CHAPTER - 2

Quasi-Essential Submodules and Minimal \( h \)-pure
Submodules of QTAG-module

Section-1

§ 2.1. Introduction

The concept of quasi-essential submodules has been studied in [23] and different characterizations were obtained in terms of center of \( h \)-purity. A submodule \( N \) of a QTAG-module \( M \) is called quasi-essential if \( M = T + K \) for a complement \( K \) of \( N \) and \( T \) an \( h \)-pure submodule of \( M \) containing \( N \).

In section 2, we extend the study of quasi-essential submodules. First of all we generalize a theorem of L. Fuchs [11], which is of very interesting nature. Here we characterize quasi-essential submodules i.e. We proved that a submodule \( N \) of a QTAG-module \( M \) is quasi-essential if and only if \( K/T \) is an absolute summand of \( M/T \) where \( K \) is an \( h \)-pure submodule of \( M \) containing \( N \) and \( T \) is a complement of \( K \) (Proposition 2.2.10). Further we established various conditions under which \( h \)-pure submodules are direct summands. We also introduced the concept of essentially finitely indecomposable QTAG-module and prove that every \( h \)-pure submodule containing \( M \) is essentially finitely indecomposable. In the end of this section after imposing one more condition on \( M \) many results have been proved to see the relation between center of \( h \)-purity and quasi-essential submodules. It has been seen in [23] that all subsocles of \( M \) are quasi-essential and condition has been obtained under which every quasi-essential subsocle is center of \( h \)-purity. So here we obtain a similar characterization.
Section 3 is devoted to the study of minimal $h$-pure submodules of QTAG-modules. In this section we obtained a necessary and sufficient condition for an $h$-pure submodule to be a minimal $h$-pure submodule containing a given submodule (Theorem 2.3.4). Further we prove that a minimal $h$-pure submodule containing a submodule of a basic submodule of a QTAG-module becomes a direct sum of uniserial submodules (Theorem 2.3.5).

Section-2

§ 2.2. Quasi-Essential Submodules

First of all we restate the following result from [27].

**Lemma A:** If $A$ and $B$ are any two uniserial submodules of a QTAG-module $M$ such that $A \cap B \neq 0$ and $d(A) \leq d(B)$. Then there exists a monomorphism $\sigma : A \rightarrow B$, which is identity on $A \cap B$.

**Proof:** As $d(A) \leq d(B)$, $A + B = B \oplus C$. Now the restriction of the projection $\rho : B \oplus C \rightarrow B$, to $A$ is a desired map.

Now we generalize [Theorem 66.3, 11], which itself is of interesting nature.

**Theorem 2.2.1:** If $M$ is a QTAG-module then every $h$-dense subsocle of $M$ supports an $h$-pure and $h$-dense submodule.

**Proof:** Let $S$ be a subsocle of $M$ and $S$ be $h$-dense; then $\text{Soc}(M) = S + \text{Soc}(H_k(M))$ for all $k \in \mathbb{Z}^+$. Let $N$ be maximal with the property $\text{Soc}(N) = S$. Firstly we show that $N$ is $h$-neat submodule of $M$. Let $x$ be a uniform element in $N \cap H_1(M)$,
then for a uniform element \( y \in M \), we have \( d(yR/xR) = 1 \). If \( y \in N \), then \( x \in H_1(N) \). Let \( y \not\in N \) then \( S \subseteq \text{Soc}(N + yR) \). Hence, there exists a uniform element \( z \in \text{Soc}(N + xR) \) such that \( z \not\in S \) and \( z = u + yr \) where \( u \in N \) and \( r \in R \).

Trivially \( yrR = yR \), hence without any loss of generality we can assume \( z = u + y \).

Define a map \( \eta : yR \rightarrow uR \) such that \( \eta(yr) = ur \). Let \( yr = 0 \), then \( zr = ur \).

If \( zrR = zR \) then \( z \in S \), a contradiction, therefore \( zr = 0 \) and we get \( ur = 0 \), consequently \( \eta \) is a well defined epimorphism. Therefore, \( uR \) is a uniform submodule.

Since \( u + y \in \text{Soc}(M) \), \( H_1(uR) = H_1(yR) \), but \( xR \) is a maximal submodule of \( M \); hence \( H_1(yR) = xR \) and we get \( x \in H_1(N) \). Thus, \( N \cap H_1(M) = H_1(N) \). Now suppose \( N \cap H_n(M) = H_n(N) \) and let \( x \) be a uniform element in \( N \cap H_{n+1}(M) \); then \( d(yR/xR) = 1 \) for some uniform element \( y \in H_n(M) \). Since \( N \) is \( h \)-neat in \( M \), there is a uniform element \( y' \in N \) such that \( d(y'R/xR) = 1 \). Hence by Lemma A, there exists an isomorphism \( \sigma : yR \rightarrow y'R \) which is identity on \( xR \).

