CHAPTER - 4
CENTER OF \( h \)-PURITY IN QTAG-MODULES

Section-1

§ 4.1. Introduction

The concept of center of \( h \)-purity in QTAG-modules was defined by Khan [18]. The purpose of this chapter is essentially to study center of \( h \)-purity and their characterizations. We have further studied subsocles of QTAG-modules and their interesting properties about height and range establishing various facts about the same.

Section 2 is devoted to the study of center of \( h \)-purity. It has been seen in [22] that every submodule of \( M^1 \) is center of \( h \)-purity and for any \( k \geq 1 \), \( H_k(M) \) is center of \( h \)-purity in \( M \). In this section we generalize some of results of Ried [41] and Pierce [40] for QTAG-modules.

Section 3 deals with the study of subsocles and their some properties about height and range. Here we also define a term open subsocle \( S \) of a QTAG-module \( M \). We have proved that if \( S \) is a open subsocle of \( M \) with height \( k \), then \( \text{range}(S) \leq n + 1 \) if and only if \( \text{range}(\text{Soc}(M)/T) \leq n \) (Theorem 4.3.6).

In section 4, we define a new concept of \( n \)-\( h \)-purity, where \( n \) is a non-negative integer. The concept of \( n \)-\( h \)-purity generalizes the concept of \( h \)-purity. It is evident that if \( n = 0 \) then \( n \)-\( h \)-purity is simply \( h \)-purity. We have established that a subsocle \( S \) of a QTAG-module \( M \) becomes center of \( n \)-\( h \)-purity if and only if either \( h(S) = \infty \) or \( S \) is open such that \( \text{range}(S) \leq n + 2 \) (Theorem 4.4.6).
In section 5, we discuss about a special type of QTAG-module obtained by laying down some restrictions on heights of elements of QTAG-module and some characterizations in this regard has been obtained. We have established that such restrictions on a QTAG-module make the module either \( h \)-divisible or decomposable.


**Section-2**

§ 4.2. Center of \( h \)-purity

Before we start, we recall the following definition from [18].

**Definition 4.2.1**: Let \( M \) be a QTAG-module and \( N \) be a submodule of \( M \) then \( N \) is called center of \( h \)-purity in \( M \) if every complement of \( N \) in \( M \) is \( h \)-pure submodule of \( M \).

[Theorem 7, 22] shows that every submodule of \( M^1 \) is center of \( h \)-purity. Also [Corollary 10, 22] shows that for any \( k \geq 1 \), \( H_k(M) \) is center of \( h \)-purity in \( M \).

Now using the similar technique of Lemma 3.2.1 we can easily prove the following:

**Proposition 4.2.2**: If \( M \) is a QTAG-module and \( x, y \) are uniform elements in \( M \) then following hold:

(i) \( x - y \in \text{Soc}^n(M) \) if and only if \( H_n(xR) = H_n(yR) \).
(ii) For every element \( t \in \text{Soc}(M) \), \( H_1((x+t)R) = H_1(xR) \).

Now we prove the following theorem which generalizes [Theorem 2.1, 41].

**Theorem 4.2.3**: If \( M \) is a QTAG-module and \( N \) is a submodule of \( M \). Then there exists a submodule \( K \) of \( M \) such that \( K \) is maximal with respect to \( K \cap N = 0 \) and \( K \) is not \( h \)-pure in \( M \) if and only if the following condition is satisfied:

\[
(*) \text{ there exists uniform element } u \in N \text{ and } v \in M \text{ such that } u + v \text{ is uniform and }
\]

(i) \( e(v) > e(u) = 1 \)

(ii) \( H(v) = H(u) < H(u + v) \)

(iii) \( vR \cap N = 0 \)

**Proof**: Let \( K \) be a submodule of \( M \), maximal with respect to \( K \cap N = 0 \) and \( K \) be not \( h \)-pure in \( M \). Let \( n \) be the least positive integer such that \( K \cap H_n(M) \neq H_n(K) \), then appealing to Proposition 1.3.21, we have \( n \geq 2 \). Let \( x \) be a uniform element in \( K \cap H_n(M) \), then there exists a uniform element \( y \in M \) such that \( y \notin K \), \( x \in yR \) and \( d(yR/xR) = n \). Let \( zR/xR \) be a submodule of \( yR/xR \) such that \( d(zR/xR) = 1 \), then \( d(yR/zR) = n - 1 \). By \( h \)-neatness of \( K \), there exists a uniform element \( t \in K \) such that \( x \in tR \) and \( d(tR/xR) = 1 \).

