CHAPTER 2

INDEPENDENCE SATURATION NUMBER
OF A GRAPH

2.1 INTRODUCTION

Subramanian (2004) introduced the concept of independence saturation number. They have obtained several basic results on this concept. In this chapter, we investigate the independence saturation number of some classes of graphs. Let $G = (V, E)$ be a graph and let $v \in V$. Let $IS(v)$ denote the maximum cardinality of an independent set in $G$ which contains $v$. Then $IS(G) = \min \{IS(v) : v \in V\}$ is called the independence saturation number of $G$. Thus $IS(G)$ is the largest positive integer $k$ such that every vertex of $G$ lies in an independent set of cardinality $k$. A vertex $v \in V$ is called an $IS$-vertex if $IS(v) = IS(G)$. Let $v \in V$ be such that $IS(v) = IS(G)$. Then any maximal independent set of cardinality $IS(G)$ containing $v$ is called an $IS$-set of $G$. Thus $IS$-set is a maximal independent set and hence is obviously a dominating set. Hence $i(G) \leq IS(G) \leq \beta_0(G)$. From Definition 1.3.14, the independence saturation number $IS(G)$ can also be denoted by $\beta_0^{m,M}(G)$. 

We start with the following example (see Figure 2.1) to illustrate this:

![Graph](image)

Figure 2.1. Graph with $IS(G) = 3$

For the above graph, $u$ and $r$ are the $IS$- vertices; $\{u, v, w\}$ is an $IS$-set containing $u$ and $\{r, s, t\}$ is an $IS$-set containing $r$. Hence $IS(u) = IS(r) = IS(G) = 3$. But $IS(v) = IS(w) = IS(s) = IS(t) = IS(z) = 5$. Hence $IS(G) = \min \{3, 5\} = 3$.

### 2.2 INDEPENDENCE SATURATION NUMBER OF SOME CLASSES OF GRAPHS

**Proposition 2.2.1.** Let $G$ be any bipartite graph with no isolated vertices. Then $IS(G) = \beta_0(G)$ if and only if $G$ has a perfect matching.

*Proof.* $IS(G) = \beta_0(G)$ means that every vertex lies in a maximum independent set and so $G$ is 1-extendable. Hence, from Theorem 1.3.28, the result follows.

In the following propositions, we investigate the independence saturation number of the central graph of star graph $K_{1,n}$,
double star $K_{1,n,n}$ and cycle graph $C_n$. Also, we compute
independence saturation number for the total graph, line graph of
star graph $K_{1,n}$ and the double star graph families $K_{1,n,n}$.

**Proposition 2.2.2.** For any star graph $K_{1,n}$, $n \geq 2$, we have

(i) $IS(C(K_{1,n})) = 2,$

(ii) $IS(T(K_{1,n})) = 1,$

(iii) $IS(L(K_{1,n})) = 1.$

**Proof.** (i) By the definition of central graph, each edge $vv_i$ in $K_{1,n}$ is
subdivided by the vertex $e_i$ in $C(K_{1,n})$ and the vertices $v_1, v_2, \ldots, v_n$
induce a clique of order $n$ in $C(K_{1,n})$. i.e. $V(C(K_{1,n})) = \{v\} \cup$
$\{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Since $v_i$ $(1 \leq i \leq n)$ induce a
clique of order $n$ and $v_i$ is adjacent to $e_i$, $\{v_i\} \cup$
$\{e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}$ is a maximum independent set containing
$v_i$ $(1 \leq i \leq n)$. Hence, $IS(v_i) = n$. Also $\{e_1, e_2, \ldots, e_i, \ldots, e_n\}$ is
a maximum independent set containing $e_i$ $(1 \leq i \leq n)$. Hence
$IS(e_i) = n$. Moreover $\{v, v_i\}$ is a maximum independent set of
$C(K_{1,n})$ containing $v$. Hence, $IS(v) = 2$ and so $IS(C(K_{1,n})) = 2.$