The map \( \eta : yR \rightarrow (y - y')R \) where \( \sigma(y) = y' \) is an epimorphism with \( xR \subseteq \text{Ker} \eta \). Hence, \( e(y - y') \leq 1 \) and we get \( y - y' \in \text{Soc}(M) = S + \text{Soc}(H_n(M)) \). Therefore, \( y - y' = s + t \) for some \( s \in S, t \in H_n(M) \). Consequently, \( y - t = y' + s \in N \cap H_n(M) = H_n(N) \). Since \( y - y' - s \in \text{Soc}(M) \), \( H_1(yR) = H_1((y' + s)R) \subseteq H_{n+1}(N) \). Hence, \( x \in H_{n+1}(N) \).

Therefore, \( N \) is \( h \)-pure submodule of \( M \).

Now let \( \bar{x} \in \text{Soc}(M/N) = (\text{Soc}(M) + N)/N \) be a uniform element; then by Lemma 1.2.47 there exists a uniform element \( x' \in M \) such that \( \bar{x} = \bar{x}' \) and \( e(x') = 1 \). Since \( \text{Soc}(M) = S + \text{Soc}(H_k(M)) \) for all \( k \), we get \( \bar{x} \in H_k(M/N) \) for every \( k \). Hence, \( \bar{x} \in \bigcap_{k=1}^{\infty} H_k(M/N) \) and appealing to Theorem 1.2.48, we get \( M/N \) is \( h \)-divisible. Hence, \( N \) is \( h \)-dense in \( M \).

Now we state the following lemmas. Since their proofs are of set theoretic nature, therefore the same is omitted.
Lemma 2.2.2: If $M$ is QTAG-module and $K \subseteq N \subseteq M$ and $T$ is a complement of $K$ then $T \cap N$ is complement of $K$ in $N$. Conversely, if $L$ is complement of $K$ in $N$, then $L = T \cap K$ whenever $T$ is complement of $K$ of $M$ containing $L$.

Lemma 2.2.3: If $M$ is QTAG-module and $K \subseteq N \subseteq M$. If $T$ is a complement of $K$, then every complement of $T \cap N$ in $T$ is a complement of a complement of $N$ in $M$.

Lemma 2.2.4: If $M$ is QTAG-module and $K \subseteq N \subseteq M$ and $T$ is a complement of $K$ in $N$. Then a submodule $L$ containing $T$ is a complement of $K$ in $M$ if and only if $L/T$ is a complement of $N/T$ in $M/T$.

Lemma 2.2.5: If $M$ is QTAG-module and $N, K$ are submodules of $M$ such that $N \cap K = 0$, then a submodule $T$ containing $K$ is a complement of $N$ in $M$ if and only if $T/K$ is a complement of $(N \oplus K)/K$ in $M/K$.

Now we prove few lemmas which are used later and are of independent interest.

Lemma 2.2.6: If $M$ is QTAG-module and $K \subseteq N \subseteq T$ are submodules of $M$ and $N$ is an $h$-pure submodules of $M$. Then $T/K$ is $h$-pure in $M/K$ if and only if $T$ is $h$-pure in $M$.

Proof: If $T$ is $h$-pure in $M$ then trivially $T/K$ is $h$-pure in $M/K$.

Conversely, let $T/K$ be $h$-pure in $M/K$ and let $f$ be the canonical map defined as $f : M/K \rightarrow M/N$ such that $f(x + K) = x + N$ then $Ker f \subseteq T/K$ and $f(T/K) = T/N$, therefore $T/N$ is $h$-pure in $M/N$. Since $N$ is $h$-pure in $M$, so $T$ is $h$-pure in $M$. 

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Lemma 2.2.7: If \( M \) is QTAG-module, \( N \) is a submodule of \( M \) and \( B \) is an \( h \)-pure, \( h \)-dense submodule of \( N \). Then there exists an \( h \)-pure, \( h \)-dense submodule \( K \) of \( M \) such that \( K \cap N = B \).

Proof: Since \( B \) is \( h \)-dense in \( N \), we have \( M/B = N/B \oplus K/B \) for some submodule \( K \) of \( M \), then by Proposition 1.3.13, \( K \) is \( h \)-pure in \( M \) and trivially \( K \cap N = B \).

Proposition 2.2.8: Let \( M \) be a QTAG-module and \( S \) be a subsocle of \( \text{Soc}(M) \) such that \( S \not\subset M^1 \). Let \( K \) be a maximal \( h \)-pure submodule of \( M \) such that \( \text{Soc}(K) \subseteq S \). Then \( (S + K)/K \) is contained in the \( h \)-reduced part of \( (M/K)^1 \).

