Chapter 2

Revisiting commutativity degree of finite groups

In this chapter we obtain a characterization for all finite groups of odd order having commutativity degree at least $\frac{11}{15}$. We also obtain a lower bound for commutativity degree and derive some necessary and sufficient conditions for attaining this bound. Finally, we compute the value of commutativity degree for certain standard families of finite groups.

This chapter is based on the papers [37] and [39].

2.1 Some auxiliary results

Let $G$ be a finite group. Let $p$ be the smallest prime divisor of $|G|$. Then, from (1.1.c), we have

$$p \leq |\text{Cl}_G(x)| \leq |G'| \iff x \in G - Z(G).$$  \hspace{1cm} (2.1.a)
Clearly, if \(|\text{Cl}_G(x)| = p\), then \(G/C_G(x)\) forms an abelian group (in fact, a cyclic group of order \(p\)), and so, \(G' \subseteq C_G(x)\), which means that \(x \in C_G(G')\). So, for all \(x \in G - C_G(G')\), we have

\[ p < |\text{Cl}_G(x)|. \]  

(2.1.b)

Note that \(C_G(G') = \bigcap_{x \in G'} C_G(x) \leq G\), using Result 1.1.1. So, by (1.1.c), \(|\text{Cl}_G(x)|\) divides \(|\frac{G}{C_G(G')}|\) for all \(x \in G'\). Moreover, \(\frac{G}{C_G(G')}\) is abelian if and only if \(G'\) is abelian, and in either case \([C_G(G'), x]\) is a subgroup of \(G'\) for all \(x \in G\), which can be easily deduced from the commutator identities in (1.1.a). Also, using the Jacobi identity (1.1.b), we have

\[ (C_G(G'))' \subseteq Z(G) \subseteq Z(C_G(G')). \]  

(2.1.c)

In view of the above discussion, we have

Lemma 2.1.1. Let \(G\) be a finite group and \(p\) be a prime. Then

(a) \(\frac{G}{C_G(G')}\) can be embedded in \(\text{Aut}(G')\), and so, \(|\text{Cl}_G(x)|\) divides \(|\text{Aut}(G')|\) for all \(x \in G'\). In particular, if \(\gcd(p - 1, |G'|) = 1\) and \(|G'| = p\), then \(G' \subseteq Z(G)\).

(b) If \(\frac{G}{C_G(G')} \cong C_p\), then \(\langle x, C_G(G') \rangle = G\), \([G, x] = [C_G(G'), x]\) and \(|\text{Cl}_G(x)|\) divides \(|G'\) for all \(x \in G - C_G(G')\).

Proof. Part (a) follows from Result 1.1.1, noting, for the particular case, that \(|\text{Aut}(C_p)| = p - 1\). Part (b) follows from (1.1.c) and the identities in (1.1.a), noting that \([C_G(G'), x]\) is a subgroup of \(G'\). \(\Box\)
In view of Result 1.5.4, the following generalization of Result 1.1.4 simplifies our task considerably.

**Lemma 2.1.2.** Let $G$ be a finite non-abelian group and $p$ be a prime such that $\gcd(p - 1, |G|) = 1$. If $|G'| = p^2$ and $|G' \cap Z(G)| = p$, then $G$ is nilpotent of class 3; in particular, $G \cong P \times A$, where $A$ is an abelian group and $P$ is a $p$-group such that $|P'| = p^2$ and $|P' \cap Z(P)| = p$.