(ii) By the definition of total graph, we have $V(T(K_{1,n})) = \{v\} \cup$
$\{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$, in which the vertices
$v, e_1, e_2, \ldots, e_n$ induce a clique of order $n + 1$. Since $v$ is adjacent
to all the vertices of $T(K_{1,n})$, $IS(v) = 1$ and so $IS(T(K_{1,n})) = 1.$

(iii) Since $L(K_{1,n}) \cong K_n$, we have $IS(L(K_{1,n})) = 1.$

**Proposition 2.2.3.** For any double star graph $K_{1,n,n}$, $n \geq 2$, we have
(i) \( IS(C(K_{1,n,n})) = n + 1, \)
(ii) \( IS(T(K_{1,n,n})) = n + 1, \)
(iii) \( IS(L(K_{1,n,n})) = n. \)

Proof. (i) By the definition of central graph, each edge \( vv_i \) and \( v_iu_i \)
\((1 \leq i \leq n)\) in \( K_{1,n,n} \) are subdivided by the vertices \( e_i \) and \( s_i \) in \( C(K_{1,n,n}) \). The vertices \( v, u_1, u_2, ..., u_n \) induce a clique of order \( n+1 \)
(say \( K_{n+1} \)) and the vertices \( v_i(1 \leq i \leq n) \) induce a clique of order \( n \) in \( C(K_{1,n,n}) \). i.e. \( V(C(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\} \)
Now \( \{v\} \cup \{e_i \} \cup \{s_i \} \) is a maximum independent set containing \( v \). Hence \( IS(v) = n + 1 \). Then \( \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\} \) is a maximum independent set containing \( e_i \) (or \( s_i \)). Hence, \( IS(e_i) = 2n \) and \( IS(s_i) = 2n \). Also note that \( \{v_i\} \cup \{u_i\} \cup \{e_1, e_2, ..., e_{i-1}, e_{i+1}, ..., e_n\} \cup \{s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n\} \) is a maximum independent set containing \( v_i \) (or \( u_i \)). Hence, \( IS(v_i) = 2n \) and \( IS(u_i) = 2n \). Hence, \( IS(G) = \min \{n+1, 2n\} = n + 1 \).

(ii) By the definition of total graph, we have \( V(T(K_{1,n,n})) = \{v\} \)
\( \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\} \) in which the vertices \( v, e_1, e_2, ..., e_n \) induce a clique of order \( n+1 \). Note that \( \{v\} \cup \{s_i : 1 \leq i \leq n\} \) is a maximum independent set containing \( v \) (or \( s_i \)). Hence \( IS(v) = n+1 \) and \( IS(s_i) = n+1 \).
Then for any \( j = 1 \) to \( n \), \( \{e_j\} \cup \{u_i : 1 \leq i \leq n\} \) is a maximum independent set containing \( e_i \) (or \( u_i \)), \( i = 1 \) to \( n \). Hence, \( IS(e_i) = n + 1 \) and \( IS(u_i) = n + 1 \). Also \( \{v_i\} \cup (\cup_{j \neq i} u_j) \cup \{e_j : j \neq i\} \) is a maximum independent set containing \( v_i \). Hence, \( IS(v_i) = 1 + n - 1 + 1 = n + 1 \).
Therefore, $IS(T(K_{1,n,n})) = n + 1$.

(iii) By the definition of line graph, each edge of $K_{1,n,n}$ is taken to be as vertex in $L(K_{1,n,n})$. The vertices $e_1, e_2, ..., e_n$ induce a clique of order $n$ and the vertices $s_1, s_2, ..., s_n$ are all pendant in $L(K_{1,n,n})$, i.e.

$V(L(K_{1,n,n})) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. We observe that

$\{e_i\} \cup (\cup_{j \neq i} s_j)$ is a maximum independent set containing $e_i$. Hence, $IS(e_i) = n$. Then $\{s_i : 1 \leq i \leq n\}$ is a maximum independent set containing $s_i$. Hence, $IS(s_i) = n$ and so $IS(L(K_{1,n,n})) = n$.