Proof: Trivially \( S \) has at least one element of finite height, therefore, there exists at least one \( h \)-pure submodule \( T \) of \( M \) such that \( \text{Soc}(T) \subseteq S \). The existence of a maximal element is ensured by Zorn’s Lemma, therefore we get a maximal \( h \)-pure submodule \( K \) of \( M \) such that \( \text{Soc}(K) \subseteq S \). Trivially \( (S + K)/K \subseteq \text{Soc}(M/K) \). If \( (S + K)/K \) has an element of finite height then \( M/K = K'/K \oplus L/K \) such that \( \text{Soc}(K'/K) \subseteq (S + K)/K \), hence \( \text{Soc}(K') \subseteq S \) and since \( K' \) is \( h \)-pure in \( M \), we get a contradiction to the maximality of \( K \).

Therefore, \( (S + K)/K \subseteq (M/K)^1 \). Since \( h \)-divisible submodules are absolute summands, hence we ultimately get \( (S + K)/K \) contained in the \( h \)-reduced part of \( (M/K)^1 \).

Proposition 2.2.9: If \( M \) is a QTAG-module such that \( M = B \oplus D \) where \( B \) is bounded and \( D \) is \( h \)-divisible, then every \( h \)-pure submodule \( K \) of \( M \) is the direct sum of bounded and \( h \)-divisible submodule.
Proof: Let $M = B \oplus D$ where $B$ is bounded and $D$ is $h$-divisible. Let $K$ be an $h$-pure submodule of $M$, then $K \cap D = K^1$. Let $T$ be a complement of $K^1$ in $K$, then $T \cap D = 0$ and therefore $T$ is bounded. Hence, $K = T \oplus (K \cap D)$ where $(K \cap D) \cong K/T$ is $h$-divisible.

Proposition 2.2.10: If $M$ be a QTAG-module and $N \subseteq M$, then $N$ is quasi-essential submodule of $M$ if and only if $K/T$ is an absolute summand of $M/T$ whenever $K$ is an $h$-pure submodule of $M$ containing $N$ and $T$ is a complement of $K$.

Proof: Let $A/T$ be a complement of $K/T$ in $M/T$, then by Lemma 2.2.5, $A$ is a complement of $N$ and if $N$ is quasi-essential, then we get $M = A + K$. Therefore, $M/T = A/T \oplus K/T$.

Conversely, let $A$ be a complement of $N$ in $M$, then by Lemma 2.2.3, $A \cap K$ is a complement of $N$ in $K$. Hence, $K/(A \cap K)$ is an absolute summand of $M/(A \cap K)$ and by Lemma 2.2.5, $A/(A \cap K)$ is a complement of $K/(A \cap K)$ in $M/(A \cap K)$. Therefore, $M/(A \cap K) = A/(A \cap K) \oplus K/(A \cap K)$ and we get $M = A + K$. Therefore, $N$ is quasi-essential submodule of $M$.

Theorem 2.2.11: If $M$ is a QTAG-module and $S$ is a subsocle of $M^1$. Then every $h$-pure submodule of $M$ containing $S$ is summand of $M$ if and only if $M$ is a direct sum of a bounded submodule and $h$-divisible submodule.

Proof: Let $K$ be a complement of $M^1$, then $K$ is $h$-pure and $M/K$ is $h$-divisible Proposition 1.3.26 and Proposition 1.3.27. If $K$ is unbounded then $K$ contains a proper basic submodule $B$ of $K$ and hence $M/B = K/B \oplus T/B$ where $T$ can be
chosen to contain $M^1$ as $K \cap M^1 = 0$. Appealing to Proposition 1.3.13, $T$ is an $h$-pure submodule of $M$ and $S \subseteq T$. Therefore, $M = T \oplus A$ and $A$ is $h$-divisible, which is a contradiction. Hence, $K$ is bounded and therefore $K$ is a summand of $M$ i.e. $M = K \oplus D$ where $D$ is $h$-divisible.

For the converse we refer to Proposition 2.2.9.

**Theorem 2.2.12:** If $M$ is a QTAG-module and $S$ is a subsocle of $M$. Then the following are equivalent:

(i) $S \supseteq Soc(M^1)$ and every $h$-pure submodule of $M$ containing $S$ is a summand of $M$.

(ii) Every $h$-pure submodule of $M$ containing $S$ is a cobounded summand of $M$.

(iii) $S \supseteq Soc(H_n(M))$, for some positive integer $n$.

