CHAPTER - 1

PRELIMINARIES

Section-1

§ 1.1. Introduction

The principal purpose of this introductory chapter is to recall some necessary definitions, notations and other background informations needed for the subsequent chapters. This is being done only to fix up the terminology and notations for subsequent use, and no originality is claimed. The concept of pure subgroups, neat subgroups, divisible subgroups, basic subgroups and high subgroups are quite important objects in abelian groups. Most of these concepts have been generalized by R.B. Warfield [48], H. Marubayashi [37,38] and S. Singh [44,45,46] etc. for modules. Later on Khan [15,16,17,18,19,20,21,22,23] generalized various results for a special type of module, which is called $S_2$-module. S. Singh [45] called them TAG-modules and proved that the results which are true for TAG-modules are also true for QTAG-modules. In section 2, some definitions and elementary properties of $S_2$-module and QTAG-module have also been given. In section 3, we have given some very useful definitions and results on $h$-pure, $h$-neat and high submodules as done in [15,17,18,22,45]. In section 4, we have recalled some of the results of $h$-divisible and basic submodules from [20,19]. Section 5 deals with the some characterizations of $h$-pure submodules as done in [23] and [32]. This chapter is thus only intended to make the thesis as much self contained as possible.

Throughout the thesis we shall consider all the rings $R$ as associative with unity and the modules as torsion and unital right $R$-modules.
§ 1.2. Some Elementary Concepts

Definition 1.2.1: A module $M_R$ is called simple if $M$ has no proper submodules.

Definition 1.2.2: Let $M_R$ be a module, then the sum of all simple submodules of $M$ is called socle of $M$ and is denoted by $Soc(M)$.

It is easy to see that for any submodule $K$ of $M$, $Soc(K) = K \cap Soc(M)$ and $Soc(Soc(M)) = Soc(M)$.

Proposition 1.2.3 [5, Page 121]: If $\{M_\alpha\}_{\alpha \in \Delta}$ is an indexed set of submodule of $M$ with $M = \bigoplus_{\alpha \in \Delta} M_\alpha$ then $Soc(M) = \bigoplus_{\alpha \in \Delta} Soc(M_\alpha)$.

Definition 1.2.4: Let $M$ be a module, then a submodule of $Soc(M)$ is called sub-socle of $M$.

Definition 1.2.5: Let $N$ be a submodule of $M$, then $N$ is called essential submodule of $M$ if $N \cap T \neq 0$ for every non-zero submodule $T$ of $M$. It is denoted by $N \subseteq' M$.

Definition 1.2.6: A module $M$ extending $N$ is called an essential extension provided every non-zero submodule of $M$ has non-zero intersection with $N$. In other words if $N \subseteq M$, $M$ is an essential extension of $N$ if and only if $N$ is essential submodule of $M$.

Proposition 1.2.7: If $N$ is essential submodule of $M$, then $Soc(N) = Soc(M)$.

Definition 1.2.8: If $N$ and $K$ are submodules of a module $M$, then $N$ is called a
complement of $K$ if $N$ is maximal with respect to the property $N \cap K = 0$.

**Definition 1.2.9:** A submodule $T$ of $M$ is called complement submodule if $T$ is a complement of some submodule $U$ of $M$.

**Definition 1.2.10:** A submodule $N$ of $M$ is called closed in $M$ if $N$ has no proper essential extension in $M$.

**Definition 1.2.11:** A submodule $N$ of $M$ is called direct summand of $M$ if there exists a submodule $K$ of $M$ such that $M = N \oplus K$, $K$ is called the complementary summand of $M$.

**Definition 1.2.12:** A submodule $N$ of $M$ is called absolute direct summand of $M$ if for every complement $K$ of $N$ in $M$, $M = N \oplus K$.

**Definition 1.2.13:** A module $M$ is called uniform if intersection of any two of its non-zero submodule is non-zero.

**Definition 1.2.14:** Let $M$ be a non-zero module. Then a finite chain of submodules of $M$, $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$ is called a composition series of length $n$ for $M$ provided $M_{i-1}/M_i$ is simple for every $i$. If the length of a module $M$ is $n$, then we write $d(M) = n$.