**Proposition 2.2.4.** For any cycle $C_n = (v_1, v_2, ..., v_n)$, $n \geq 4$, we have $IS(C(C_n)) = n - 1$.

**Proof.** By the definition of central graph, each edge $v_iv_j$ ($i < j$ and $1 \leq i, j \leq n$) in $C_n$ is subdivided by the vertex $e_i$ in $C(C_n)$ and $\deg v_i = n - 1$, $\deg e_i = 2$. Note that $\{v_i, v_{i+1}, e_{i+2}, e_{i+3}, ..., e_{i+n-2}\}$ is a maximum independent set of $C(C_n)$ containing $v_i$ ($1 \leq i \leq n$). Hence, $IS(v_i) = n - 2 + 1 = n - 1$. Also $\{e_1, e_2, ..., e_n\}$ is a maximum independent set containing $e_i$. Hence, $IS(e_i) = n$ ($1 \leq i \leq n$) and so $IS(C(C_n)) = n - 1$.

In the following proposition, we determine the independence saturation number of expansion and corona of graphs.

**Proposition 2.2.5.** For any graph $G$,

(i) $IS(\text{exp}(G, r)) = r.IS(G)$.

(ii) $IS(\text{cor}(G, r)) = r |V(G)| - r + 1$.

**Proof.** (i) Let $D$ be any independent set of $\text{exp}(G, r)$. Note that each set $I_v$ in $D$ corresponds to a vertex $v$ in $G$. Then $\{v : I_v \subseteq D\}$ is an
independent set of $G$. Let $z$ be any $IS$-vertex of $G$ and $S$ be any $IS$-set of $G$ containing $z$ in $G$. Then $IS(G) = |S| = \min \{IS(v)\}$. Obviously, for every two vertices $v, w$ in $S$, there corresponds two sets $I_v, I_w$ in $\text{exp}(G, r)$. Since $v$ and $w$ are non-adjacent, all the vertices of $I_v \cup I_w$ are independent. Hence, $\cup I_v, v \in S$ is a maximum independent set containing $w; w \in I_z$. Hence, $IS(\text{exp}(G, r)) = r.IS(G)$.

(ii) Let $D$ be any maximum independent set of $\text{cor}(G, r)$. For every vertex $v$ of $G$, $D$ contains all of $r$ leaves adjacent to $v$. Let $v \in \text{cor}(G, r)$ and $I_v$ be any maximum independent set containing $v$. If $v$ is not a leaf, then $D$ contains $|V(G)| - 1$ leaves and $v$. Hence, $IS(v) = (|V(G)| - 1)r + 1 = r|V(G)| - r + 1$. If $v$ is a leaf, then $IS(v) = r|V(G)|$. Therefore, $IS(G) = \min \{r|V(G)| - r + 1, r|V(G)|\} = r|V(G)| - r + 1$.

In Figures 2.2(a) and 2.2(b) below we illustrate Proposition 2.2.5.

**Example 2.2.6.**

In Figure 2.2(a) $IS(\text{exp}(C_4, 2)) = 4$ and in Figure 2.2(b) $IS(\text{cor}(C_4, 2)) = 7$.  

Figure 2.2(a). $\text{exp}(C_4, 2)$  
Figure 2.2(b). $\text{cor}(C_4, 2)$
In Theorem 2.2.7, we investigate the independence saturation number $IS(G)$ of Mycielskian graph $G$. A collection of articles related to Mycielskian graphs can be found in [Lin et al. (2006), Mojdeh & Rad (2008) and Mycielski (1955)].

Recall that Mycielskian of $G$ is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and is disjoint from $V$, and edge set $E' = E \cup \{xy', x'y : xy \in E\} \cup \{x'u : x' \in V'\}$. The vertices $x$ and $x'$ are called twins of each other and $u$ is called the root of $\mu(G)$. Also the graph $\mu(G) - u$ is called the shadow graph of $G$ and is denoted by $Sh(G)$.

