(G) = \{u_1, u_2, ..., u_n\}$ and $E(G) = \{e_1, e_2, ..., e_m\}$. Let $v_i$ be the vertex in $S(G)$ subdividing the edge $e_i$, $1 \leq i \leq m$. It is clear that $V(G)$ and $\{v_1, v_2, ..., v_m\}$ are the maximal independent
sets in \( S(G) \). Hence, \( is(S(G)) \leq \max \{n, m\} \). Since any maximal independent set \( I \) of \( G \) is an independent set of \( S(G) \), any maximal independent set of \( S(G) \) is of greater cardinality than \( I \). Hence, \( is(G) < is(S(G)) \).

Hence, the following question naturally arises: What is the minimum number of edges of \( G \) to be subdivided, so as to increase the independent dom-saturation number of \( G \), where each edge is subdivided at most once?

**Definition 4.5.7.** Let \( G \) be a connected graph. The *is-subdivision number* \( \zeta(G) \) is defined to be the minimum number, that can be the cardinality of a set of edges, such that subdividing each of them exactly once results in a graph whose independent dom-saturation number is greater than that of \( G \).

**Proposition 4.5.8.** For the complete graph \( K_n \), \( n \geq 2 \), \( \zeta(K_n) = 1 \).

**Proof.** Let \( V(K_n) = \{u_1, u_2, ..., u_n\} \). Let \( G \) be the graph obtained from \( K_n \) by subdividing the edge \( u_1u_2 \) and let \( w \in V(G) \) be the vertex that subdivides \( u_1u_2 \). Then \( \{u_1, u_2\} \) and \( \{w, u_i\} \), where \( i \geq 3 \), are the maximal independent sets in \( G \). Hence, \( is(G) = 2 > is(K_n) = 1 \). Therefore, \( \zeta(K_n) = 1 \).

**Proposition 4.5.9.** For the cycle graph \( C_n \), \( n \geq 3 \),
\[
\zeta(C_n) = \begin{cases} 
1 & \text{if } n \equiv 0 \pmod{3} \\
3 & \text{if } n \equiv 1 \pmod{3} \\
2 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** Let \( G \) be the graph obtained from \( C_n \) by subdividing exactly one edge. Then \( G = C_{n+1} \). Since \( is(C_n) = \lfloor n/3 \rfloor \), it follows that

\[
\zeta(C_n) = \begin{cases} 
1 & \text{if } n \equiv 0 \pmod{3} \\
3 & \text{if } n \equiv 1 \pmod{3} \\
2 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

**Proposition 4.5.10.** For the path \( P_n; n \geq 3 \),

\[
\zeta(P_n) = \begin{cases} 
2 & \text{if } n \equiv 0 \pmod{3} \\
1 & \text{if } n \equiv 1 \pmod{3} \\
3 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** Let \( G \) be the graph obtained from \( P_n \) by subdividing exactly one edge. Then \( G = P_{n+1} \). From Proposition 4.2.1, the result follows.

**Theorem 4.5.11.** For the complete bipartite graph \( K_{m,n} \), \( m, n \geq 1 \),

\[
\zeta(K_{m,n}) = \max \{m, n\}.
\]

**Proof.** Let \( X = \{x_1, x_2, ..., x_m\} \) and \( Y = \{y_1, y_2, ..., y_n\} \) be the bipartition of \( K_{m,n} \) and let \( m \leq n \). Then \( is(K_{m,n}) = n \). Now let \( G \) be the graph obtained from \( K_{m,n} \) by subdividing the edges \( x_1y_1, x_1y_2, ..., x_1y_n \) and let \( w_i \) be the vertex of \( G \) that subdivides the
edge \( x_iy_i, 1 \leq i \leq n \). Then \( \{x_1, y_1, y_2, \ldots, y_n\} \) and \( \{x_i, w_1, w_2, \ldots, w_n\} \), where \( 2 \leq i \leq m \), are the maximal independent sets of cardinality \( n + 1 \) in \( G \) and hence \( is(G) = n + 1 \). Therefore, \( \zeta(K_{m,n}) \leq n \).

Now, let \( H \) be any graph obtained from \( K_{m,n} \) by subdividing \( k \) edges, so that \( \zeta(H) = n + 1 \). We claim that \( k \geq n \). Let \( A \) be a maximal independent set in \( H \) of cardinality \( n + 1 \) containing \( y_1 \). Let \( r = |A \cap X|, s = |A \cap Y| \) and \( t = |A \cap (V(H) - X \cup Y)| \), so that \( r, s, t \geq 0 \) and \( r + s + t = n + 1 \). Since \( y_1 \in A \) and \( |A| = n + 1 \), we have \( r, s > 0 \). Since the \( r \) vertices in \( A \cap X \) and the \( s \) vertices in \( A \cap Y \) form a maximal independent set in \( H \), the \( rs \) edges in \( G \) having one end in \( A \cap X \) and other end in \( A \cap Y \) are subdivided. Further, corresponding to each of the \( t \) vertices in \( A \cap (V(H) - (X \cup Y)) \) at least one edge is subdivided. Hence \( k \geq rs + t \geq r + s - 1 + t \geq n \).

Thus \( \zeta(K_{m,n}) = \max \{m, n\} \).

### 4.6 CONCLUSION AND SCOPE

In this chapter, we have initiated the study of independent dom-saturation number of a graph and have established some basic results on this parameter. We conclude with a brief list of unsolved problems.

**Problem 4.6.1.** Characterize the class of graphs for which 
\( is(G) = n - \delta \). Find properties of such graphs.

**Problem 4.6.2.** Given seven positive integers \( a \leq b \leq c \leq d \leq e \leq
$f \leq g$, find a necessary and sufficient conditions for the existence of a graph $G$ with $\text{ir}(G) = a$, $\gamma(G) = b$, $i(G) = c$, $\text{is}(G) = d$, $\beta_0(G) = e$, $\Gamma(G) = f$ and $\text{IR}(G) = g$.

**Problem 4.6.3.** Characterize the class of graphs for which $\text{is}(G) + \chi(G) = n$.

**Problem 4.6.4.** Find necessary and sufficient conditions for graphs with $E = E^+$ and $E = E^0$.

**Problem 4.6.5.** Characterize is-excellent graphs.

**Problem 4.6.6.** Characterize the class of graphs for which $\text{eis}(G) = \gamma'$.

**Problem 4.6.7.** Determine the upper and lower bounds for the is-bondage number $\text{ib}(G)$.

**Problem 4.6.8.** Study characteristics of graphs having a given $\text{ib}(G)$.

**Problem 4.6.9.** Given six positive integers $a \leq b \leq c \leq d \leq e \leq f$, obtain necessary and sufficient conditions for the existence of a graph $G$ with $\text{ir}^i(G) = a$, $\gamma'(G) = b$, $\text{eis}(G) = c$, $\beta_1(G) = d$, $\Gamma'(G) = e$ and $\text{IR}^i(G) = f$. 