**Definition 1.2.15:** A module $M$ is called uniserial if it has a unique composition series of finite length.

From the definition it follows that uniserial modules are totally ordered.
Definition 1.2.16: A module $M$ is said to decomposable if it is a direct sum of uniserial modules.

Definition 1.2.17: A torsion module $M$ is called indecomposable if it is not a direct sum of any two of its non-zero submodules.

Definition 1.2.18: Let $N$ be a submodule of $M$ then $\{ r \in R/\forall x \in N \}$ is called annihilator of $N$ and is denoted by $\text{ann}(N)$.

Definition 1.2.19: A module $M$ is called divisible if $Mr = M$ for all regular elements $r \in R$.

Definition 1.2.20: A module $M$ is called projective if given any diagram,

\[
\begin{array}{ccc}
M & \xrightarrow{g} & A \\
\downarrow{h} & & \downarrow{f} \\
0 & = & B
\end{array}
\]

of $R$-modules with exact row, it is always possible to find an $R$-homomorphism $h : M \rightarrow A$ such that $f \circ h = g$.

Definition 1.2.21: A module $M$ is called injective if given any diagram,

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow{g} & & \downarrow{f} \\
M & = & B
\end{array}
\]

of $R$-modules with exact row, it is always possible to find an $R$-homomorphism $h : B \rightarrow M$ such that $h \circ f = g$. 
Definition 1.2.22: The minimal injective right $R$-module $E$ containing $M$ is called injective hull of $M$ and is denoted by $E(M)$.

Remark 1.2.23: If $E$ is the injective hull of $M$ then $Soc(M) = Soc(E)$.

Definition 1.2.24: A module $M$ satisfies ascending chain condition (a.c.c) [descending chain condition (d.c.c)] if every properly ascending (descending) chains of submodules of $M$ terminates after a finite number of steps.

Definition 1.2.25: A module $M$ is called Noetherian (Artinian) if every ascending (descending) chain of submodules becomes stationary after a finite number of steps.

Definition 1.2.26: A subset $\{x_1, x_2, \cdots, x_m\}$ of a module $M$ is called linearly independent set if $\sum x_i r_i = 0, r_i \in R$ implies $x_i r_i = 0, i = 1, \cdots, m$. An infinite subset $A$ of $M$ is linearly independent if and only if every finite subset of $A$ is linearly independent.

Now we shall define some different types of rings.

Definition 1.2.27: A ring $R$ in which every strictly descending chain of right (left) ideals is finite is called right (left) artinian ring.

Definition 1.2.28: A ring $R$ is called right (left) hereditary if every right (left) ideal is projective.

Definition 1.2.29: A ring $R$ is called hereditary if it is both right as well as left hereditary.
**Definition 1.2.30:** A ring $R$ is called prime ring if $(0)$ is a prime ideal.

**Definition 1.2.31:** A prime ring $R$ which is right hereditary, left hereditary, right noetherian and left noetherian is called $(hnp)$-ring.

**Definition 1.2.32:** A ring $R$ is called right (left) bounded if each of its essential right (left) ideal contains a non-zero two sided ideal.

**Definition 1.2.33:** In a module $M$, an element $x$ is said to be a torsion element if $xr = 0$ for some regular element $r \in R$. The set of all torsion elements $T(M)$ forms a submodule and is called torsion submodule of $M$. A module $M$ is said to be torsion module if $T(M) = M$. Equivalently if every non-zero element $M$ is torsion.

**Proposition 1.2.34** [Lemma 1 & 2, 42]: Let $R$ be a bounded $(hnp)$-ring then the following hold:

(a) Every finitely generated torsion $R$-module is a direct sum of finitely many uniserial modules.

(b) Any uniform torsion $R$-module is either of finite length and uniserial or is injective and of finite length.

(c) Let $U$ and $V$ be two uniform torsion right $R$-modules and $b(\neq 0) \in U$. If $f : bR \to V$ is a non-zero $R$-homomorphism and length, $d(U/bR) \leq d(V/f(bR))$, then $f$ can be extended to an $R$-homomorphism $g : U \to V$ and $U/bR \cong g(U)/g(bR)$.