**Proof.** We have

$$\left| \frac{G}{Z(G)} \right| = \left| \frac{G'Z(G)}{Z(G)} \right| = \left| \frac{G'}{G' \cap Z(G)} \right| = p.$$

So, it follows from Lemma 2.1.1(a) that $G/Z(G)$ is nilpotent of class 2, and hence, $G$ is nilpotent of class 3. Thus, in particular, $G$ is the direct product of its Sylow subgroups. The proof follows considering $P$ to be the $p$-Sylow subgroup and $A$ to be the product of the rest of the Sylow subgroups. \qed

Given $H \subseteq G$, consider the set $H^* = \{x \in G : [G, x] \subseteq H\}$. Then, as observed in [44, Section I], we have

**Lemma 2.1.3.** Let $G$ be a finite group. If $H, H_1, H_2 \trianglelefteq G$, then

(a) $(G' \cap H)^* = H^*$, $\{1\}^* = Z(G)$, and $(G')^* = G$,

(b) $(H_1 \cap H_2)^* = H_1^* \cap H_2^*$, and $H_1^*H_2^* \subseteq (H_1H_2)^*$,

(c) $H_1 \subseteq H_2 \implies H_1^* \subseteq H_2^*$,

(d) $H \trianglelefteq H^* \trianglelefteq G$, and $Z(G/H) = H^*/H$,

(e) $G/H^*$ is never a non-trivial cyclic group.
Proof. Parts (a), (b), (c) and (d) follow from the definition of ( )* operation. For part (e), note that \( G/H^* \cong \frac{G\langle H \rangle}{Z(G\langle H \rangle)} \).

### 2.2 Groups with \( |G'| = p^2 \) and \( |G' \cap Z(G)| = p \)

Let \( G \) be a finite group. Recall that the commutativity degree of \( G \), denoted by \( Pr(G) \), is defined as the ratio

\[
Pr(G) = \frac{|\{(x, y) \in G \times G : [x, y] = 1\}|}{|G \times G|}.
\]

As noted in Result 1.5.8 and Result 1.5.12, D. J. Rusin has computed the values of \( Pr(G) \) when \( G' \subseteq Z(G) \), and when \( G' \cap Z(G) = \{1\} \). In this section we determine the value of \( Pr(G) \) and the size of \( \frac{G}{Z(G)} \) when \( |G'| = p^2 \) and \( |G' \cap Z(G)| = p \), where \( p \) is a prime such that \( \gcd(p - 1, |G|) = 1 \). It may be noted here that, on a number of occasions, the structure of \( \frac{G}{Z(G)} \) is determined by its size. For example, using GAP [51] and the notion of semidirect product (see Section 1.1), we have

**Result 2.2.1.** If \( G \) is a finite group with \( G' \not\subseteq Z(G) \), then \( \frac{G}{Z(G)} \) is isomorphic to \( C_7 \times C_3 \), \( (C_3 \times C_3) \times C_3 \), \( C_{13} \times C_3 \), \( C_{19} \times C_3 \), \( C_3 \times (C_7 \times C_3) \) or \( (C_5 \times C_3) \times C_3 \) according as \( |\frac{G}{Z(G)}| \) is 21, 27, 39, 57, 63 or 75.

We begin with a few lemmas.

**Lemma 2.2.2.** Let \( G \) be a finite group and \( p \) be a prime. If \( |G' \cap Z(G)| = p \) and \( C_G(G') \) is non-abelian, then \( \frac{C_G(G')}{Z(C_G(G'))} \cong (C_p \times C_p)^s \) and

\[
Pr(C_G(G')) = \frac{1}{p} \left( 1 + \frac{p - 1}{p^{2s}} \right),
\]

for some positive integer \( s \).
Proof. In view of (2.1.c), the lemma follows from Result 1.5.9. 

Lemma 2.2.3. Let $p$ be a prime and $G$ be a finite group such that $G' \nsubseteq Z(G)$ and one of the following conditions holds:

(a) $G' \cong C_{p^2}$ and $\gcd(p - 1, |G|) = 1,$

(b) $G' \cong C_p \times C_p$ and $\gcd(p^2 - 1, |G|) = 1.$

Then $|\frac{G}{C_G(G')}| = |G' \cap Z(G)| = |C\ell_G(x)| = p$ for all $x \in G' - Z(G)$.

