CHAPTER 5

INDEPENDENT OPEN IRREDUNDANT COLORINGS OF GRAPHS

5.1 INTRODUCTION

In this chapter, we introduce and study the concept of independent open irredundant colorings which combines the domination parameter namely independent open irredundant set and coloring parameter. Cockayne (1999) introduced the study of a large class of generalised irredundant sets in graphs. Cockayne (1999) identifies 12 types of generalised irredundant sets. Each type of a generalised irredundant set $S \subseteq V$ is defined by the types of private neighbors (i.e. self, internal or external) that each vertex in the set must have. A set $S$ is called an independent open irredundant set or ioir-set if $S$ is an independent set and every vertex in $S$ has an external private neighbor. Perhaps the most interesting of these are the ioir-sets. The minimum cardinality of a maximal independent open irredundant set in $G$ is called the independent open irredundance number of $G$ and is denoted by $ioir(G)$. The maximum cardinality of a maximal independent open irredundant set in $G$ is called the upper independent open irredundance number of $G$ and is denoted
by $IOIR(G)$. These generalised irredundant sets are also studied by Finbow (2003) and Cockayne & Finbow (2004).

Given a property $P$ concerning subsets of $V$, a $P$-coloring induces a partition $\Pi = \{V_1,V_2,\ldots,V_k\}$ of $V$ into sets, such that each $V_i$ has the property $P$. If the property $P$ is independence, the $P$-coloring is the usual coloring and if the property $P$ is domination, the corresponding $P$-coloring gives the concept of domatic partition. Haynes et al. (2008) introduced the concept of irredundant colorings and open irredundant colorings of graphs. Arumugam et al. (2011) initiate a study of open irredundant colorings and obtained several results on irredundant colorings and open irredundant colorings. Motivated by the work on Haynes et al. (2008) and Arumugam et al. (2011), we initiate a study of independent open irredundant colorings. An independent open irredundant coloring of a graph $G$ is a partition of $V$ into the nonempty independent open irredundant sets. The independent open erratic number is the minimum order of an independent open irredundant coloring of $G$ and it is denoted by $\chi_{ioir}(G)$. In this chapter, we present some results on independent open irredundant colorings.

5.2 INDEPENDENT OPEN IRREDUNDANT COLORINGS

Throughout this chapter we assume that $G$ is a graph without isolated vertices. In this section, we present independent open irredundant coloring of some basic graphs and also obtain certain
bounds. The following observations are immediate from the definition.

Observation 5.2.1. Since any ioir-coloring of \( G \) is an oir-coloring and \( \chi \)-coloring of \( G \), it follows that \( \chi_{ir}(G) \leq \chi_{oir}(G) \leq \chi_{ioir}(G) \) and \( \chi_{ir}(G) \leq \chi(G) \leq \chi_{ioir}(G) \).

Observation 5.2.2. Since \( V(G) \) is not an ioir-set of \( G \), it follows that \( 2 \leq \chi_{ioir}(G) \leq n \).

Theorem 5.2.3. For any graph \( G \), \( \chi_{ioir}(G) = 2 \) if and only if \( G \cong nK_2 \), where \( n \geq 1 \) is a positive integer.

Proof. Assume that \( \chi_{ioir}(G) = 2 \). Let \( \{V_1, V_2\} \) be a partition of \( V(G) \) into independent open irredundant sets. We now prove that every vertex \( v \) in \( G \) has degree 1. Suppose there exists a vertex \( v \in V_1 \) such that \( \text{degv} = 2 > 1 \). Then there exist two vertices \( u, w \in V_2 \) such that \( u \) and \( w \) are adjacent to \( v \). It contradicts the fact that \( V_2 \) is open irredundant. Hence \( G \) is an 1-regular graph and \( G \cong nK_2 \). The converse is obvious.

Theorem 5.2.4. Let \( G \) be a graph of order \( n \). Then \( \chi_{ioir}(G) = n \) if and only if for any independent set \( S \subset V \), there exist \( v, w \in S \) such that \( N(v) \subseteq N(w) \) or \( N(w) \subseteq N(v) \).