Hence, there exists an isomorphism \( \sigma : zR \to tR \) which is the identity on \( xR \). Trivially \( e(z - \sigma(z)) \leq 1 \), so \( z - \sigma(z) = u + w \) where \( u \in \text{Soc}(N) \) and \( w \in \text{Soc}(K) \). It is easy to see that \( u \) and \( w \) are uniform. Let \( H(u) \geq n - 1 \), then we can find a uniform element \( s \in M \) such that \( d(sR/uR) = n - 1 \). Now \( z - u = w + \sigma(z) \in K \) and \( z - u \in H_{n-1}(M) \), so \( z - u = w + \sigma(z) \in K \cap H_{n-1}(M) = H_{n-1}(K) \). Since \( (w + \sigma(z))R \) is homomorphic image of \( zR \), \( w + \sigma(z) \) is an uniform element.

Now we can find a uniform element \( w' \in K \) such that \( w + \sigma(z) \in w'R \) and
\[ d(w'R/(w + \sigma(z))R) = n - 1. \] Trivially \( d(w + \sigma(z))R > 1 \), so we can find a submodule \( gR \subseteq (w + \sigma(z))R \) such that \( d((w + \sigma(z))R/gR) = 1 \). Now appealing to Proposition 3.2.1 and 4.2.2 we get, \( H_1(zR) = xR \) and

\[
H_1((w + \sigma(z))R) = gR
\]

\[
= H_1(\sigma(z)R)
\]

\[
= H_1(zR)
\]

\[
= xR
\]

which in turn gives \( x \in H_n(K) \), a contradiction. Hence \( H(u) \leq n - 1 \). Let \( v = w + \sigma(z) \) then \( c(v) > c(u) = 1 \) and \( H(u) = H(v) < H(z) = H(u + v) \), since \( v \in K, vR \cap N = 0 \).

Therefore the conditions of the theorem are satisfied.

Conversely, suppose that the conditions are satisfied. Let for some natural number \( n, H(v) < n \leq H(u + v) \) and \( T_n = \text{Soc}(H_u(M)) \). Since \( e(v) > e(u) = 1, e(v) \geq 2 \). Let \( zR = \text{Soc}(vR) \), then \( d(vR/zR) \geq 1 \) and we get \( zR \subseteq H_1(vR) \). Also \( H_1((u + v)R) = H_1(vR) \supseteq zR \), consequently \( z \in T_n \). Since \( vR \cap N = 0 \) and \( z \notin N \). Let \( T_n = S \oplus T_n \cap \text{Soc}(N) \) and \( z \in S \). Also (ii) gives \( u \notin T_n \cap \text{Soc}(N) \), so \( \text{Soc}(N) = T \oplus (T_n \cap \text{Soc}(N)), u \in T \). Now \( T_n + \text{Soc}(N) = S \oplus T \oplus (T_n \cap \text{Soc}(N)) \).

Similarly, we get \( \text{Soc}(M) = L \oplus (T_n + \text{Soc}(N)) \) for some subsocle \( L \). Let \( T_0 = L \oplus S \) then \( \text{Soc}(M) = T_0 \oplus \text{Soc}(N) \), with \( z \in T_0 \).

Let \( \pi \) be the projection of \( \text{Soc}(M) \) onto \( \text{Soc}(N) \) then \( \pi(T_n) = (T_n \cap \text{Soc}(N)) \).

Let \( U = T_0 + vR \), then

\[
\text{Soc}(U) = T_0 + \text{Soc}(vR)
\]

\[
= T_0 + zR
\]

\[
= T_0
\]

Therefore, \( \text{Soc}(U) \cap \text{Soc}(N) = 0 \) and we get \( U \cap N = 0 \). Now we embed \( U \) into a complement \( K \) of \( N \). Let \( tR \) be a submodule of \( vR \) such that \( d(vR/tR) = 1 \). As
\( H_1((v + u)R) = H_1(vR) = tR \), we get \( H(t) \geq n + 1 \).

Now we show that \( H_K(t) \leq n \). Let \( H_K(t) \geq n + 1 \) then there exists a uniform element \( y \in K \) such that \( t \in yR \) and \( d(yR/tR) = n+1 \). Let \( wR/tR \) be a submodule of \( yR/tR \) such that \( d(wR/tR) = 1 \) and \( d(yR/wR) = n \).

Hence, there exists an isomorphism \( \sigma : vR \rightarrow wR \) which is the identity on \( tR \). The map \( \eta : vR \rightarrow (v - \sigma(v))R \) is an \( R \)-epimorphism with \( tR \leq \text{Ker} \eta \). Hence, \( e(v - \sigma(v)) \leq 1 \) and we get \( v - \sigma(v) \in \text{Soc}(M) \). Since, \( H(u+v) \geq n \), \( u+v \in H_n(M) \).

Therefore, \( u + v - \sigma(v) \in H_n(M) \), consequently

\[
u + v - \sigma(v) \in \text{Soc}(M) \cap H_n(M) = T_n.\]

Also \( v - \sigma(v) \in K \), so \( v - \sigma(v) \in K \cap \text{Soc}(M) = K \cap (T_0 + \text{Soc}(N)) = T_0.\)

Therefore,

\[
u = \pi(u + v - \sigma(v)) \in \pi(T_n) = T_n \cap \text{Soc}(N)\]

and we get \( H(u) \geq n \) but \( H(u) = H(v) < n \).