Proof. Since $|\text{Aut}(C_{p^2})| = p(p - 1)$ and $|\text{Aut}(C_p \times C_p)| = p(p + 1)(p - 1)^2,$ the result follows from Lemma 2.1.1(a), noting that $G' - Z(G)$ is a union of conjugacy classes of $G$. 

Lemma 2.2.4. Let $G$ be a finite group and $p$ be the smallest prime divisor of $|G|$. If $G' \nsubseteq Z(G)$ and $|\frac{G}{C_G(G')}| = |G' \cap Z(G)| = p$, then $Z(G)^* \subset C_G(G')$ and $\frac{C_G(G')}{Z(G)^*}$ can be embedded in $\frac{G'}{G' \cap Z(G)}$.

Proof. By Lemma 2.1.3(d), we have $Z(G)^* \neq C_G(G')$. Let $z \in Z(G)^*$. Then, using (1.1.c) and the definition of $Z(G)^*$, we have $|C\ell_G(z)| \leq |G' \cap Z(G)| = p$. This, in view of (2.1.b), implies that $z \in C_G(G')$. Thus, $Z(G)^* \subset C_G(G')$. Again, let $x \in G - C_G(G')$. Since $\frac{G}{C_G(G')}$, and hence, $G'$ is abelian, it can be easily seen, using (1.1.a), that the mapping $f : C_G(G') \longrightarrow G'Z(G)/Z(G)$, defined by $f(z) = [z, x]Z(G)$, is a homomorphism. Also, using the definition of $Z(G)^*$ together with (2.1.c) and Lemma 2.1.1(b), we have $\ker f = Z(G)^*$. Thus, it follows that $C_G(G')/Z(G)^*$ is isomorphic to a subgroup of $\frac{G'Z(G)}{Z(G)}$. This completes the proof. 

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Lemma 2.2.5. Let \( p \) be a prime. If \( G \) is a finite \( p \)-group such that \( |G'| = p^2 \) and \( |G' \cap Z(G)| = p \), then \( Z(G)^* \cap Z(C_G(G')) = G'Z(G) \).

Proof. By Lemma 2.2.3, \( \frac{G}{C_G(G')} \) is abelian. Also, by Lemma 2.1.2, \( G \) is nilpotent of class 3. So, it follows that \( G'Z(G) \subseteq Z(G)^* \cap Z(C_G(G')) \). For the converse, we first fix \( x \in G - C_G(G') \) and \( y \in G' - Z(G) \). Then, by Lemma 2.1.1(b), \( C_G(x) \cap Z(C_G(G')) = Z(G) \). Therefore, since \( y \notin Z(G) \), we have \([y^{-1}, x] \neq 1\). Now, let \( z \in Z(G)^* \cap Z(C_G(G')) \). Note that both \( z \) and \( y \) are in \( Z(G)^* \), and so, \([z, x], [y^{-1}, x] \in G' \cap Z(G) \), which is cyclic of order \( p \). Hence, it follows that there is a positive integer \( i \) with \( 1 \leq i \leq p - 1 \) such that

\[
[y^{-1}, x]^i = [z, x]
\]

\[
\implies [y^i z, x] = 1
\]

\[
\implies y^i z \in C_G(x)
\]

\[
\implies y^i z \in Z(G), \text{ since } y^i z \in Z(C_G(G'))
\]

\[
\implies z \in G'Z(G), \text{ since } y^i \in G'.
\]

This completes the proof. \( \Box \)

The main result of this section is given as follows.