Proof. Assume that \( \chi_{ioir}(G) = n \). Suppose there is an independent set \( S \subset V \) such that \( N(v) \not\subseteq N(w) \) and \( N(w) \not\subseteq N(v) \) for all \( v, w \in S \). Then there exists a vertex \( z_1 \in N(v) \) such that \( z_1 \) is not adjacent to \( w \) and there exists a vertex \( z_2 \in N(w) \) such that \( z_2 \) is not adjacent to \( v \). Hence \( \{v, w\} \) is an ioir-set and \( IOIR(G) \geq 2 \).
Then \( \{v, w\} \) and all the singleton sets of \( V(G) - \{v, w\} \) form an \( ioir \)-coloring. Therefore \( \chi_{ioir}(G) \leq n - 1 \) which is a contradiction. The converse is obvious.

**Observation 5.2.5.** For the complete graph \( K_n \) and the complete bipartite graph \( K_{m,n} \), we have \( \chi_{ioir}(K_n) = n \) and \( \chi_{ioir}(K_{m,n}) = m + n \).

**Observation 5.2.6.** For any tree \( T \) of order \( n \), \( \chi_{ioir}(T) = n \) if and only if \( T \) is a star.

**Theorem 5.2.7.** For the path \( P_n = (v_1, v_2, ..., v_n) \), \( n \geq 3 \), we have \( \chi_{ioir}(P_n) = 3 \).

*Proof.* Let \( V_1 = \{v_1, v_4, v_7, v_{10}, \ldots\} \), \( V_2 = \{v_2, v_5, v_8, v_{11}, \ldots\} \) and \( V_3 = \{v_3, v_6, v_9, v_{12}, \ldots\} \). Clearly \( \{V_1, V_2, V_3\} \) is a partition of \( V(G) \) into independent open irredundant sets. Hence \( \chi_{ioir}(P_n) \leq 3 \). By Theorem 5.2.3 , \( \chi_{ioir}(P_n) \geq 3 \) and so \( \chi_{ioir}(P_n) = 3 \).

**Theorem 5.2.8.** For the cycle \( C_n = (v_1, v_2, ..., v_n) \), \( n \geq 3 \), we have

\[
\chi_{ioir}(C_n) = \begin{cases} 
4 & \text{if } n = 4 \text{ or } n = 7 \\
3 & \text{otherwise.}
\end{cases}
\]

*Proof.* We can easily observe that \( \chi_{ioir}(C_4) = 4 \). We now prove that \( \chi_{ioir}(C_n) = 3 \) for \( n \neq 4 \) or 7. By Theorem 5.2.3, \( \chi_{ioir}(C_n) \geq 3 \). There are three cases.

Case i. \( n \equiv 0 \pmod{3} \).

Let \( V_1 = \{v_1, v_4, v_7, v_{10}, \ldots v_{n-2}\} \), \( V_2 = \{v_2, v_5, v_8, v_{11}, \ldots v_{n-1}\} \), and \( V_3 = \{v_3, v_6, v_9, v_{12}, \ldots v_n\} \). Clearly \( \{V_1, V_2, V_3\} \) is a partition of \( V(G) \)
into independent open irredundant sets since any three consecutive
vertices in the cycle receive distinct colors. Hence \( \chi_{ioir}(C_n) \leq 3 \).

Case ii. \( n \equiv 1 \pmod{3} \), when \( n \geq 10 \)
Let \( V_1 = \{v_1, v_3, v_6, v_8, v_{11}, v_{14}, v_{17}, \ldots, v_{l-3}, v_l, v_{l+3}, \ldots v_{n-2}\} \),
\( V_2 = \{v_2, v_4, v_7, v_9, v_{12}, v_{15}, v_{18}, \ldots, v_{l-3}, v_l, v_{l+3}, \ldots, v_{n-1}\} \),
\( V_3 = \{v_5, v_{10}, v_{13}, v_{16}, v_{19}, \ldots, v_{l-3}, v_l, v_{l+3}, \ldots, v_n\} \).

We now prove that \( \{V_1, V_2, V_3\} \) is a partition of \( V(G) \) into indepen-
dent open irredundant sets. Clearly the sets \( V_i, i = 1, 2, 3 \) are independent. Hence, it is enough to prove that every vertex in the
set \( V_i \) has an external private neighbor with respect to \( V_i, i = 1, 2, 3 \).
Note that \( v_1, v_5, v_6 \) are the external private neighbors of \( v_2, v_4, v_7 \)
respectively and \( v_n, v_4, v_7 \) and \( v_{10} \) are the external private neighbors
of \( v_1, v_3, v_8 \) and \( v_9 \) respectively. All other remaining vertices \( v_i \) have external private neighbor \( v_{i-1} \).