Hence, we reach at a contradiction. This shows that \( H_K(t) \leq n \). Therefore, \( K \) is not \( h \)-pure in \( M \).

Using the above theorem we prove the following generalization of [Theorem 1, 40]. It may be noticed that the proof given below has similarity with the corresponding proof in [Theorem 1, 40].

**Theorem 4.2.4:** Let \( M \) be a QTAG-module and \( T_n = \text{Soc}(H_n(M)) \), \( T_\infty = \text{Soc}(M^1) \) and \( T_{\infty+1} = T_{\infty+2} = 0 \). Let \( N \) be a submodule of \( M \) then \( N \) is center of \( h \)-purity in \( M \), if and only if there exists \( k \) with \( 0 \leq k \leq \infty \) such that \( T_k \supseteq \text{Soc}(N) \supseteq T_{k+2} \).

**Proof:** Let for some \( n \), \( T_n \supseteq \text{Soc}(N) \supseteq T_{n+2} \). Suppose \( N \) is not center of \( h \)-purity in \( M \). Now if \( n = \infty \) then there does not exist any uniform element in \( \text{Soc}(N) \)
satisfying condition (ii) of Theorem 4.2.3. Suppose \( n \) is finite. Let \( u \in \text{Soc}(N) \), \( v \in M \) be uniform elements satisfying conditions of Theorem 4.2.3. Let \( H(u) = k \) then as \( u \in T_n \), \( n \leq k < H(u + v) \). Since \( e(v) > e(u) = 1 \), we can find a submodule \( tR \) of \( vR \) such that \( d(vR/tR) = 1 \). Let \( w = u + v \) then
\[
H_1((u + v)R) = H_1(vR) = tR
\]
Let \( zR = \text{Soc}(vR) \) then as \( vR \) is totally ordered \( zR \leq tR \). Hence \( H(z) \geq n + 2 \). This shows that \( z \in T_{n+2} \supseteq \text{Soc}(N) \) and we get a contradiction to the fact that \( vR \cap N = 0 \). Therefore, \( N \) is center of \( h \)-purity in \( M \).

Conversely, suppose \( T_n \supseteq \text{Soc}(N) \supseteq T_{n+2} \) is not true for any \( n \). Then \( \text{Soc}(N) \not\subseteq M^1 \), so \( \text{Soc}(N) \not\subseteq T_m \) for some \( m \). Let \( k \) be the greatest natural number such that \( \text{Soc}(N) \subseteq T_k \). Then the maximality of \( k \) and the assumption yield \( \text{Soc}(N) \not\subseteq T_{k+1} \) and \( T_{k+2} \not\subseteq \text{Soc}(N) \). Hence, there exist uniform elements \( u \in \text{Soc}(N) \) and \( s \in T_{k+2} \) such that \( H(u) = k \) and \( s \notin \text{Soc}(N) \).

Now we can find a uniform element \( y \in M \) such that \( s \in yR \) and \( d(yR/sR) = k + 2 \). Let \( xR/sR \) be a submodule of \( yR/sR \) such that \( d(xR/sR) = 1 \), then \( d(yR/xR) = k + 1 \), \( e(x) = 2 \) and we get \( H(x) \geq k + 1 \). Let \( v = x - u \), then
\[
H_1((x - u)R) = H_1(vR) = H_1(xR) = sR
\]
Consequently, \( s \in (x - u)R \). Hence, \( s = (x - u)r \) for some \( r \in R \). If \( xr = 0 \), then \( ur = 0 \) otherwise, \( s \in \text{Soc}(N) \).

Define a map \( \eta : xR \rightarrow (x - u)R \) given as \( xr \rightarrow (x - u)r \) then \( \eta \) is a well defined onto homomorphism, consequently \( v = x - u \) is a uniform element. Trivially \( H(v) = k \) and \( H(u + v) = H(x) \geq k + 1 \). Since \( e(x) = 2 \) and \( e(u) = 1 \), \( e(v) = 2 > e(u) \). Now suppose \( vR \cap N \neq 0 \) then there exists a uniform element \( x' \in vR \cap N \) and \( x' = vr \)
for some \( r \in R \). Now \( x' = vr = xr - ur \). Trivially \( xr \neq 0 \), so either \( xrR = xR \) or \( xrR = sR \) and in each case we get \( s \in N \) which is a contradiction. Therefore, \( vR \cap N = 0 \). Hence, by Theorem 4.2.3, \( N \) is not a center of \( h \)-purity in \( M \). This completes the proof of the theorem.

**Section-3**

§ 4.3. Height of Subsocles

In this section we talk about subsocle and their some properties about height and range. We introduce here open subsocles of QTAG-module. Firstly we give the following definitions:

**Definition 4.3.1:** Let \( S \) be a subsocle of a QTAG-module \( M \), then height of \( S \) is defined as a non-negative integer \( k \) such that \( S \subseteq H_k(M) \) but \( S \nsubseteq H_{k+1}(M) \) and we write \( h(S) = k \).