Theorem 2.2.6. Let \( G \) be a finite group and \( p \) be a prime such that \( \gcd(p - 1, |G|) = 1 \). If \( |G'| = p^2 \) and \( |G' \cap Z(G)| = p \), then

(a) \( \Pr(G) = \begin{cases} \frac{2p^2 - 1}{p^4} & \text{if } C_G(G') \text{ is abelian} \\ \frac{1}{p^4} \left( \frac{p-1}{p^{2x-1}} + p^2 + p - 1 \right) & \text{otherwise}, \end{cases} \)
(b) \[ \left| \frac{G}{Z(G)} \right| = \begin{cases} p^3 & \text{if } C_G(G') \text{ is abelian} \\ p^{2s+2} \text{ or } p^{2s+3} & \text{otherwise} \end{cases} \]

where \( p^{2s} = |C_G(G') : Z(C_G(G'))| \). Moreover,

\[ \left| \frac{G}{G' \cap Z(G)} : Z(\frac{G}{G' \cap Z(G)}) \right| = \left| \frac{G}{Z(G)} : Z(\frac{G}{Z(G)}) \right| = p^2. \]

Proof. In view of Lemma 2.1.2, we can assume that \( G \) is a \( p \)-group. So, by Lemma 2.2.3, we have \( |G : C_G(G')| = p \).

(a) If \( x \in G - C_G(G') \), then it follows from Lemma 2.1.1(b) and (2.1.b) that \( |C_G(x)| = p^2 \), and hence, \( |C_G(x)| = |G|/p^2 \). Thus, the result follows from Result 1.5.13 and Lemma 2.2.2.

(b) By Lemma 2.2.4, we have \( |C_G(G') : Z(G')^*| = p \). Therefore, using the second isomorphism theorem [43, page 25] and Lemma 2.2.5, we have

\[ \left| Z(C_G(G')) : G'Z(G) \right| = \begin{cases} p & \text{if } C_G(G') \text{ is abelian} \\ 1 \text{ or } p & \text{otherwise} \end{cases} \]

Hence, using the first part of Lemma 2.2.2, the result follows from the normal series \( Z(G) \subseteq G'Z(G) \subseteq Z(C_G(G')) \subseteq C_G(G') \subseteq G \).

The final statement follows from Lemma 2.2.4 and Lemma 2.1.3(a)(c). \( \square \)

### 2.3 Groups of odd order with \( Pr(G) \geq \frac{11}{75} \)

In [1, Theorem 4.12], F. Barry, D. MacHale and Á. Ní Shé have proved that if \( G \) is a finite group with \( |G| \) odd and \( Pr(G) > \frac{11}{75} \), then \( G \) is supersolvable.
It may be recalled that a group $G$ is said to be supersolvable if there is a series of the form

$$\{1\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r = G,$$

where $A_i \triangleleft G$ and $A_{i+1}/A_i$ is cyclic for each $i$ with $0 \leq i \leq r - 1$.

In this section we obtain a characterization (similar to the one mentioned in Table 1.5.3 obtained by Rusin) for all finite groups $G$ of odd order with $\Pr(G) \geq \frac{14}{15}$. We also point out a few small but significant lacunae in the work of Rusin. As usual, we begin with a few lemmas.

**Lemma 2.3.1.** Let $G$ be a finite group such that $|G|$ is odd.

(a) If $G' \cong C_{15}$, then $G' \subseteq Z(G)$.

(b) If $G' \cong C_{21}$ and $G' \nsubseteq Z(G)$, then $|\frac{G}{C_G(G')}| = |G' \cap Z(G)| = 3$ and exactly one of the following conditions holds:

(i) $|\text{Cl}_G(x)| = 21$ for all $x \in G - C_G(G')$.

(ii) There exists a subset $X$ of $G - C_G(G')$ such that

$$|X| = 2|Z(C_G(G'))|$$

and

$$|\text{Cl}_G(x)| = \begin{cases} 7 & \text{if } x \in X \\ 21 & \text{if } x \in G - (C_G(G') \cup X). \end{cases}$$

**Proof.** Note that $|\text{Aut}(C_{15})| = 8$ and $|\text{Aut}(C_{21})| = 12$. Therefore, (a) and the first part of (b) follow from Lemma 2.1.1(a).