**Proof:** We establish (ii) $\rightarrow$ (i) $\rightarrow$ (iii) $\rightarrow$ (ii)

(ii) $\rightarrow$ (i) Let $x$ be a uniform element in $Soc(M^1)$ and $x \notin S$, then $xR \cap S = 0$. Embedding $S$ into a complement $K$ of $xR$. Then $K$ is an $h$-pure submodule of $M$ and $M/K$ is $h$-divisible, which is a contradiction. Therefore, $x \in S$ and we get $Soc(M^1) \subseteq S$.

(i) $\rightarrow$ (iii) Let $S = M^1$, then by Theorem 2.2.11, $M = B \oplus D$ where $B$ is bounded and $D$ is $h$-divisible. Let $H_n(B) = 0$, then clearly $Soc(H_n(M)) \subseteq S$. Let $S \neq M^1$ and $K$ be a maximal $h$-pure submodule of $M$ such that $Soc(K) \subseteq S$, then by Proposition 2.2.8, $(K + S)/K \subseteq (M/K)^1$. Now every $h$-pure submodule $A/K$ of $M/K$ containing $(K + S)/K$ is a summand of $M/K$ as $A$ is $h$-pure submodule of $M$ containing $S$. Hence, $M/K$ is a direct sum of a bounded submodule and a $h$-divisible submodule. Thus, $M/K$ is $h$-pure complete, which is a contradiction. Therefore,
$Soc(K) = S$ and $M/K$ is bounded. Hence, for some $n$, $H_n(M/K) = 0$ and we get $Soc(H_n(M)) \subseteq S$.

(iii) $\rightarrow$ (ii) Let $K$ be an $h$-pure submodule of $M$ such that $S \subseteq K$, then $H_n(M) \subseteq K$ and hence $K$ is a cobounded summand of $M$.

Corollary 2.2.13: If $M$ is a $h$-reduced QTAG-module and $S$ is a subsocle of $M$, then every $h$-pure submodule $K$ of $M$ containing $S$ is summand of $M$ if and only if $S \supseteq Soc(H_n(M))$ for some $n$.

Proof: Due to above Theorem it is sufficient to show that $Soc(M^1) \subseteq S$. Let $x$ be a uniform element in $Soc(M^1)$ and let $x \notin S$. Let $K$ be a complement of $xR$ and $S \subseteq K$ then by Proposition 1.3.26 and Proposition 1.3.27, $K$ is $h$-pure submodule of $M$ and $M = K \oplus D$ where $M/K \cong D$ is $h$-divisible, which is a contradiction as $M$ is $h$-reduced. Therefore, $x \in S$ and we get $Soc(M^1) \subseteq S$.

Proposition 2.2.14: If $M$ is QTAG-module and $N$ is a submodule of $M$ such that no proper $h$-pure submodule contains $N$. Then every $h$-pure submodule containing $Soc(N)$ is a cobounded summand of $M$.

Proof: Let $T$ be a submodule of $M$ such that $T \cap N = 0$, then $T$ is bounded, since otherwise $T$ will contain a proper basic submodule $B$ and we will have $M/B = T/B \oplus K/B$. Appealing to Proposition 1.3.13, we get $K$ to be $h$-pure submodule containing $N$, which is a contradiction. Now let $A$ be an $h$-pure submodule of $M$ such that $Soc(N) \subseteq A$, then $M/A$ has a bounded basic submodule. Otherwise, if $B/A$ is unbounded basic submodule of $M/A$, then $B = A \oplus L$ where $L \cong B/A$ and $A \cap N = 0$, which is a contradiction as $L$ is unbounded. Therefore, $M/A = B/A \oplus D/A$ where
$B/A$ is bounded and $D/A$ is $h$-divisible.

Now we show that $D/A = 0$. Let $D/A \neq 0$, then $M/B$ is $h$-divisible and $B$ is $h$-pure submodule of $M$. This implies that $Soc(B)$ is proper dense in $Soc(M)$ and $Soc(N) \subseteq Soc(B)$, which is a contradiction. Hence, $M/A$ is bounded. As $A$ is $h$-pure in $M$, $A$ is a summand of $M$.

Corollary 2.2.15: If $M$ is QTAG-module and $N$ is a submodule of $M$ and $T$ is a minimal $h$-pure submodule of $M$ containing $N$. Then $T = B \oplus K$ where $B$ is bounded and $Soc(K) = Soc(N)$.

Proof: Appealing to Proposition 2.2.14 and Theorem 2.2.12, we see that $Soc(N)$ supports an $h$-pure submodule $K$ of $T$ and $T/K$ is bounded. Therefore, $T = B \oplus K$.