If no such \( k \) is possible then we write \( h(S) = \infty \), so \( S \subseteq M^1 \).

**Definition 4.3.2:** A subsocle \( S \) of a QTAG-module \( M \) is called open if \( \text{Soc}(H_n(M)) \subseteq S \) for some non-negative integer \( n \).

**Definition 4.3.3:** If \( S \) is open subsocle of a QTAG-module \( M \) with \( h(S) = k \) then the range of \( S \) is the least non-negative integer \( n \) such that \( \text{Soc}(H_{k+n}(M)) \subseteq S \) and we write \( \text{range}(S) = n \).

Now from Theorem 4.2.4, it is evident that a subsocle \( S \) of finite height is center of \( h \)-purity if and only if \( \text{range}(S) \leq 2 \).
**Proposition 4.3.4:** Let $S$ be a subsocle of a QTAG-module $M$ and $n$ be any non-negative integer then

1. $S \cap H_{n+1}(M) = 0$ if and only if $\text{Soc}(H_n(M/S)) \subseteq \text{Soc}(M)/S$.

2. $S + \text{Soc}(H_n(M)) = \text{Soc}(M)$ if and only if $\text{Soc}(M)/S \subseteq H_n(M/S)$.

**Proof:** (1) Let $S \cap H_{n+1}(M) = 0$ and $\bar{x} \in \text{Soc}(H_n(M/S)) = \text{Soc}((H_n(M) + S)/S)$, then $x \in H_n(M)$ and $H_1(\bar{x}R) = 0$ which in turn implies $H_1(xR) \subseteq S$, so

$$H_1(xR) \subseteq S \cap H_{n+1}(M) = 0$$

Therefore, $x \in \text{soc}(M)$ and we get $\text{Soc}(H_n(M/S)) \subseteq \text{Soc}(M)/S$.

Conversely, suppose $S \cap H_{n+1}(M) \neq 0$. Let $x$ be a uniform element in $S \cap H_{n+1}(M)$, then there is a uniform element $y \in M$ such that $d(yR/xR) = n + 1$. Let $zR/xR = \text{Soc}(yR/xR)$, then $d(yR/zR) = n$ and $d(zR/xR) = 1$, so $z \in H_n(M)$ and $H_1(zR) = xR \subseteq S$. Now $H_1(\bar{z}R) = 0$, so we get $\bar{z} \in \text{Soc}(H_n(M/S)) \subseteq \text{Soc}(M)/S$, which gives $z \in \text{Soc}(M)$ but this is not possible. Therefore $S \cap H_{n+1}(M) = 0$.

(2) Let $\text{Soc}(M) = S + \text{Soc}(H_n(M))$ and $\bar{x} \in \text{Soc}(M)/S$, then $\bar{x} = y + S$, where $y \in \text{Soc}(H_n(M))$, consequently $\bar{x} \in H_n(M/S)$.

Conversely, if we take $x \in \text{Soc}(M)$ then $x + S = z + S$ where $z \in H_n(M)$. Hence $x = z + s$, $s \in S$ and we get $\text{Soc}(M) = S + \text{Soc}(H_n(M))$.

**Proposition 4.3.5:** Let $S$ be a subsocle of a QTAG-module $M$ such that $h(S) = k$ and $\text{Soc}(H_{k+n+1}(M)) \nsubseteq S$ for some integer $n \geq 0$. Then there exists a complementary subsocle $T$ of $S$ in $\text{Soc}(M)$ such that $h(\text{Soc}(M)/T) = k$ and $\text{Soc}(H_{k+n}(M/T)) \nsubseteq$
Proof: Trivially \( S \cap \text{Soc}(H_{k+1}(M)) \subseteq \text{Soc}(H_{k+n+1}(M)) \). Since \( \text{Soc}(H_{k+n+1}(M)) \) is bounded, we shall have

\[
\text{Soc}(H_{k+n+1}(M)) = T_0 \oplus S \cap \text{Soc}(H_{k+n+1}(M)).
\]

It is easy to see that \( T_0 \cap S = 0 \) and \( T_0 \subseteq H_{k+1}(M) \). As \( S \cap H_{k+1}(M) \oplus T_0 \subseteq \text{Soc}(H_{k+1}(M)) \), we can find a subsocle \( T_1 \) such that

\[
\text{Soc}(H_{k+1}(M)) = S \cap H_{k+1}(M) \oplus T_0 \oplus T_1.
\]

Now using the definition of height of \( S \), we will have \( S \cap H_{k+1}(M) \subseteq S \).