(d) Any non-zero homomorphic image of a uniform, torsion $R$-module is uniform.
Let \( R \) be an associative ring with identity and \( M \) be a unital right \( R \)-module. Consider the following conditions of \( M_R \) as introduced by Singh [44].

(I) Every finitely generated submodule of every homomorphic image of \( M \) is a direct sum of uniserial modules.

(II) Given any two uniserial submodules \( U \) and \( V \) of a homomorphic image of \( M \), for any submodule \( W \) of \( U \), any non-zero homomorphism \( f : W \to V \) can be extended to a homomorphism \( g : U \to V \) provided the composition length \( d(U/W) \leq d(V/f(W)) \).

**Definition 1.2.35 [22]:** A module \( M \) satisfying condition (I) and (II) is called an \( S_2 \)-module.

**Definition 1.2.36 [45]:** A module \( M \) satisfying only the condition (I) is said to be the QTAG-module.

Now we give some elementary definitions and results as introduced in [15,17,22,45].

**Definition 1.2.37:** Let \( M \) be an \( S_2 \)-module, then an element \( x \neq 0 \) of \( M \) is called uniform if \( xR \) is a uniform module.

**Definition 1.2.38:** Let \( M \) be an \( S_2 \)-module, then an uniform element \( x \in M \) is called of exponent \( n \) (denoted by \( e(x) \)) if \( d(xR) = n \); and the \( \sup \{ d(yR/xR)/yR \text{ is uniserial submodule of } M \text{ containing } x \} \) is called the height of \( x \) and is denoted by \( H_M(x) \) (or simply \( H(x) \)).

**Definition 1.2.39:** An \( S_2 \)-module \( M \) is called bounded if there exists a positive integer \( k \) such that \( H(x) \leq k \) for all uniform elements \( x \in M \).
Proposition 1.2.40 [Lemma 4, 42]: Let $M$ be a module and $x_1, x_2, \ldots, x_n$ be finitely many uniform element of $M$ such that for some positive integer $k$, $H(x_i) \geq k$ for all $i$. Then for every uniform element $x$ of $M$ in $\sum_i x_i R$, $H(x) \geq k$.

Definition 1.2.41: Let $M$ be an $S_2$-module, then for every $k \geq 0$, $H_k(M)$ will denote the submodule of $M$ generated by the uniform elements of $M$, which are of height $\geq k$.

Definition 1.2.42: Let $M$ be a $S_2$-module, then for every $k \geq 0$, $H^k(M)$ will denote the submodule of $M$ generated by the uniform elements of exponent at most $k$.

Definition 1.2.43: Let $N$ be a submodule of $S_2$-module $M$ then for any integer $k \geq 0$, we define $H^k(N)$ to be submodule of $M$ generated by those elements $x \in M$ for which the elements $\bar{x} = x + N$ in $M/N$ has exponent $\leq k$.

In other words $H^k(N)$ is the submodule generated by those uniform elements $x \in M$ for which $d(xR/(xR \cap N)) \leq k$ i.e. there exists at least a uniform element $y \in xR \cap N$ such that $d(xR/yR) \leq k$ and we denote $H^k_0(0)$ by $Soc^k(N)$.

Proposition 1.2.44 [Corollary 1, 44]: Any bounded $S_2$-module $M$ is a direct sum of uniserial modules.

Proposition 1.2.45 [Lemma 6, 42]: Let $M = A + B$ be a torsion $R$-module and $A$, $B$ be its submodules. Then for any non-negative integer $k$, $H_k(M) = H_k(A) + H_k(B)$.

Lemma 1.2.46 [Lemma 2.3, 45]: Let $A$ and $B$ be any two uniserial submodules
of a QTAG-module $M$ such that $A \cap B \neq 0$ and $d(A) \leq d(B)$. Then there exists a monomorphism $\sigma : A \rightarrow B$, which is identity on $A \cap B$.