For the second part of (b), we first observe, using (2.1.b) and Lemma 2.1.1(b), that $|\text{Cl}_G(x)| = 7$ or 21 for all $x \in G - C_G(G')$. Now, assume
that the condition (i) fails. Then there is an \( x_0 \in G - C_G(G') \) such that \( |C_G(x_0)| = 7 \). Since \( |C_G(G')| = 3 \), it is easy to see that

\[
G - C_G(G') = x_0C_G(G') \sqcup x_0^{-1}C_G(G').
\]  

(2.3.a)

Let \( X = x_0Z(C_G(G')) \sqcup x_0^{-1}Z(C_G(G')) \). Clearly, \( X \subseteq G - C_G(G') \). Since \( |C_G(x_0)| = |C_G(x_0)\rangle \), it follows from (1.1.c), (1.1.a) and Lemma 2.1.1(b) that \( |C_G(x)| = 7 \) for all \( x \in X \). On the other hand, suppose that \( x \in G - (C_G(G') \cup X) \). Then, by (2.3.a), we have \( x = x_0w \) or \( x_0^{-1}w \) for some \( w \in C_G(G') - Z(C_G(G')) \). Choose \( w_1 \in C_G(G') - Z(C_G(G')) \) such that \( [w_1, w] \neq 1 \). Then, by (1.1.a) and (2.1.c), we have

\[
[w_1, x] = \begin{cases} 
[w_1, x_0][w_1, w] & \text{if } x = x_0w \\
[w_1, x_0^{-1}][w_1, w] & \text{if } x = x_0^{-1}w.
\end{cases}
\]

Note that \( o([w_1, w]) = 3 \) and \( o([w_1, x_0]) = o([w_1, x_0^{-1}]) = 1 \) or 7, and hence, it follows that \( o([w_1, x]) = 3 \) or 21. Thus, using (1.1.c), we have \( |C_G(x)| = 21 \). This completes the proof.

\[\square\]

**Lemma 2.3.2.** Let \( G \) be a finite group. If \( |G| \equiv 3 \pmod{6} \), \( G' \cong C_5 \times C_5 \) and \( |G' \cap Z(G)| = 1 \), then

(a) \( |C_G(x)| = 3 \) for all \( x \in G' - Z(G) \),

(b) \( \frac{G}{C_G(G')} = 3 \).

**Proof.** Note that no non-identity element of \( G \) is conjugate to its inverse, and also that \( G' - Z(G) \) is a union of conjugacy classes of \( G \).

(a) Let \( x \in G' - Z(G) \). Since \( |G' - Z(G)| = 24 \), we have \( |C_G(x)| \leq 12 \). This, in view of Lemma 2.1.1(a), implies that \( |C_G(x)| = 3 \) or 5. On the
other hand, since \( x^5 = 1 \), it is a routine matter to see that the classes \( C\ell_G(x) \), \( C\ell_G(x^2) \), \( C\ell_G(x^3) \) and \( C\ell_G(x^4) \) are all distinct and have the same size. Thus, considering the possible partitions of 24 with 3 and 5 as summands, we have \( |C\ell_G(x)| = 3 \) for all \( x \in G' - Z(G) \).

(b) Suppose that there exist two elements \( x, y \in G' - Z(G) \) such that \( C_G(x) \neq C_G(y) \). Then, as seen above, \( |G : C_G(x)| = |G : C_G(y)| = 3 \), and so

\[
\left| \frac{C_G(x)}{C_G(x) \cap C_G(y)} \right| = \left| \frac{C_G(x)C_G(y)}{C_G(y)} \right| = \left| \frac{G}{C_G(y)} \right| = 3.
\]

Therefore, using the series

\[
C_G(G') \subseteq C_G(x) \cap C_G(y) \subseteq C_G(x) \subseteq G,
\]

we see that 9 divides \( \frac{G}{C_G(G')} \). However, in view of Lemma 2.1.1(a), we have \( \frac{G}{C_G(G')} = 3, 5, \) or 15. Hence, it follows that \( C_G(x) = C_G(y) \) for all \( x, y \in G' - Z(G) \). This completes the proof.