Case iii. \( n \equiv 2 \pmod{3} \).

Let \( V_1 = \{v_1, v_4, v_7, v_{10}, \ldots v_{n-1}\} \), \( V_2 = \{v_2, v_5, v_8, v_{11}, \ldots v_n\} \) and \( V_3 = \{v_3, v_6, v_9, v_{12}, \ldots v_{n-2}\} \).
Since \( v_2, v_{n-1}, v_{n-2} \) are the external private neighbors of \( v_1, v_n, v_{n-1} \) respectively and remaining vertices \( v_i \) have external private neighbor \( v_{i+1} \), \( \{V_1, V_2, V_3\} \) is a partition of \( V(G) \)
into independent open irredundant sets. Hence \( \chi_{ioir}(C_n) \leq 3 \). Now
we prove that \( \chi_{ioir}(C_7) = 4 \). Since any independent open irredu-
dant set of \( C_7 \) has at most two vertices, minimum four colors are
required to color the vertices of \( C_7 \). Let \( V_1 = \{v_1, v_3\}, V_2 = \{v_2, v_6\},
V_3 = \{v_3, v_5\} \) and \( V_4 = \{v_7\} \). Clearly \( \{V_1, V_2, V_3, V_4\} \) is an ioir-
coloring of \( C_7 \). Hence \( \chi_{ioir}(C_7) = 4 \).
In the following theorem, we investigate the independent open erratic number of the corona of cycle graph.

**Theorem 5.2.9.** For the cycle \( C_n = (v_1, v_2, ..., v_n) \), \( n \geq 3 \),
we have \( \chi_{ioir}(C_n \circ K_1) = \begin{cases} 
4 & \text{if } n \text{ is odd} \\
3 & \text{if } n \text{ is even.}
\end{cases} \)

**Proof.** Let \( v_1, v_2, ..., v_n \) be the vertices of \( C_n \) and \( u_1, u_2, ..., u_n \) be pendant vertices adjacent to \( v_1, v_2, ..., v_n \) respectively.

Case i. \( n \) is odd

From Theorem 5.2.3, \( \chi_{ioir}(C_n \circ K_1) \geq 3 \). First, we prove that \( \chi_{ioir}(C_n \circ K_1) = 4, n \neq 7 \). From Theorem 5.2.8, \( \chi_{ioir}(C_n) = 3 \). Hence there exists a vertex \( v \in C_n \) such that the vertices of \( N[v] \) receive three distinct colors \( c_1, c_2 \) and \( c_3 \). Let \( u \) be the pendant vertex adjacent to \( v \). Since \( v \) is the only external private neighbor of \( u \) with respect to any subset of \( V(G) \), \( v \) is colored with \( c_4 \) which is different from \( c_1, c_2 \) and \( c_3 \). Hence \( \chi_{ioir}(C_n \circ K_1) \geq 4 \).

From Theorem 5.2.8, vertices of \( C_n \) are colored with three distinct colors \( c_1, c_2 \) and \( c_3 \). Then the pendant vertices \( u_1, u_2, ..., u_n \) are colored with \( c_4 \). It is clear that \( v_1, v_2, ..., v_n \) are the external private neighbors of \( u_1, u_2, ..., u_n \) respectively. Hence \( \chi_{ioir}(C_n \circ K_1) = 4, n \neq 7 \). Now we prove that \( \chi_{ioir}(C_7 \circ K_1) = 4 \). From Theorem 5.2.8, \( \chi_{ioir}(C_7) = 4 \). Let \( c_1, c_2, c_3 \) and \( c_4 \) be the four distinct colors. The vertices \( v_1, v_2, ..., v_n \) are colored with \( c_1, c_2, c_3 \) and \( c_4 \) in the manner that any three consecutive vertices in \( c_7 \) receive three distinct colors. Hence, let \( v_i \) be the vertex in \( c_7 \) such that vertices of \( N[v_i] \) receive three distinct colors \( c_1, c_2 \) and \( c_3 \). Then we color the vertex \( u_i \) with
$c_4$ which is different from $c_1$, $c_2$ and $c_3$. Hence $\chi_{ioir}(C_7 \circ K_1) = 4$.  