Hence, \( S = S \cap H_{k+1}(M) \oplus S' \) for some subsocle \( S' \). Trivially \( S' \subseteq H_{k}(M) \) and \( S' \cap H_{k+1}(M) = 0 \). Since \( \text{Soc}(H_{k+1}(M)) \oplus S' \subseteq \text{Soc}(H_{k}(M)) \), we get a subsocle \( T_2 \) such that \( \text{Soc}(H_{k}(M)) = \text{Soc}(H_{k+1}(M)) \oplus S' \oplus T_2 \). Trivially \( S \cap (T_0 \oplus T_1 \oplus T_2) = 0 \).

Let \( \text{Soc}(M) = \text{Soc}(H_{k}(M)) \oplus T_3 \) and \( T = T_0 \oplus T_1 \oplus T_2 \oplus T_3 \) then

\[
\text{Soc}(M) = \text{Soc}(H_{k}(M)) \oplus T_3 = \text{Soc}(H_{k+1}(M)) + S' + T_2 + T_3 = S \cap \text{Soc}(H_{k+1}(M)) \oplus T_0 \oplus T_1 \oplus S' \oplus T_2 \oplus T_3 = S \oplus T
\]

Hence, \((S+T)/T = \text{Soc}(M)/T \subseteq H_{k}(M)/T\). Now since \( T_0 \neq 0 \), \( T \cap H_{k+n+1}(M) \neq 0 \) and consequently, by Proposition 4.3.4, \( \text{Soc}(H_{k+n}(M/T)) \subseteq \text{Soc}(M/T) \). Also as \( \text{Soc}(M) \neq T + \text{Soc}(H_{k+1}(M)) \), appealing to Proposition 4.3.4, we get \( \text{Soc}(M)/T \subseteq H_{k+1}(M/T) \). Hence \( h(\text{Soc}(M)/T) = k \).

Theorem 4.3.6: Let \( S \) be a open subsocle of a QTAG-module \( M \) such that \( h(S) = k \) and \( n \) be a non-negative integer. Then \( \text{range}(S) \leq n + 1 \) if and only if \( \text{range}(\text{Soc}(M)/T) \leq n \), for every subsocle \( T \) of \( M \) such that \( \text{Soc}(M) = T \oplus S \).
Proof: Let range(S) ≤ n + 1, then Soc(H_{k+n+1}(M)) ⊆ S ⊆ (H_k(M)). Trivially T ∩ H_{k+n+1}(M) = 0. Hence, by Proposition 4.3.4, Soc(H_{k+n}(M/T)) ⊆ Soc(M)/T.

It is trivial to see that Soc(M) = Soc(H_k(M)) + T, so by Proposition 4.3.4, we get Soc(M)/T ⊆ H_k(M/T). Therefore, range(Soc(M)/T) ≤ n.

Conversely, let range(Soc(M)/T) ≤ n. Now we show that Soc(H_{k+n+1}(M)) ⊆ S.

Let Soc(H_{k+n+1}(M)) ∉ S, then by Proposition 4.3.5, we find a subsocle T such that Soc(M) = T ⊕ S such that h(Soc(M)/T) = k and Soc(H_{k+n}(M/T)) ∉ Soc(M)/T and hence range(Soc(M)/T) ∉ n. Which is a contradiction.

Therefore, Soc(H_{k+n+1}(M)) ⊆ S and we get range (S) ≤ n + 1.

Section 4

§ 4.4. Center of n-h-purity

In this section we define a new concept of n-h-purity which generalizes the concept of h-purity and obtain a characterization of center of n-h-purity.

Definition 4.4.1: A submodule N of a QTAG-module M is called n-h-pure in M if N/Soc^n(N) is h-pure in M/Soc^n(N), where n is a non-negative integer.

It is evident that if n = 0 then n-h-purity is simply h-purity.

Definition 4.4.2: A subsocle S of a QTAG-module M is center of n-h-purity if all complements of S in M are n-h-pure submodules of M.

Firstly we prove the following:
**Theorem 4.4.3:** If $N$ is a submodule of a QTAG-module $M$, then there is a complement of $N$ which is $h$-pure in $M$.

**Proof:** It is sufficient to consider $\text{Soc}(N) \neq \text{Soc}(M)$. Suppose every uniform element of $\text{Soc}(M)$ is of infinite height then trivially $N \subseteq M^1$. Now appealing to Theorem 1.4.8, we get a complement $K$ of $N$, which is $h$-pure in $M$.

On the other hand if there is a uniform element $x \in \text{Soc}(M)$ such that $x \notin \text{Soc}(N)$ and $H(x) < \infty$. As if $y \in \text{Soc}(M)$ such that $y \notin \text{Soc}(N)$ and $H(y) = \infty$, then $H(x + y) = H(x) < \infty$.

Hence, appealing to Proposition 1.3.5, we shall get a summand $K$ such that $\text{Soc}(K) = (x + y)R$ and $K \cap N = 0$. Hence, $K$ is $h$-pure in $M$.

**Theorem 4.4.4:** If $S \subseteq \text{Soc}(M)$ then there exists a $h$-neat submodule $K$ of $M$ which is $1-h$-pure with $\text{Soc}(K) = S$.