We are now in a position to characterize all finite groups of odd order with commutativity degree at least \( \frac{11}{75} \), essentially replicating the technique used by Rusin in [44, Section IV].

**Theorem 2.3.3.** Let \( G \) be a finite group. If \( |G| \) is odd and \( \Pr(G) \geq \frac{11}{75} \), then the possible values of \( \Pr(G) \) and the corresponding structures of \( G', G' \cap Z(G) \) and \( G/Z(G) \) are given as follows:
Table 2.3.4: Groups with $|G|$ odd and $Pr(G) \geq \frac{11}{75}$

**Proof.** Without any loss, we can assume that $G$ is non-abelian. Now, by Result 1.5.2, we have $|G'| \leq 25$. Also, by Result 1.1.5, $G'$ is not isomorphic to the unique non-abelian group of order 21. So, it follows that $G'$ is isomorphic to $C_3 \times C_3$, $C_9$, $C_{21}$, $C_5 \times C_5$, $C_{25}$ or $C_p$, where $p$ is an odd prime with $p \leq 23$. We analyze these possibilities as follows:

**Case 1.** $G' \subseteq Z(G)$.

If $G' \cong C_p$, with $p$ as above, then, by Result 1.5.9, there is a positive...
integer $s$ such that $\frac{G}{Z(G)} \cong (C_p \times C_p)^s$ and
\[
\Pr(G) = \frac{1}{p} \left(1 + \frac{p - 1}{p^s}\right).
\]
It follows that $s$ can have infinitely many values for $p = 3$ and 5, whereas $s = 1$ is the only possibility if $p = 7$. For the rest of the values of $p$, we have $\Pr(G) < \frac{11}{75}$.

If $G' \cong C_3 \times C_3$, then, by Result 1.5.8, we have
\[
\Pr(G) = \frac{1}{9} \left(1 + \frac{2}{3^{2m_1}} + \frac{2}{3^{2m_2}} + \frac{2}{3^{2m_3}} + \frac{2}{3^{2m_4}}\right)
\]
with $3^{2m_i} = |G/K^*_i|$, $1 \leq i \leq 4$, where $K_1$, $K_2$, $K_3$ and $K_4$ are the proper non-trivial subgroups of $G'$. Note that if $1 \leq i \neq j \leq 4$, then $K_i \cap K_j = \{1\}$, and so, by Lemma 2.1.3, we have $K^*_i \cap K^*_j = Z(G)$. Without any loss, we may assume that $1 \leq m_1 \leq m_2 \leq m_3 \leq m_4$. Thus, for $m_2 \geq 2$, we have
\[
\Pr(G) \leq \frac{1}{9} \left(1 + \frac{2}{9} + \frac{2}{3^4} + \frac{2}{3^4} + \frac{2}{3^4}\right) < \frac{11}{75}.
\]
Also, we have
\[
|K^*_1||K^*_2| = |K^*_1K^*_2||K^*_1 \cap K^*_2| = |K^*_1K^*_2||Z(G)| \leq |G||K^*_4|,
\]
which implies that $|\frac{G}{K^*_1}| |\frac{G}{K^*_2}| \geq |\frac{G}{K^*_4}|$. This, in turn, gives $m_1 + m_2 \geq m_4$. Hence, it follows that, for $\Pr(G) \geq \frac{11}{75}$, we must have $m_1 = m_2 = 1$ and $1 \leq m_3 \leq m_4 \leq 2$. Now, by Result 1.5.8, $\frac{G}{K^*_1}$ and $\frac{G}{K^*_2}$ are elementary abelian 3-groups. Therefore, given $g \in G$, we have $g^3 \in K^*_1 \cap K^*_2 = Z(G)$, and thus $\frac{G}{Z(G)}$ is also an elementary abelian 3-group. Moreover, by Lemma 2.1.3, we have
\[
\left|\frac{G}{Z(G)}\right| = \left|\frac{G}{K^*_1}\right| \left|\frac{K^*_1}{Z(G)}\right| = 9 \left|\frac{K^*_1K^*_2}{Z(G)}\right| \leq 9 \left|\frac{G}{K^*_2}\right| = 81.
\]
On the other hand, for \( m_4 = 2 \), we have

\[
\frac{|G|}{Z(G)} = \frac{|G|}{K_4^*} \geq 81,
\]

and hence \( \frac{|G|}{Z(G)} = 81 \).