**Lemma 1.2.47** [Lemma 3.9, 45]: Let $N$ be a submodule of a QTAG-module $M$. Then $N$ is an $h$-pure submodule of $M$ if and only if for every uniform element $\bar{x} = x + N$ of $M/N$, there exists a uniform element $x' \in M$ such that $x + N = x' + N$ and $e(x') = e(x)$.

**Theorem 1.2.48** [Theorem 3.11, 45]: (a) If every element in $\text{Soc}(M)$ is of infinite height, then $M$ is a direct sum of serial modules, each of infinite length.
(b) Any QTAG-module $M$ admits a uniform summand, which can be chosen to be of finite length in case not all uniform element in $\text{Soc}(M)$ are of infinite heights.

**Section-3**

§ 1.3. $h$-pure and $h$-neat Submodules

This section is significant in the sense that some of the result mentioned here have been very often used in the subsequent chapters. As it is obvious from the heading of this section, $h$-pure, $h$-neat and high submodules are given here.

**Proposition 1.3.1** [Theorem 2, 44]: Let $M$ be an $S_2$-module and $N$ be a submodule of $M$ such that $N$ is a direct sum of uniserial modules of same length $k$. Then the following are equivalent:

(a) $N$ is a direct summand of $M$.
(b) $H_n(N) = N \cap H_n(M)$ for all $n$.
(c) $N$ satisfies $H_k(M) \cap N = 0$. 

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Definition 1.3.2 [44]: A submodule $N$ of an $S_2$-module $M$ is called $h$-pure if $H_k(N) = N \cap H_k(M)$ for all non-negative integer $k$.

Proposition 1.3.3: Every direct summand of an $S_2$-module is $h$-pure.

Proposition 1.3.4 [Theorem 3, 44]: Every bounded $h$-pure submodule of an $S_2$-module $M$ is a direct summand of $M$.

Proposition 1.3.5 [Lemma 1, 44]: Let $x$ be a uniform element in $Soc(M)$ such that $H(x)$ is finite. If $u \in M$ is a uniform element such that $x \in uR$ and $d(uR/xR) = H(x)$, then $uR$ is a $h$-pure submodule of $M$ and hence a summand of $M$.

Proposition 1.3.6 [Lemma 2, 44]: Let $N$ be a submodule of an $S_2$-module $M$, then the following hold:

(i) If $N$ is $h$-pure in $M$, given any uniform element $\bar{x} \in M/N$ there exists a uniform element $x' \in N$ such that $e(\bar{x}) = e(x')$ and $\bar{x} = \bar{x'}$.

(ii) If $N$ is $h$-pure in $M$, and $K$ is any submodule of $N$, then $N/K$ is $h$-pure $M/K$.

(iii) If $K$ is $h$-pure submodule of $M$ such that $K \subseteq N$ and $N/K$ is $h$-pure in $M/K$, then $N$ is $h$-pure in $M$.

Proposition 1.3.7 [Theorem 4, 44]: Let $M$ be an $S_2$-module. If every uniform element of $Soc(M)$ is of infinite height, then $M$ is a direct sum of infinite length uniform submodules.

Proposition 1.3.8 [Proposition 2, 15]: If $M$ is a $S_2$-module and $N$ is $h$-pure submodule of $M$ with same socle then $N = M$. 
Proposition 1.3.9 [Lemma 2, 15]: If \( N \) is a \( h \)-pure submodule of a \( S_2 \)-module \( M \) such that \( \text{Soc}(H_k(M)) \subseteq N \) for some non-negative integer \( k \), then \( H_k(M) \subseteq N \).

Proposition 1.3.10 [Lemma 3, 15]: If \( K \) is \( h \)-pure submodule of a \( S_2 \)-module \( M \) then \( \text{Soc}(H_n(M/K)) = (\text{Soc}(H_n(M)) + K)/K \).

Proposition 1.3.11 [Lemma 2, 22]: If \( N \) is a submodule of a \( S_2 \)-module \( M \) and for every uniform element \( x \in \text{Soc}(N) \), \( H_N(x) = H_M(x) \), then \( N \) is \( h \)-pure submodule of \( M \).