Let $M$ be a QTAG-module satisfying the following:

\[(\ast) \quad M/K = B/K \oplus D/K \text{ where } B/K \text{ is bounded and } D/K \text{ is } h\text{-divisible, whenever } K \text{ is } h\text{-pure submodule of } M \text{ containing } M^1.\]

Definition 2.2.16: A QTAG-module $M$ is called essentially finitely indecomposable (e.f.i) if it has no unbounded direct sum of uniserial submodules summand.

Theorem 2.2.17: If $M$ is a QTAG-module and if $M$ satisfies $(\ast)$, then every $h$-pure submodule of $M$ containing $M^1$ is e.f.i.

Proof: Let $A$ be an $h$-pure submodule of $M$ containing $M^1$, then $A$ satisfies $(\ast)$, because if $K$ is $h$-pure submodule of $A$ containing $A^1 = M^1$, then $A/K$ is $h$-pure submodule of $M/K$ and the assertion follows from Proposition 2.2.9. Therefore, $A$ satisfies $(\ast)$. Now let $A$ be not e.f.i., then $A = S \oplus T$ where $S$ is unbounded direct
sum of uniserial submodules. Therefore, \( T \) is \( h \)-pure submodule of \( A \) containing \( A^1 \) and \( A/T \) is unbounded, a contradiction. Hence, \( A \) is e.f.i.. 

Let us consider one more condition on \( M \) introduced by S. Singh (unpublished) as mentioned below:

(A) For any finitely generated submodule \( N \) of \( M \), \( R/\text{ann}(N) \) is right artinian.

Now we prove the following result which is of independent interest.

**Theorem 2.2.18:** If \( M \) is a QTAG-module satisfying condition (A) and \( N \) is a quasi-essential submodule of \( M \) such that \( \text{Soc}(N) \not\subseteq M^1 \). Then every \( h \)-pure submodule \( K \) of \( M \) containing \( N \) is a cobounded summand of \( M \).

**Proof:** Let \( K \) be \( h \)-pure submodule of \( M \) with \( N \subseteq K \), then by Proposition 2.2.10, \( K/T \) is an absolute summand of \( M/T \) where \( T \) is any complement of \( N \) in \( K \). Since \( \text{Soc}(N) \not\subseteq M^1 \), then Proposition 1.4.9 implies that \( K/T \) is not \( h \)-divisible for some complement \( T \) of \( N \) in \( K \), as \( K^1 \subseteq M^1 \). Now appealing to Theorem 1.5.4, there exists a positive integer \( n \) such that

\[
\text{Soc}(H_n(M/T)) \subseteq \text{Soc}(K/T) \subseteq \text{Soc}(H_n(M/T))
\]

Therefore, \( \text{Soc}(H_{n+1}(M)) \subseteq K \) and as \( K \) is \( h \)-pure, then appealing to Proposition 1.3.9, we get \( H_{n+1}(M) \subseteq K \). Hence, \( K \) is cobounded summand of \( M \).

Now we state the following lemma, since the proof is of set theoretic nature, therefore it is omitted.

**Lemma 2.2.19:** If \( M \) is a QTAG-module such that \( M = N \oplus K \) such that \( N_0 \subseteq N \) and \( K_0 \subseteq K \) are submodules, if \( N' \) is a complement of \( N_0 \) in \( N \) and \( K' \) is a comple-
ment of \( K_0 \) in \( K \), then \( N' \oplus K' \) is a complement of \( K_0 \oplus N_0 \) in \( M \).

**Proposition 2.2.20:** If \( S \) is a quasi-essential subsocle of a QTAG-module \( M \) and \( N \) is an \( h \)-pure submodule of \( M \) with \( \text{Soc}(N) = \text{Soc}(H_n(M)) \). Then \( S \cap H_n(M) \) is a quasi-essential subsocle of \( N \).

**Proof:** Let \( N_0 = S \cap H_n(M) \) and \( S = N_0 \oplus K_0 \), then trivially \( K_0 \cap H_n(M) = 0 \).

Let \( K \) be a complement of \( N \) in \( M \) containing \( K_0 \); then since \( N \) is \( h \)-pure and \( M/N \) is bounded, we get \( M = K \oplus N \). Now let \( N' \) be a complement of \( N_0 \) in \( N \) and \( T \) be an \( h \)-pure submodule of \( N \) containing \( N_0 \). If \( K' \) is complement of \( K_0 \) in \( K \), then \( N' \oplus K' \) is complement of \( S \) in \( M \) by Lemma 2.2.19. Now

\[
(T \oplus K) \cap H_n(M) = (T \oplus K) \cap (H_n(K) \oplus H_n(N))
\]

\[
= H_n(K) + (T \oplus K) \cap H_n(N)
\]