For example, $IS(\mu(C_4)) = 3$. In Figure 2.3, the Mycielskian graph of $C_4$ is shown below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{mycielskian_graph.png}
\caption{The Mycielskian graph $\mu(C_4)$}
\end{figure}

**Theorem 2.2.7.** For any graph $G$, we have,

$$IS(\mu(G)) = \min \{\beta_0(G) + 1, 2IS(G)\}.$$ 

**Proof.** Let $u$ be the root of $\mu(G)$. Since $u$ is adjacent to every $v', v \in V(G)$, any $\beta_0$ set of $G \cup \{u\}$ is a maximum independent set containing $u$ in $\mu(G)$. Hence $IS_{\mu(G)}(u) = \beta_0(G) + 1$. Let $v' \in \mu(G), v \in V(G)$. Let $S = \{v, v_1, v_2, ..., v_n\}$ be any $IS$- set
of $G$. Then $S' = \{v, v', v_1, v_1', v_2, v_2', ..., v_n, v_n'\}$ is a maximum independent set containing $v$ in $\mu(G)$ and so $IS_{\mu(G)}(v) = 2IS_G(v)$. We observe that $\{v' : v \in V(G)\}$ is a maximal independent set containing $v'$. Hence, $IS_{\mu(G)}(v') = \max \{n(G), 2IS_G(v)\}$ and $IS_{\mu(G)}(v) \leq IS_{\mu(G)}(v')$. Hence, $IS(\mu(G)) = \min \{\beta_0(G) + 1, 2IS(G)\}, v \in V(G)$.

Below we give an example of a graph $G$ having $IS(\mu(G)) = \beta_0(G) + 1$.

**Example 2.2.8.** In Figure 2.4, $IS(G_1) = 3$, $\beta_0(G_1) = 4$ and $IS(\mu(G_1)) = 5 = \beta_0(G_1) + 1$. Now we have $IS(z) = IS(s) = IS(x) = IS(w) = IS(t) = 4$ and $IS(u) = IS(v) = 3$. Hence $IS(\mu(G_1)) = \min \{8, 6, 5\} = 5$.

![Figure 2.4](image)

Figure 2.4. Graph with $IS(\mu(G_1)) = 5 = \beta_0(G_1) + 1$

**Example 2.2.9.** A graph $G$ having $IS(\mu(G)) = 2IS(G)$ is shown below.

In Figure 2.5, $IS(u) = IS(G_2) = 3$, $\beta_0(G_2) = 7$ and $IS(\mu(G_2)) = 6 = 2IS(u)$. Let $r$ be the root of $\mu(G_2)$. Here $IS(r) = 8 = \beta_0(G_2) + 1$, $IS_{\mu(G_2)}(u) = 6$, $IS_{\mu(G_2)}(v) = 12$, $IS_{\mu(G_2)}(v_i) = 14$; $i = 1$ to 5,
\[ IS_{\mu(G_2)}(u') = \max \{n(G_2), 2IS_{G_2}(u)\} = \max \{9, 6\} = 9, \]
\[ IS_{\mu(G_2)}(v') = \max \{n(G_2), 2IS_{G_2}(v)\} = \max \{9, 12\} = 12 \text{ and} \]
\[ IS_{\mu(G_2)}(v_i') = 14. \text{ Hence, } IS(\mu(G_2)) = \min \{6, 8, 9, 12, 14\} = 6. \]

Figure 2.5. Graph with \( IS(\mu(G_2)) = 6 = 2IS(G_2) \)

In the following theorem, we compute the independence saturation number of maximal triangle free graphs.

**Theorem 2.2.10.** Let \( G \) be a maximal triangle free graph of order \( n \geq 2 \) and minimum degree \( \delta(G) \). Then \( IS(G) \) is either \( \delta(G) \) or \( \Delta(G) \).