Proposition 1.3.13 [Proposition 2.5, 31]: If \( M \) is a \( S_2 \)-module such that \( M/K = N/K \oplus T/K \), where \( N \), \( T \) and \( K \) are the submodules of \( M \) and \( K \) is \( h \)-pure in \( N \), then \( T \) is also \( h \)-pure in \( M \).

Definition 1.3.14 [18]: If \( M \) is a \( S_2 \)-module and \( N \) is a submodule of \( M \) then \( N \) is called center of \( h \)-purity in \( M \) if every complement of \( N \) in \( M \) is \( h \)-pure in \( M \).

Proposition 1.3.15 [Corollary 5, 18]: If \( M \) is a \( S_2 \)-module then for every \( k \geq 0 \), \( H_k(M) \) is center of \( h \)-purity.

Definition 1.3.16 [4]: If \( M \) is an \( S_2 \)-module then a submodule \( S \) of \( \text{Soc}(M) \) is called subsocle.

Definition 1.3.17 [2]: A subsocle \( S \) of an \( S_2 \)-module \( M \) is said to support a submodule \( N \) of \( M \) if \( S = \text{Soc}(N) \).
Definition 1.3.18 [17]: An $S_2$-module is called $h$-pure complete if every subsocle of $M$ supports an $h$-pure submodule.

Definition 1.3.19 [17]: A submodule $N$ of an $S_2$-module $M$ is called $h$-neat if $H_1(N) = N \cap H_1(M)$.

Proposition 1.3.20 [Theorem 3, 17]: A submodule $N$ of a $S_2$-module $M$ is $h$-neat in $M$ if and only if $N$ has no proper essential extension in $M$.

Proposition 1.3.21 [Proposition 4, 22]: If $M$ is an $S_2$-module and $N$ is a submodule of $M$ then any complement $T$ of $N$ is $h$-neat.

Definition 1.3.22 [4]: If $N$ is a submodule of an $S_2$-module $M$, then $h$-neat hull of $N$ is defined as the minimal $h$-neat submodule $K$ of $M$ such that $N \subseteq K$.

Definition 1.3.23 [27]: If $N$ is a submodule of a QTAG-module $M$, then $N$ is called kernel of $h$-purity if $h$-neat hulls of $N$ are $h$-pure submodule of $M$.

Definition 1.3.24: The submodule of $M$ generated by the uniform elements of infinite height is denoted by $M^1$. Equivalently $M^1 = \bigcap_{k=0}^{\infty} H_k(M)$.

Definition 1.3.25 [22]: A submodule $N$ of an $S_2$-module $M$ is called high submodule if it is a complement of $M^1$.

Proposition 1.3.26 [Theorem 7, 22]: If $N$ is a submodule of an $S_2$-module $M$ such that $N \subseteq M^1$. Then any complement $T$ of $N$ is $h$-pure submodule of $M$. 

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Proposition 1.3.27 [Proposition 13, 22]: If \( M \) is an \( S_2 \)-module and \( N \subseteq M^1 \neq 0 \) then for any complement \( T \) of \( N \) in \( M \), \( M/T \) is direct sum of infinite length uniform submodules.

Theorem 1.3.28 [Lemma 4, 1]: Let \( N \) be a submodule of an \( S_2 \) module \( M \) such that for some \( n \), \( N + H_n(M) \nsubseteq Soc(H_{n-1}(M)) \). Then there exists a proper \( h \)-pure submodule \( H \) of \( M \) such that \( H \supseteq N + H_n(M) \) and \( N + H_n(M) \supseteq Soc(H_{n-1}(M)) \).

Theorem 1.3.29 [Theorem 6, 1]: Let \( K \) be an \( h \)-pure submodule of an \( S_2 \) module \( M \), containing a submodule \( N \) of \( M \). If \( K \) is a minimal \( h \)-pure submodule of \( M \) containing \( N \), then \( N + H_n(K) \supseteq Soc(H_{n-1}(K)) \) for all \( n \). On the converse, if \( N + H_n(K) \supseteq Soc(H_{n-1}(K)) \) for all \( n \) and \( N \supseteq Soc(H_m(K)) \) for some \( m \), then \( K \) is a minimal \( h \)-pure submodule of \( M \), containing \( N \).