Now let \( x \in (T \oplus K) \cap H_n(N) \) then \( x = a + b, a \in T, b \in K \) and \( x \in H_n(N) \), then \( x - a = b \in K \cap N = 0 \), so \( x \in T \cap H_n(N) = H_n(T) \). Hence, we get

\[
(T \oplus K) \cap H_n(M) = H_n(K) \oplus H_n(T)
\]

\[
= H_n(K \oplus T)
\]

So \( T \oplus K \) is an \( h \)-pure submodule of \( M \). Trivially \( S \subseteq T \oplus K \). Since \( S \) is quasi-essential submodule of \( M \), we get \( M = T \oplus K + N' \oplus K' = (T + N') \oplus K \). Hence, \( N = T + N' \). Therefore, \( S \cap H_n(M) \) is quasi-essential in \( N \).

**Proposition 2.2.21:** If \( S \) be a quasi-essential subsocle of a QTAG-module \( M \) satisfying condition (A) and if \( \text{Soc}(H_n(M)) \neq (S \cap H_n(M)) + \text{Soc}(H_{n+1}(M)) \) for some \( n \in \mathbb{Z}^+ \), then \( S \subset \text{Soc}(H_n(M)) \).

**Proof:** Let \( A_0 = S \cap H_{n+1}(M) \) and \( S = A_0 \oplus B_0 \). Let \( \text{Soc}(H_{n+1}(M)) \) support an \( h \)-pure submodule \( A \) of \( M \). Let \( B \) be a complement of \( A \) in \( M \) such that \( B_0 \subset B \). Then
as done in Proposition 2.2.20, \( M = A \oplus B \). Let \( K \) be an \( h \)-pure submodule of \( B \) such that \( \text{Soc}(K) = B_0 \) and \( B' \) be a complement of \( K \) in \( B \). Then \( B' \) is also a complement of \( B_0 \). Let \( A' \) be a complement of \( A_0 \) in \( A \), then \( A' \oplus B' \) is complement of \( S \) in \( M \). Since \( S \) is quasi-essential in \( M \) and as done in Proposition 2.2.20, \( A \oplus K \) is an \( h \)-pure submodule of \( M \) containing \( S \). Therefore, \( M = A \oplus K + A' \oplus B' = A \oplus (K \oplus B') \).

Thus, we get \( B = K \oplus B' \), so \( K \) is an absolute direct summand of \( B \). Now appealing to Theorem 1.5.4, we get \( \text{Soc}(H_{k+1}(B)) \subseteq B_0 \subseteq \text{Soc}(H_k(B)) \) for some \( k \in \mathbb{Z}^+ \).

Since \( \text{Soc}(H_n(M)) = \text{Soc}(A) \oplus \text{Soc}(H_n(B)) \) and \( \text{Soc}(H_n(M)) \neq (S \cap H_n(M)) + \text{Soc}(H_{n+1}(M)) \), we get \( \text{Soc}(H_n(B)) \subseteq B_0 \). Thus \( n \leq k \), so \( B_0 \subseteq \text{Soc}(H_n(B)) \).

Hence, \( S = A_0 + B_0 \subseteq \text{Soc}(H_{n+1}(M)) \oplus \text{Soc}(H_n(B)) = \text{Soc}(H_n(M)) \).

**Proposition 2.2.22:** If \( S \) is quasi-essential subsocle of a QTAG-module \( M \) satisfying condition (A) and is \( h \)-dense in \( M \). Then either \( S \subseteq M^1 \) or \( S = \text{Soc}(M) \).

**Proof:** Appealing to Theorem 2.2.1, we see that \( S \) supports an \( h \)-pure submodule and is quasi-essential. Now if \( S \not\subseteq M^1 \), then by Theorem 1.5.4, \( \text{Soc}(H_{k+1}(M)) \subseteq S \subseteq \text{Soc}(H_k(M)) \) for some \( k \in \mathbb{Z}^+ \). Since \( \text{Soc}(M) = S + \text{Soc}(H_{k+1}(M)) \) and as \( \text{Soc}(H_{k+1}(M)) \subseteq S \), we get \( S = \text{Soc}(M) \).

**Proposition 2.2.23:** If \( S \) be a quasi essential subsocle of a QTAG-module \( M \) satisfying condition (A) and if \( \text{Soc}(H_k(M)) = (S \cap H_k(M)) + \text{Soc}(H_{k-1}(M)) \) for every \( k > n \), then either \( H_{n+1}(M) \) is \( h \)-divisible or \( \text{Soc}(H_{n+1}(M)) \subseteq S \).