**Proof:** Applying Theorem 4.4.3 for $M/S$, we get an $h$-pure submodule $K/S$ in $M/S$, which is a complement of $\text{Soc}(M)/S$. Since $(K/S) \cap (\text{Soc}(M)/S) = 0$, for every uniform element $x \in \text{Soc}(K)$, $x + S = S$, so $x \in S$ and hence $\text{Soc}(K) = S$. Therefore, $K$ is $1-h$-pure in $M$.

Now we show that $K$ is $h$-neat. Let $x$ be a uniform element in $K \cap H_1(M)$, then we get a uniform element $y \in M$ such that $d(yR/xR) = 1$. Now if $y \in K$ we get $K$ to be $h$-neat submodule. Let $y \notin K$ then $((K + yR)/S) \cap (\text{Soc}(M)/S) \neq 0$ implies $k + y + S = z + S$ for some $z \in \text{Soc}(M)$, $k \in K$.

Hence, $0 = H_1(zR) = H_1((k + y)R = 0$, so $k + y \in \text{Soc}(M)$. Therefore, $H_1(kR) = H_1(yR) = xR$ and $x \in H_1(K)$. Hence, $K$ is $h$-neat.

**Proposition 4.4.5:** Let $S$ be a subsocle of a QTAG-module $M$ such that $S$ is center
of $n$-$h$-purity for $n \geq 1$. Then $\text{Soc}(M)/T$ is center of $(n-1)$-$h$-purity in $M/T$ for every complementary subsocle $T$ of $S$ in $\text{Soc}(M)$.

**Proof:** Let $K/T$ be a complement of $\text{Soc}(M)/T$ in $M/T$. Then trivially $K \cap S = 0$. Now we show that $N \cap S \neq 0$ for $K \subseteq N$.

Let $N \cap S = 0$ then we show that $N/T \cap (S \oplus T)/T = 0$. Let on contrary $N/T \cap (S \oplus T)/T \neq 0$, then $x + T = s + T$ where $x \in N$, $s \in S$ and we get $x - s \in T \subseteq K \subseteq N$, consequently, $s \in N \cap S = 0$ and $x + T = T$. Which is a contradiction. Therefore, $K$ is a complement of $S$. Hence $K/\text{Soc}^n(K)$ is $h$-pure in $M/\text{Soc}^n(K)$. Now we show that $(K/T)/(\text{Soc}^{n-1}(K/T))$ is $h$-pure in $(M/T)/(\text{Soc}^{n-1}(K/T))$. It is easy to see that $\text{Soc}(K) = T$ and $\text{Soc}^{n-1}(K/T) \subseteq \text{Soc}^n(K)/T$. Now for any uniform element $x \in \text{Soc}^n(K)$, let $yR = \text{Soc}(xR)$ then $H_{n-1}(xR) = yR$. Hence

$$H_{n-1}(xR) = H_{n-1}((xR + T)/T)$$

$$= (H_{n-1}(xR) + T)/T$$

$$= 0$$

Therefore, $\text{Soc}^{n-1}(K/T) = \text{Soc}^n(K)/T$. Further, under the canonical isomorphism

$$(M/T)/(\text{Soc}^{n-1}(K/T)) = (M/T)/(\text{Soc}^n(K)/T)$$

$$\cong M/\text{Soc}^n(K)$$

Therefore, $(K/T)/(\text{Soc}^{n-1}(K/T))$ is mapped onto $K/\text{Soc}^n(K)$. Hence, $K/T$ is $(n-1)$-$h$-pure in $M/T$ and we get the result.

Now we prove the main result of this section:

**Theorem 4.4.6:** A subsocle $S$ of a QTAG-module $M$ is center of $n$-$h$-purity for some $n \geq 0$ if and only if either $h(S) = \infty$, or $S$ is open subsocle of $M$ such that $\text{range}(S) \leq n + 2$. 64
Proof: Let $S$ be a center of $n$-$h$-purity and $h(S) < \infty$. Suppose $h(S) = k$, then we show that $\text{Soc}(H_{k+n+2}(M)) \subseteq S$, which in turn will imply $\text{range}(S) \leq n + 2$. Let $\text{Soc}(H_{k+n+2}(M)) \not\subseteq S$, then appealing to Proposition 4.3.5, we will find a subsocle $T$ such that $\text{Soc}(M) = S \oplus T$, $h(\text{Soc}(M)/T) = k$ and $\text{Soc}(H_{k+n+1}(M/T)) \not\subseteq \text{Soc}(M)/T$. As remarked in section 3, for $n = 0$, $\text{range}(S) \leq 2$, so we use induction. However, appealing to Proposition 4.4.5, we get $\text{Soc}(M)/T$ as center of $(n-1)$-$h$-purity. Therefore, $\text{range}(\text{Soc}(M)/T) \leq n - 1 + 2 = n + 1$, consequently,

$$\text{Soc}(H_{k+n+1}(M/T)) \subseteq \text{Soc}(M)/T$$

Which is a contradiction. Hence, $\text{range}(S) \leq n + 2$.