Section-4

§ 1.4. \( h \)-divisible and Basic Submodules

In this section, we recall some definitions and properties of \( h \)-divisible and basic submodules for \( S_2 \)-modules as introduced by Khan in [20] and [19] respectively.

Definition 1.4.1 [20]: Let \( M \) be a \( S_2 \)-module, then \( M \) is called \( h \)-divisible if \( H_1(M) = M \).

Remark 1.4.2: An \( S_2 \)-module \( M \) is \( h \)-divisible if and only if every uniform element of \( M \) is of infinite height.
Proposition 1.4.3 [Lemma 1, 20]: Let $M$ be a $S_2$-module and $M = \bigoplus \sum M_\alpha$ then $M$ is $h$-divisible if and only if each $M_\alpha$ is $h$-divisible.

Proposition 1.4.4 [Lemma 2, 20]: Let $M$ be a $S_2$-module, then $M$ is $h$-divisible if and only if every uniform element of $Soc(M)$ is of infinite height.

Theorem 1.4.5 [Theorem 3, 20]: If $M$ is a $S_2$-module then $M$ is $h$-divisible if and only if $M$ is a direct sum of infinite length uniform submodules.

Theorem 1.4.6 [Theorem 4, 20]: Let $M$ be a $S_2$-module and $N$ be a $h$-divisible submodule of $M$ then $N$ is a direct summand of $M$.

Proposition 1.4.7 [Proposition 6, 28]: If $N$ is $h$-pure submodule of a QTAG-module $M$ such that $M/N$ is $h$-divisible then $Soc(M) = Soc(N) + Soc(H_n(M))$ for all $n$.

Theorem 1.4.8 [Corollary 8, 28]: If $M$ is a QTAG-module and $N$ is a submodule of $M^1$, then every complement $K$ of $N$ is $h$-pure in $M$ and $M/K$ is $h$-divisible.

Theorem 1.4.9 [Corollary 10, 29]: If $M$ is a QTAG-module and $N$ is a submodule of $M$ then $M/K$ is $h$-divisible for every complement $K$ of $N$ if and only if $Soc(N) \subseteq M^1$.

Definition 1.4.10: Let $M$ be a QTAG-module. The divisible hull of $M$ is the intersection of all divisible QTAG-modules containing $M$. In other words, it is the smallest divisible QTAG-module containing $M$. 


Definition 1.4.11: An $S_2$-module $M$ is said to be reduced if it is free from the elements of infinite height. Equivalently $\{0\}$ is the only $h$-divisible submodule of $M$.

Definition 1.4.12 [3]: A submodule $N$ of an $S_2$-module $M$ is called $h$-dense in $M$ if and only if $M/N$ is $h$-divisible.

Proposition 1.4.13 [Proposition 2, 3]: A submodule $N$ an $S_2$-module $M$ is $h$-dense in $M$ if and only if $M = N + H_n(M)$ for all non negative integers $n$.

Definition 1.4.14 [4]: A submodule $N$ of an $S_2$ module $M$ is said to be almost $h$-dense in $M$ if for every $h$-pure submodule $K$ of $M$ containing $N$, $M/K$ is $h$-divisible.

Theorem 1.4.15 [Theorem 5, 1]: A submodule $N$ of an $S_2$ module $M$ is almost $h$-dense in $M$ if and only if $N + H_n(M) \supseteq \text{Soc}(H_{n-1}(M))$ for all $n$.

Definition 1.4.16 [19]: Let $M$ be an $S_2$-module. A submodule $B$ of $M$ is called a basic submodule of $M$ if the following conditions hold:

(i) $B$ is a direct sum of uniserial submodules

(ii) $B$ is $h$-pure in $M$

(iii) $M/B$ is $h$-divisible

The following theorem shows the existence of basic submodules.

Theorem 1.4.17 [Theorem 1, 19]: Let $M$ be a QTAG-module then $M$ possesses a basic submodule.
Theorem 1.4.18 [Theorem 4, 19]: A submodule $N$ of a QTAG-module can be extended to a basic submodule $B$ of $M$ if and only if $N = \cup_i C_i$ where $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \subseteq \cdots$, such that the height of uniform elements of $C_n$ (taken in $M$) are bounded.