**Proof:** Let \( K \) be an \( h \)-pure submodule supported by \( \text{Soc}(H_{n+1}(M)) \), then \( \text{Soc}(H_k(M)) = \text{Soc}(H_k(K)) \) and \( S \cap H_k(M) = S \cap H_k(K) \) for \( k > n \), consequently \( \text{Soc}(H_k(K)) = (S \cap H_k(K)) + \text{Soc}(H_{k+1}(K)) \) for every \( k > n \). Since \( K \) is \( h \)-pure and \( \text{Soc}(H_{n+1}(M)) = \text{Soc}(K) \), we get \( \text{Soc}(K) = \text{Soc}(H_{n+1}(K)) \). Using induction
it is easy to see that $Soc(H_{n+1}(K)) = (S \cap H_{n+1}(K)) + Soc(H_{n+m}(K))$ for all $m \geq 1$. Thus $S \cap H_{n+1}(K)$ is $h$-dense in $Soc(K)$ and is quasi-essential in $Soc(K)$ (see Proposition 2.2.20). Now by Proposition 2.2.22, either $S \cap H_{n+1}(K) \subseteq K^1$ or $S \cap H_{n+1}(K) = Soc(K)$. If $S \cap H_{n+1}(K) \subseteq K^1$, then as $S \cap H_{n+1}(K)$ is $h$-dense in $K$, therefore $K$ is $h$-divisible; consequently $H_{n+1}(M)$ is $h$-divisible. If $S \cap Soc(H_{n+1}(K)) = Soc(K)$ then $S \cap Soc(H_{n+1}(M)) = Soc(H_{n+1}(M))$ and we get $Soc(H_{n+1}(M)) \subset S$.

Now we state and prove the main result of this section.

**Theorem 2.2.24:** If $M$ is a QTAG-module satisfying condition (A) and $S$ is a subsocle of $M$, then $S$ is quasi-essential if and only if one of the following conditions holds:

(i) $S \subseteq M^1$.

(ii) $Soc(H_{n+1}(M)) \subseteq S \subseteq Soc(H_n(M))$ for some $n \geq 0$.

**Proof:** The sufficiency follows from Theorem 1.5.2 and Theorem 1.5.3.

Conversely, suppose $S$ is quasi-essential. Now if $Soc(H_n(M)) \neq (S \cap H_n(M)) + Soc(H_{n+1}(M))$ for arbitrarily large $n$, then by Proposition 2.2.21, $S \subseteq M^1$. If not so, then there exists $n \in Z^+$ such that $Soc(H_n(M)) \neq (S \cap H_n(M)) + Soc(H_{n+1}(M))$ and equality holds for every $k > n$. Thus $S \subseteq Soc(H_n(M))$ by Proposition 2.2.21 and either $Soc(H_{n+1}(M)) \subseteq S$ or $H_{n+1}(M)$ is $h$-divisible by Proposition 2.2.23. If $Soc(H_{n+1}(M)) \subseteq S$, then the condition (ii) is satisfied.

If $H_{n+1}(M)$ is $h$-divisible then every subsocle of $M$ will support an $h$-pure submodule. Thus $S$ supports an absolute direct summand. Therefore, appealing to Theorem 1.5.4, we see that either (i) or (ii) is satisfied.
Appealing to above theorem, the following immediately follows:

**Corollary 2.2.25:** If $M$ is a QTAG-module satisfying condition (A) then a subsocle $S$ of $M$ supports an absolute direct summand if and only if $S$ is quasi-essential and $S \subseteq M^1$ implies $S \subseteq D$, where $D$ is the maximal $h$-divisible submodule of $M$.

### Section-3

§ 2.3 Minimal $h$-pure Submodules

Firstly we recall the following definition from chapter 1.

**Definition 2.3.1:** A submodule $N$ of a QTAG-module $M$ is called almost dense in $M$ if for every $h$-pure submodule $K$ of $M$ containing $N$, $M/K$ is $h$-divisible.

**Definition 2.3.2:** Let $K$ be a submodule of a QTAG-module $M$, then an $h$-pure submodule $N$ of $M$ containing $K$ is called minimal $h$-pure submodule of $M$.

**Theorem 2.3.3:** Let $N$ be a submodule of a QTAG-module $M$. Then there is no proper $h$-pure submodule of $M$ containing $N$ if and only if $N$ is almost dense in $M$ and $Soc(H_n(M)) \subseteq N$ for some $n$.

**Proof:** Let $N$ be almost dense in $M$ and $Soc(H_n(M)) \subseteq N$. Let $K$ be an $h$-pure submodule of $M$ such that $N \subseteq K$, then $Soc(H_n(M)) \subseteq K$ and hence by Proposition 1.3.9, $H_n(M) \subseteq K$, consequently $M/K$ is bounded but it is also $h$-divisible which is not possible and we get $M/K = 0$ i.e. $M = K$. 