Case ii. $n$ is even  

From Theorem 5.2.3, $\chi_{ioir}(C_n \circ K_1) \geq 3$. We show that  

$\chi_{ioir}(C_n \circ K_1) \leq 3$ by exhibiting an $ioir$ coloring. Let $c_1$, $c_2$ and $c_3$ be three distinct colors. Assign $ioir$ coloring with $c_1$, $c_2$ and $c_3$ as follows: The vertices $v_1, v_2, \ldots, v_n$ are alternately colored with two distinct colors $c_1$ and $c_2$. The pendant vertices $u_1, u_2, \ldots, u_n$ are all colored with $c_3$. It is clear that $v_i$ is the external private neighbor of $u_i$ and vice-versa. Hence $\chi_{ioir}(C_n \circ K_1) = 3$.  

**Proposition 5.2.10.** For any graph $G$ with order $n$,  

$$\frac{n}{IOIR(G)} \leq \chi_{ioir}(G) \leq n - IOIR(G) + 1, \text{ where } IOIR(G) \text{ is the upper independent open irredundance number of } G.$$  

**Proof.** Let $\chi_{ioir}(G) = k$. Let $\{V_1, V_2, \ldots, V_k\}$ be an $ioir$-coloring of $G$. Since $|V_i| \leq IOIR(G)$, it follows that $n = \sum_{i=1}^{k} |V_i| \leq k.IOIR(G)$. Hence $n/IOIR(G) \leq \chi_{ioir}(G)$. Now, let $S$ be an independent open irredundant set of $G$ with $|S| = IOIR(G)$. Then $\{S\} \cup \{\{v\} : v \in V - S\}$ is an $ioir$-coloring of $G$. Hence $\chi_{ioir}(G) \leq n - IOIR(G) + 1$.  

**Proposition 5.2.11.** Let $G \neq K_{1,n}$ be a connected graph with $\delta = 1$ and let $r$ denote the maximum number of leaves adjacent to a support vertex $v$ of $G$. Then $\chi_{ioir}(G) \geq r + 2$.  

**Proof.** Let $v_1, v_2, \ldots, v_r$ be the leaves adjacent to $v$. Since any independent open irredundant set in $G$ contains at most one of the leaves $v_i$, the result follows.
Corollary 5.2.12. Let $T \neq K_{1,n}$ be any tree and let $r$ denote the maximum number of leaves adjacent to a support vertex $v$ of $T$. Then $\chi_{iioir}(T) \geq r + 2$.

5.3 IOIR-COLORING ON DOUBLE STAR GRAPH FAMILIES

A study of harmonious, achromatic coloring on middle graph, central graph, total graph, line graph of various classes of graphs can be found in [Venkatachalam et al. (2012), Vernold Vivin (2007), Vernold Vivin et al. (2007) and Vernold Vivin et al. (2009)]. Motivated by the above, we investigate the independent open erratic number for the central graph, middle graph, total graph, line graph of the star graph $K_{1,n}$ and the double star graph $K_{1,n,n}$.

Proposition 5.3.1. For the star graph $K_{1,n}$, $n \geq 2$, we have

(i) $\chi_{iioir}(M(K_{1,n})) = n + 2$,
(ii) $\chi_{iioir}(C(K_{1,n})) = n + 1$,
(iii) $\chi_{iioir}(T(K_{1,n})) = n + 2$,
(iv) $\chi_{iioir}(L(K_{1,n})) = n$.

Proof. (i) By the definition of middle graph, each edge $vv_i$ in $K_{1,n}$ is subdivided by the vertex $e_i$ in $M(K_{1,n})$ and the vertices $v, e_1, e_2, ..., e_n$ induce a clique of order $n + 1$ in $M(K_{1,n})$. i.e. $V(M(K_{1,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Hence $n + 1$ distinct colors are required to color the vertices $v, e_1, e_2, ..., e_n$. Note that $e_i$ is the only external private neighbor of $v_i$ with respect to any subset
$S \subseteq V$. Therefore, we assign the color which is different from the already assigned colors to $v_i$. Hence $\chi_{ioir}(M(K_{1,n})) \geq n + 2$. Assign $ioir$-coloring as follows: For $1 \leq i \leq n$, assign the color $c_i$ for $e_i$ and assign the color $c_{n+1}$ to $v$. For $1 \leq i \leq n$, assign the color $c_{n+2}$ to all the vertices $v_1, v_2, \ldots, v_n$. Thus $\chi_{ioir}(M(K_{1,n})) = n + 2$.