The following result gives the uniqueness of basic submodules.

Theorem 1.4.19 [Theorem 5, 19]: If $M$ is a QTAG-module, then any two basic submodules are isomorphic.

Theorem 1.4.20 [Theorem 4, 30]: If $M$ is a QTAG-module, then $M$ has only one basic submodule if and only if it is either $h$-divisible or bounded.

Section-5

§ 1.5. Some Characterizations of $h$-pure Submodules

Here we state some important definitions and results of [23] and [32] which are general in nature but significant for next chapters.

Definition 1.5.1 [23]: Let $M$ be an $S_2$-module and $N$ be a submodule of $M$, then $N$ is called quasi-essential in $M$ if $M = T + K$ where $K$ is complement of $N$ and $T$ is any $h$-pure submodule of $M$ containing $N$.

Let $M$ be an $S_2$-module satisfying the following condition introduced by Singh (unpublished):

(A) For any finitely generated submodule $N$ of $M$, $R/\text{ann}(N)$ is right artinian.
Theorem 1.5.2 [Corollary 2, 23]: Let $M$ be a $S_2$-module, $N$ be a submodule of $M$ and $K$ be any $h$-pure submodule of $M$ containing $N$. Then for any complement $T$ of $N$ in $M$, $M = T + K$.

Theorem 1.5.3 [Corollary 8, 23]: Let $M$ be an $S_2$-module satisfying the condition (A) and $S \subseteq Soc(M)$ such that $Soc(H_n(M)) \supseteq S \supseteq Soc(H_{n+1}(M))$ for some $n$, then $S$ is quasi-essential in $M$.

Theorem 1.5.4 [Theorem 12, 23]: Let $M$ be an $S_2$-module satisfying the condition (A) and $S \subseteq Soc(M)$ such that $S \not\subseteq M$, then the following are equivalent:

(i) $S$ is both a center of $h$-purity in $M$ and a quasi-essential subsocle of $M$.

(ii) $S$ supports an absolute summand.

(iii) There exists a natural number $n$ such that $Soc(H_n(M)) \supseteq S \supseteq Soc(H_{n+1}(M))$.

Notation 1.5.5 [32]: For any non-negative integer $t$ and for a submodule $N$ of a QTAG-module $M$, we denote by $N_{t}(M)$ the submodule $(N + H_{t+1}(M)) \cap Soc(M)$ and by $N_{t}(M)$ the submodule $(N \cap Soc(M)) + Soc(H_{t+1}(M))$ and by $Q_{t}(M,N) = N_{t}(M)/N_{t}(M)$.

Definition 1.5.6 [32]: A submodule $N$ of a QTAG-module $M$ is quasi $h$-pure in $M$ if $Q_{n}(M,N) = 0$ for all $n \geq 0$.

Proposition 1.5.7 [Proposition 4.5, 32]: If $N$ is $h$-pure submodule of $M$ of if $N$ is a subsocle of $M$, then $N$ is quasi $h$-pure.

Theorem 1.5.8 [Theorem 4.6, 32]: If $N$ is a submodule of a QTAG-module $M$, then the following are equivalent:
(a) $N$ is quasi $h$-pure in $M$

(b) $\text{Soc}(N + H_n(M)) = \text{Soc}(N) + \text{Soc}(H_n(M))$ for all $n \geq 1$

(c) $H_1(N \cap H_n(M)) = H_1(N) \cap H_{n+1}(M)$ for all $n \geq 1$

**Theorem 1.5.9** [Theorem 4.7, 32]: If $N$ is a submodule of a $M$, then $N$ is $h$-pure in $M$ if and only if $N$ is $h$-neat and quasi $h$-pure in $M$.

**Theorem 1.5.10** [Theorem 4.12, 32]: If $K$ is $h$-pure submodule of $H_n(M)$, where $n \geq 0$. Then every submodule $T$ of $M$ maximal with respect to $T \cap H_n(M) = K$, is $h$-pure in $M$. 

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