Conversely, if $h(S) = \infty$ then by Theorem 1.4.8, $S$ is center of $h$-purity and hence for $n = 0$, $S$ is center of $n$-$h$-purity. Suppose $\text{range}(S) \leq n + 2$ and $\text{Soc}(H_{k+n+2}(M)) \subseteq S \subseteq H_k(M)$. Let $K$ be a complement of $S$ in $M$. Now we prove that

$$\text{Soc}(H_{k+2}(M/\text{Soc}^n(K)) \subseteq (\text{Soc}(M) + \text{Soc}^n(K)))/\text{Soc}^n(K) \subseteq H_k(M/\text{Soc}^n(K))$$

For any uniform element $x \in H_{k+2}(M)$, let $\bar{x} \in \text{Soc}(H_{k+2}(M/\text{Soc}^n(K))$. Then $H_1(\bar{x}R) = 0$, hence $H_1(xR) \subseteq K$, but due to Proposition 1.3.21, $K$ is $h$-neat and so there is a uniform element $t \in K$ such that $H_1(xR) = H_1(tR) = zR$. Now as $x \in H_{k+2}(M)$, there is a uniform element $y \in M$ such that $d(yR/xR) = k + 2$, consequently, $H_{k+3}(yR) = H_1(tR) = zR$ and we get

$$H_{k+3+n-1}(yR) = H_n(tR)$$

$$= H_{n-1}(zR)$$

but

$$H_{k+n+2}(yR) = H_n(tR) \subseteq K \cap H_{k+n+2}(M) = 0$$

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Hence, \( t \in \text{Soc}^n(K) \). Further, as \( H_t(xR) = H_t(tR) \), we get \( x - t \in \text{Soc}(M) \).

Therefore, \( x - t + \text{Soc}^n(K) = x + \text{Soc}^n(K) = \bar{x} \in (\text{Soc}(M) + \text{Soc}^n(K))/\text{Soc}^n(K) \)
and we get the first inclusion. Trivially,
\[
H_k(M/\text{Soc}^n(K)) = (H_k(M) + \text{Soc}^n(K))/\text{Soc}^n(K)
\]
and as \( K \) is complement of \( S \), \( \text{Soc}(M) = S + \text{Soc}(K) \). Therefore, the second inclusion also follows. Hence,
\[
\text{range} \left( (\text{Soc}(M) + \text{Soc}^n(K))/\text{Soc}^n(K) \right) \leq 2
\]
and we get \( (\text{Soc}(M) + \text{Soc}^n(K))/\text{Soc}^n(K) \) as center of \( h \)-purity in \( M/\text{Soc}^n(K) \).

Further, it is easy to see that \( K/\text{Soc}^n(K) \) is complement of \( (\text{Soc}(M) + \text{Soc}^n(K))/\text{Soc}^n(K) \) in \( M/\text{Soc}^n(K) \) and hence \( K/\text{Soc}^n(K) \) is \( h \)-pure submodule of \( M/\text{Soc}^n(K) \).

Therefore, \( S \) is center of \( n-h \)-purity.

Section-5

§ 4.5. Special QTAG-Module

In this section we study the implications of the consequence of imposition of some restrictions on heights of the elements of QTAG-module and some characterizations in this regard has been obtained.

First of all we defining the following:

**Definition 4.5.1:** A module \( M \) is said to be *special* if and only if
\[
H(x + y) \leq H(x) + H(y),
\]
for each uniform element \( x, y \in M \). Such that \( x + y \neq 0 \).
To start with we need the following lemmas.

**Lemma 4.5.2:** If $M$ is special, then every uniform element of zero height is of exponent 1.

**Proof:** Let $y \in M$ such that $H(y) = 0$. Suppose $e(y) \neq 1$, therefore there exists $x \in yR$ such that $d(yR/xR) = 1$. Let $x = yr$, for $r \in R$, then

\[
1 \leq H(x) = H(yr) = H(yr - y + y) = H(y(r - 1) + y) \leq H(y(r - 1)) + H(y)
\]

Since $yR$ is totally ordered, either $y(r - 1)R = yR$ or $y(r - 1)R = xR$. But $y(r - 1)R \neq xR$ yields $y(r - 1)R = yR$. Consequently,

\[
1 \leq H(x) \leq H(y(r - 1)) + H(y) = 2H(y) = 0.
\]

Which shows a contradiction. Hence, $e(y) = 1$. Thus every element of zero height is of exponent 1.