(ii) By the definition of central graph, each edge $vv_i$ in $K_{1,n}$ is subdivided by the vertex $e_i$ in $C(K_{1,n})$ and the vertices $v_1, v_2, \ldots, v_n$ induces a clique of order $n$ in $C(K_{1,n})$. i.e. $V(C(K_{1,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Since $v_i \ (1 \leq i \leq n)$ induce a clique of order $n$, we have $\chi_{ioir}(C(K_{1,n})) \geq n$. We now prove that $\chi_{ioir}(C(K_{1,n})) \geq n + 1$. Suppose $\chi_{ioir}(C(K_{1,n})) = n$. Let $V_i$ be the set of vertices which are colored with $c_i$, $i = 1$ to $n$. Let we assign the color $c_i$ to $v_i \ (1 \leq i \leq n)$ and assign the color $c_1$ to $v$. Therefore the vertices $e_1, e_2, \ldots, e_n$ are colored by $c_2, c_3, \ldots, c_{n-1}, c_n$ in some arrangement. Hence at least two of the vertices $e_i$ and $e_j$ are colored with the same color $c_m$. Clearly, any vertex adjacent to vertices $e_i$ and $e_j$ is also joined to vertex of color $c_m$. It follows that there is no external private neighbor for the vertices $e_i$ and $e_j$ with respect to $V_m$. This is a contradiction. Hence $\chi_{ioir}(C(K_{1,n})) \geq n + 1$. Assign $ioir$-coloring as follows: For $1 \leq i \leq n$, assign the color $c_i$ for $v_i$ and assign the color $c_{n+1}$ for each $e_i$. Finally we assign the color $c_1$ to $v$. Thus $\chi_{ioir}(C(K_{1,n})) = n + 1$.

(iii) By the definition of total graph, we have $V(T(K_{1,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$, in which the vertices $v, e_1, e_2, \ldots, e_n$ induce a clique of order $n+1$. Clearly $\chi_{ioir}(T(K_{1,n})) \geq n + 1$. Let we assign the color $c_i$ to $e_i \ (1 \leq i \leq n)$ and assign the color $c_{n+1}$ to $v$. Thus $\chi_{ioir}(T(K_{1,n})) = n + 1$. 

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Since $e_i$ and $v$ are the external private neighbors of $v_i$ with respect to $V_i$ and $V_{n+1}$, we need one more color to $v_i$. Hence $\chi_{ioir}(T(K_{1,n})) \geq n + 2$. Assign $ioir$-coloring as follows: For $1 \leq i \leq n$, assign the color $c_i$ for $e_i$ and assign the color $c_{n+1}$ to $v$. Finally, we assign the color $c_{n+2}$ to each $v_i$. Thus $\chi_{ioir}(T(K_{1,n})) = n + 2$.

(iv) Since $L(K_{1,n}) \cong K_n$, $\chi_{ioir}(L(K_{1,n})) = n$ is immediate.

**Proposition 5.3.2.** For the double star graph $K_{1,n,n}$, $n \geq 2$, we have

$$
\chi_{ioir}(M(K_{1,n,n})) = \begin{cases} 
n + 1 & \text{for all } n \geq 3 \\
4 & \text{if } n = 2.
\end{cases}
$$

**Proof.** Clearly, we observe that $\chi_{ioir}(M(K_{1,2,2})) = 4$. By the definition of middle graph, each edge $vv_i$ and $v_iu_i$ ($1 \leq i \leq n$) in $K_{1,n,n}$ are subdivided by the vertices $e_i$ and $s_i$ in $M(K_{1,n,n})$ and the vertices $v, e_1, e_2, ..., e_n$ induce a clique of order $n + 1$ (say $K_{n+1}$) in $M(K_{1,n,n})$.

i.e. $V(M(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. Clearly, $\chi_{ioir}(M(K_{1,n,n})) \geq n + 1$.

Assign $ioir$-coloring as follows: For $1 \leq i \leq n$, assign the color $c_i$ for $e_i$ and assign the color $c_{n+1}$ to $v$. For $1 \leq i \leq n$, assign two distinct colors $c_l$ and $c_m$ other than $c_{n+1}$ and $c_i$ to the vertices $v_i$ and $s_i$. Furthermore, assign the color $c_{n+1}$ to each $u_i(1 \leq i \leq n)$.