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Conversely, if no proper $h$-pure submodule of $M$ contains $N$, clearly $N$ is almost $h$-dense in $M$ and by Theorem 2.2.12 and Proposition 2.2.14, we get $Soc(H_n(M)) \subseteq N$ for some positive integer $n$.

Now we prove the following useful criterion:

**Theorem 2.3.4:** Let $N$ be a submodule of a QTAG-module $M$. Then $N$ is contained in a minimal $h$-pure submodule of $M$ if and only if there exists a $h$-pure submodule $K$ of $M$ such that $Soc(H_n(M)) \subseteq N \subseteq K$ for some $n \in \mathbb{Z}^+$. 

**Proof:** If $N$ is contained in a minimal $h$-pure submodule of $M$ then the result follows from Theorem 1.3.29.

Conversely, suppose that there exists an $h$-pure submodule $K$ of $M$ such that $Soc(H_n(M)) \subseteq N \subseteq K$ for some $n \in \mathbb{Z}^+$. If $n = 0$, then trivially $K$ itself is an $h$-pure submodule containing $N$. If $n \geq 1$, then for every $h$-pure submodule $T$ of $K$ containing $N$, we define

$$E(T) = \{l \geq 1/Soc(T,_{l-1}) \not\subseteq N + H_l(T)\}$$

and set $m(T) = 0$ if $E(T) = \emptyset$ and $m(T) = \max\{m \in E(T)\}$ if $E(T) \neq \emptyset$. Trivially, $m(T) \leq n$ and therefore, there exists an $h$-pure submodule $A$ of $M$ containing $N$ for which $m(A)$ is minimal. Now by Theorem 1.3.28, we see that $m(A) = 0$ i.e. $A \supseteq N \supseteq Soc(H_n(A))$ and $Soc(H_{l-1}(A)) \subseteq N + H_l(A)$ for all $l \geq 1$. Hence, by Theorem 1.3.29, $A$ is a minimal $h$-pure submodule of $M$ containing $N$.

**Theorem 2.3.5:** If $N$ is a submodule of a QTAG-module such that $M/N$ is a direct sum of uniserial submodules. If $K$ is minimal $h$-pure submodule of $M$ containing $N$ then $M/K$ is also a direct sum of uniserial submodules.

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Proof: By Theorem 2.3.4, there exists \( n \in \mathbb{Z}^+ \) such that \( \text{Soc}(H_n(K)) \subseteq N \). Since \( K \) is \( h \)-pure in \( M \), therefore by Proposition 1.3.10, \( \text{Soc}(H_n(M/K)) = (\text{Soc}(H_n(M)) + K)/K \). It is trivial to see that the natural homomorphism \( f: M/N \rightarrow M/K \) defined by \( f(x + N) = x + K \) is onto and maps \( (\text{Soc}(H_n(M)) + N)/N \) onto \( (\text{Soc}(H_n(M)) + K)/K \). Since we know that homomorphism never decreases heights.

We show that \( f \) is height preserving. Let \( x \) be a uniform element in \( \text{Soc}(H_n(M)) \) and \( x + K \in (\text{Soc}(H_n(M)) + K)/K \), then we can find a uniform element \( y \in \text{Soc}(H_n(M)) \) such that \( x + K = y + K \), then trivially \( x - y \in \text{Soc}(K) \) and as \( K \) is \( h \)-pure, \( x - y \in \text{Soc}(H_n(K)) \subseteq N \). Hence, \( x + N = y + N \in (\text{Soc}(H_n(M)) + N)/N \) and we get \( H_{M/K}(x + K) \leq H_{M/N}(x + N) \). Since \( (\text{Soc}(H_n(M)) + N)/N \) is the union of the ascending chain of submodules of bounded height in \( M/N, (\text{Soc}(H_n(M)) + K)/K \) is also the union of an ascending chain of submodules of bounded height in \( M/K \).

Thus, \( H_n(M/K) \) is a direct sum of uniserial submodules and \( M/K \) is direct sum of uniserial submodules.

Finally we prove the following:

Theorem 2.3.6: If \( N \) is a submodule of a basic submodule \( B \) of a QTAG-module \( M \). If \( N \) is contained in a minimal \( h \)-pure submodule \( K \) of \( M \), then \( K \) is a direct sum of uniserial submodules.

Proof: Since \( N \subseteq B \) and \( K \) is an \( h \)-pure submodule of \( M \), then using Theorem 1.4.18, \( N \) can be extended to a basic submodule \( A \) of \( K \). Since \( K \) is minimal \( h \)-pure containing \( N \), \( A = K \) and therefore \( K \) is direct sum of uniserial submodules.