**Lemma 4.5.3:** If $M$ is special, then every uniform element of finite height has zero height.

**Proof:** Let $x \in M$ and $H(x) = n < \infty$. Then there exists $y \in M$ such that $x \in yR$ and $d(yR/xR) = n$ with $H(y) = 0$. As, from Lemma 4.5.2, $y$ is of exponent 1, we have $xR = yR$. Hence, $H(x) = 0$. Thus in a special QTAG-module every element of finite height has zero height.
Lemma 4.5.4: If $M$ is $h$-reduced and special, then every uniform element has zero height and is of exponent 1.

Proof: Let $M^1$ be a submodule of $M$ generated by all the uniform elements of infinite height.

Now we shall prove that $M^1$ is $h$-pure. Suppose on contrary that $M^1$ is not $h$-pure, then there exists $0 \neq x \in M^1$ such that $H_{M^1}(x) \neq H_{M}(x)$, therefore $x$ has finite height in $M^1$. Since $x \in M^1$, therefore $x$ has infinite height in $M$, then there exists $y \in M$, $y \notin M^1$, such that $d(yR/xR) = n + 1$ and consequently, $x = yr$ for some $r \in R$. Since $y \notin M^1$, it has finite height and as we did in Lemma 4.5.2,

$$\infty = H(x)$$

$$= H(yr)$$

$$= H(yr - y + y)$$

$$= H(y(r - 1) + y)$$

$$\leq H(y(r - 1)) + H(y)$$

$$= 2H(y)$$

$$\leq \infty$$

Which is a contradiction. Therefore, $M^1$ is $h$-pure.

Hence, height of every element of $M^1$ in $M^1$ is same as in $M$. Thus $M^1$ is $h$-divisible. Since $M$ is $h$-reduced therefore $M^1 = (0)$. Thus every non-zero element of $M$ has finite height. The result follows from Lemma 4.5.2 and Lemma 4.5.3.

Now we are able to prove the main result.

Theorem 4.5.5: If $M$ is special, then it is either $h$-divisible or $h$-reduced.

Proof: Let $T$ be the maximal $h$-divisible submodule of $M$, then by Theorem 1.4.6,
we have $M = T \oplus S$, where $S$ is $h$-reduced. If $S$ has no element of finite height in $M$, then $M$ has no element of finite height. Therefore, either $M$ has all elements of infinite height or $S$ has an element of finite height, hence, $M$ is $h$-divisible.

Now we consider the case where $M$ is not $h$-divisible. Suppose $T$ and $S$ both are nontrivial. For any $t \in T$, $H(t) = \infty$; also there exists $s \in S$ such that $H(s) < \infty$. Applying Lemma 4.5.2 and Lemma 4.5.3, we have $H(s) = 0$ and exponent of $s$ is 1. Let us consider $P = \text{ann}(sR)$, then $sP = 0$. Now choosing $r \in P$, we have $t = (s(r - 1) + s + t)$, and consequently,

$$\infty = H(t) = H(s(r - 1) + s + t) \leq H(s(r - 1)) + H(s + t)$$

Now suppose that $H(s + t) = n$ i.e., $s + t \in H_n(M)$ implying thereby $s \in H_n(M)$ as $t \notin H_n(M)$. Thus, $H(s) \geq n$. But $H(s) = 0$ yields $n = 0$. Hence,

$$\infty = H(t) \leq 0 + 0 = 0.$$ 

Which shows a contradiction.

Therefore, the supposition that both $T$ and $S$ are nontrivial is false, and since $M$ is not $h$-divisible, so is $h$-reduced.

**Proposition 4.5.6:** If every uniform element of $M$ is of exponent 1, then $M$ is decomposable.

**Proof:** Let $F$ be the set of all linearly independent subsets of $M$, then trivially $F$ will contain a maximal element $B$ of $F$.

Let $C(B) = \{T : T$ is a cyclic module generated by $x \in B\}$. Let $M' = \sum T$, where $T \in C(B)$. Then trivially $M' = \bigoplus \sum T$. Now suppose that $x \in M$, $x \notin M'$, then there exists $r \in R$ such that $xr \notin M'$, where $xrR = xR$, as $x$ is of exponent 1.
Consider $B \cup xR$, $B$ is a proper subset of $B \cup xR$. If for some
\[ r_1, r_2, \cdots, r_k \in R, \quad x_1 r_1 + x_2 r_2 + \cdots + x_k r_k + xr = 0, \]
where $x_i r_i R = x_i R, x_r R = x R$ and $x_i \in B$, then $xr = 0$. Otherwise, $xr \in B$, which shows a contradiction.

Hence, $xr = 0$ and $x_r = 0$, for all $i$. Thus $B \cup xR$ is linearly independent, which contradicts the maximality of $B$, hence $x \in M'$. So we get $M = M' = \sum T$, where $T$ is a cyclic module and $M = \oplus \sum T$. Hence $M$ is decomposable.

**Corollary 4.5.7:** If $M$ is special, then it is either h-divisible or decomposable.

**Proof:** The result follows from Lemma 4.5.3, Lemma 4.5.4, Theorem 4.5.5 and Proposition 4.5.6.