Let $V_i$ be the set of vertices which are colored with $c_i$, $i = 1$ to $n + 1$. Note that $v$ is the external private neighbor of all the vertices $e_i$ with respect to $V_i$, $1 \leq i \leq n$ and $e_i$'s are the external private neighbors of $v$ with respect to $V_{n+1}$. For $1 \leq i \leq n$, $s_i$ is the external private neighbor of $u_i$ and $v_i$ with respect to $V_{n+1}$ and $V_i$. Finally,
$v_i$ is the external private neighbor of $s_i$ with respect to $V_m$. Hence 
$\chi_{ioir}(M(K_{1,n,n})) \leq n + 1$.

**Example 5.3.3.** For example, the graph $M(K_{1,4,4})$ is shown below.

Now, $\chi_{ioir}(M(K_{1,4,4})) = 5$ and \{V_1, V_2, V_3, V_4, V_5\} is an ioir-coloring
of $M(K_{1,4,4})$ where $V_1 = \{v, u_1, u_2, u_3, u_4\}$,

$V_2 = \{e_1, v_2, s_3, s_4\}$,

$V_3 = \{e_2, v_1, v_3, v_4\}$,

$V_4 = \{e_3, s_1, s_2\}$ and

$V_5 = \{e_4\}$.

![Figure 5.1. The graph $M(K_{1,4,4})$](image)

**Proposition 5.3.4.** For any double star graph $K_{1,n,n}$, $n \geq 1$, we
have $\chi_{ioir}(C(K_{1,n,n})) = n + 2$.

**Proof.** By the definition of central graph, each edge $vv_i$ and $v_iu_i$
$(1 \leq i \leq n)$ in $K_{1,n,n}$ are subdivided by the vertices $e_i$ and $s_i$
in $C(K_{1,n,n})$. The vertices $v, u_1, u_2, ..., u_n$ induce a clique of order
$n + 1$ (say $K_{n+1}$) and the vertices $v_i(1 \leq i \leq n)$ induce a clique of
order $n$ in $C(K_{1,n,n})$. i.e. $V(C(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. Clearly

$\chi_{ioir}(C(K_{1,n,n})) > n+1$. We now prove that $\chi_{ioir}(C(K_{1,n,n})) \geq n+2$. Suppose $\chi_{ioir}(C(K_{1,n,n})) = n + 1$. Since $v, u_i$ ($1 \leq i \leq n$) induce a clique of order $n+1$, let us assign the color $c_{n+1}$ to $v$ and assign the color $c_i$ to $u_i$ ($1 \leq i \leq n$). Since $e_i$ has degree 2 and $v$ is adjacent to the vertex of color $c_i$ for all $i$, $v_i$ is the only external private neighbor of $e_i$. But $v_i$ is adjacent to the vertex of color $c_j$, for all $j \neq i$. Therefore $e_i$ must be colored only with $c_i$ and $v_i$ must be colored only with $c_{n+1}$. Since $v_i$ ($1 \leq i \leq n$) induce a clique of order $n$, it leads to a contradiction. Hence $\chi_{ioir}(C(K_{1,n,n})) \geq n+2$.

Consider the colors $c_1, c_2, ..., c_{n+2}$. Assign $ioir$-coloring as follows: Assign the colour $c_{n+1}$ to $v$ and assign the color $c_i$ to $u_i$, where $1 \leq i \leq n$. Assign the color $c_{n+1}$ to all the vertices $s_1, s_2, ..., s_n$ and assign the color $c_{n+2}$ to all the vertices $e_1, e_2, ..., e_n$. Finally, we assign the color $c_i$ to $v_i$ for $1 \leq i \leq n$. Let $V_i$ be the set of vertices which are colored with $c_i$, $i = 1$ to $n+2$. For $1 \leq i \leq n$, $e_i$ is the external private neighbor of $v$ with respect to $V_{n+1}$ and $v_i$ is the external private neighbor of $e_i$ with respect to $V_{n+2}$. For $1 \leq i \leq n$, $e_i$ is the external private neighbor of $v_i$ with respect to $V_i$ and $v_i$ is the external private neighbor of $s_i$ with respect to $V_{n+1}$. Finally, $v$ is the external private neighbor of all the vertices $u_i$ with respect to $V_i$. Hence $\chi_{ioir}(C(K_{1,n,n})) \leq n+2$.