**Proof.** Let \( u \in V(G) \) and \( I_u \) be any maximum independent set containing \( u \). Choose a vertex \( v \in V(G) \) such that \( u \in N(v) \). Since \( G \) is a maximal triangle free graph, \( N(v) \) is an independent set containing \( u \). We prove \( N(v) \) is maximal. Suppose there exists \( w \in V(G) - N[v] \) such that \( w \) is not adjacent to any vertex of \( N(v) \). Then \( G + uv \) is triangle free, a contradiction to the fact that \( G \) is maximal triangle free. Hence \( N(v) \) is maximal and so \( |I_u| \geq \deg v \). Since every vertex in \( G \) is of degree either \( \delta(G) \) or \( \Delta(G) \), \( |I_u| \geq \delta(G) \) or \( |I_u| \geq \Delta(G) \). Let \( I \) be any independent set of \( G \) containing \( u \). Now we show that \( I \subseteq N(v) \) for some \( v \) such
that $u \in N(v)$. Suppose not. Then there exists a $z \in I$ such that $z \notin N(v)$ for every $v$ such that $u \in N(v)$. Then $G + vz$ is triangle free, a contradiction. Hence, $|I| \leq \deg v$ for some $v$ such that $u \in N(v)$. Since every vertex in $G$ is of degree either $\delta(G)$ or $\Delta(G)$ and $|I_u| \leq \deg v$ for some $v$ such that $u \in N(v)$, $|I_u|$ is either $\delta(G)$ or $\Delta(G)$. Note that $IS(G) = \min \{|I_u| : u \in V(G)\}$. Hence, $IS(G)$ is either $\delta(G)$ or $\Delta(G)$.

Below we give an example of a graph which has $IS(G) = \delta(G)$ and one which has $IS(G) = \Delta(G)$.

**Example 2.2.11.** In Figure 2.6(a), we have $IS(G_1) = \delta(G_1) = 2$ and in Figure 2.6(b), we have $IS(G_2) = \Delta(G_2) = 3$.

![Figure 2.6(a). Graph with $IS(G_1) = \delta(G_1)$](image1)

![Figure 2.6(b). Graph with $IS(G_2) = \Delta(G_2)$](image2)

**Theorem 2.2.12.** Let $G$ be a maximal triangle free graph of order $n \geq 2$ and minimum degree $\delta(G)$. Then $IS(G) = \delta(G)$ if and only if there exists $v$ such that $\deg w = \delta(G)$ for all $w \in N(v)$.

**Proof.** Assume that $IS(G) = \delta(G)$. Suppose for every $v \in V(G)$, there exists $w \in N(v)$ such that $\deg w > \delta(G)$. Since $G$ is maximal triangle free, $N(w)$ is a maximal independent set containing $v$. 
Hence, $IS(v) \geq \delta + 1$ and so $IS(G) \geq \delta + 1$. It contradicts the assumption. Conversely, to prove that $IS(v) = IS(G) = \delta(G)$. Since $N(w)$ is a maximal independent set containing $v$, $IS(v) \geq \delta(G)$. Let $I$ be any independent set containing $v$. From the proof of Theorem 2.2.10, $I \subseteq N(w)$ for some vertex $w$ such that $w \in N(v)$. Hence $IS(v) = \delta(G)$. Since $IS(G)$ is either $\delta(G)$ or $\Delta(G)$, it follows that $IS(v) = IS(G) = \delta(G)$.

2.3 CONCLUSION AND SCOPE

In this chapter, we determined the value of independence saturation number of some classes of graphs. Since the decision problem corresponding to this parameter is NP-complete for general graphs, the problem of designing efficient algorithms for computing $IS(G)$ for special classes of graphs is an interesting direction for further research.

The following problems remain open.

**Problem 2.3.1.** Characterize the class of graphs $G$ for which $IS(G) = i(G)$.

**Problem 2.3.2.** Characterize the class of graphs $G$ for which $IS(G) = \Gamma_i(G)$.

**Problem 2.3.3.** Characterize the class of graphs $G$ for which $IS(G) = n - \Delta$. 
Problem 2.3.4. Characterize the class of graphs $G$ for which $IS(G) = n - \chi(G)$.

Problem 2.3.5. Characterize the class of graphs $G$ for which $IS(\mu(G)) = \beta_0(G) + 1$.

Problem 2.3.6. Characterize the class of graphs $G$ for which $IS(\mu(G)) = 2IS(G)$. 