**Example 5.3.5.** The graph $C(K_{1,4,4})$ is shown below.

Now, $\chi_{ioir}(C(K_{1,4,4})) = 6$ and $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ is an $ioir$-coloring of $C(K_{1,4,4})$ where $V_1 = \{v, s_1, s_2, s_3, s_4\}$, $V_2 = \{v_1, u_1\}$, $V_3 = \{v_2, u_2\}$,
$V_4 = \{v_3, u_3\}, \ V_5 = \{v_4, u_4\}$ and $V_6 = \{e_1, e_2, e_3, e_4\}$.

**Figure 5.2.** The graph $C(K_{1,4,4})$

**Proposition 5.3.6.** For the double star graph $K_{1,n,n}$, $n \geq 2$, we have $\chi_{ioir}(T(K_{1,n,n})) = n + 1$.

*Proof.* By the definition of total graph, we have $V(T(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$ in which the vertices $v, e_1, e_2, ..., e_n$ induce a clique of order $n + 1$. Clearly $\chi_{ioir}(T(K_{1,n,n})) \geq n + 1$. Consider the colors $c_1, c_2, ..., c_{n+1}$. Assign ioir-coloring as follows: Assign the color $c_{n+1}$ to $v$ and assign the color $c_i$ to $e_i$, where $1 \leq i \leq n$. For $1 \leq i \leq n$, assign two distinct colors other than $c_{n+1}$ and $c_i$ to the vertices $v_i$ and $s_i$. Finally, assign the color $c_{n+1}$ to each $u_i(1 \leq i \leq n)$. Hence, $\chi_{ioir}(T(K_{1,n,n})) \leq n + 1$.

**Example 5.3.7.** For example, the graph $T(K_{1,4,4})$ is shown below. Now, $\chi_{ioir}(T(K_{1,4,4})) = 5$ and $\{V_1, V_2, V_3, V_4, V_5\}$ is an ioir-coloring of $T(K_{1,4,4})$ where $V_1 = \{v, u_1, u_2, u_3, u_4\}$, $V_2 = \{e_1, v_2, v_3, v_4\}$, $V_3 = \{e_2, v_1, s_3, s_4\}$, $V_4 = \{e_3, s_1, s_2\}$ and $V_5 = \{e_4\}$. 
Proposition 5.3.8. For any double star graph $K_{1,n,n}$, $n \geq 1$, we have $\chi_{ioir}(L(K_{1,n,n})) = n + 1$.

Proof. By the definition of a line graph, each edge of $K_{1,n,n}$ is taken to be a vertex in $L(K_{1,n,n})$. The vertices $e_1, e_2, \ldots, e_n$ induce a clique of order $n$ and the vertices $s_1, s_2, \ldots, s_n$ are all pendant in $L(K_{1,n,n})$. i.e. $V(L(K_{1,n,n})) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

Since $\chi(L(K_{1,n,n})) = n + 1$ and $\chi(G) \leq \chi_{ioir}(G)$, we have $\chi_{ioir}(L(K_{1,n,n})) \geq n + 1$. Assign $ioir$-coloring as follows: Assign the color $c_{n+1}$ to all the vertices $s_i$, where $1 \leq i \leq n$ and assign the color $c_i$ to $e_i$, where $1 \leq i \leq n$. Hence $\chi_{ioir}(L(K_{1,n,n})) \leq n + 1$. 

Figure 5.3. The graph $T(K_{1,4,4})$
5.4 CONCLUSION AND SCOPE

In this chapter, we have introduced the concept of independent open irredudant colorings of graphs and have obtained a few results on this parameter. The following are some interesting problems for further investigation.

**Problem 5.4.1.** Determine additional upper and lower bounds for $\chi_{ioir}$.

**Problem 5.4.2.** Obtain bounds relating any two of the parameters in $\chi_{ir}(G) \leq \chi_{ioir}(G) \leq \chi_{ioir}(G)$ and in $\chi_{ir}(G) \leq \chi(G) \leq \chi_{ioir}(G)$.

**Problem 5.4.3.** Establish relationships between the parameters given in this chapter with other graph theoretic parameters.

**Problem 5.4.4.** Characterize the class of graphs $G$ for which $\chi_{ioir}(G) = \chi(G)$.

**Problem 5.4.5.** Characterize the class of graphs $G$ for which $\chi_{ioir}(G) = \chi_{oir}(G)$. 