Let \( m_1 = m_2 = m_3 = 1 \) and \( m_4 = 2 \). Let \( x \in K_1^* \) and \( y \in K_3^* \). Then, by the definition of \( (\cdot)^* \) operation, \([x, y] \in K_3 \) and \([x, y]^{-1} = [y, x] \in K_1 \). So, \([x, y] \in K_1 \cap K_3 = \{1\} \). Therefore, \( K_1^* \subseteq C_G(K_3^*) \). Similarly, \( K_2^* \subseteq C_G(K_3^*) \), and hence \( K_1^* K_2^* \subseteq C_G(K_3^*) \). If \( C_G(K_3^*) = G \), then it follows that \( K_3^* = Z(G) \), and so \( |\frac{G}{Z(G)}| = |\frac{G}{K_3^*}| = 9 \), a contradiction. Thus, we have \( K_1^* K_2^* \neq G \). But, by Lemma 2.1.3, we have

\[
\frac{|G|}{K_1^* K_2^*} = \frac{|G||Z(G)|}{|K_1^*||K_2^*|} = 1,
\]

and so \( K_1^* K_2^* = G \), a contradiction. Therefore, it is not possible to have \( m_1 = m_2 = m_3 = 1 \) and \( m_4 = 2 \). Hence, \( \frac{17}{81} \) (if \( m_1 = m_2 = m_3 = m_4 = 1 \)) and \( \frac{121}{729} \) (if \( m_1 = m_2 = 1, m_3 = m_4 = 2 \)) are the only values of \( \text{Pr}(G) \), in the interval \([\frac{11}{75}, 1] \), and, as noted above, on each occasion \( \frac{G}{Z(G)} \) is an elementary abelian 3-group with \( |\frac{G}{Z(G)}| \leq 81 \). In fact, if \( \text{Pr}(G) = \frac{121}{729} \), then we have \( |\frac{G}{Z(G)}| = 81 \). On the other hand, if \( \text{Pr}(G) = \frac{17}{81} \), then equality holds in Result 1.5.2, and so, using Result 1.2.12 and the second part of Result 1.5.13, we have \( |\frac{G}{Z(G)}| = 27 \).

For the rest of the possibilities for \( G' \), we have \( \text{Pr}(G) < \frac{11}{75} \), using Result 1.5.4 and Result 1.5.8.

**Case 2.** \( G' \cap Z(G) = \{1\} \).

In this case, by (2.1.c), \( C_G(G') \) is an abelian group. Moreover, in view of Lemma 2.1.1(a), Lemma 2.2.3 and Lemma 2.3.1, it is not possible to have
If $|G'| = 3, 5, 9, 15, 17$ or $21$, then $\text{Pr}(G)$ and $\frac{|G|}{|Z(G)|}$ are determined by Result 1.5.12. More precisely, since there is a unique odd divisor $n > 1$ of $p - 1$ for each $p \in \{7, 11, 13, 23\}$, we have $\text{Pr}(G) = \frac{5}{21}$ and $\frac{|G|}{|Z(G)|} = 21$ if $|G'| = 7$, and $\text{Pr}(G) = \frac{7}{39}$ and $\frac{|G|}{|Z(G)|} = 39$ if $|G'| = 13$, while $\text{Pr}(G) < \frac{11}{75}$ if $|G'| = 11$ or $23$. On the other hand, if $p = 19$, then there are two such odd divisors ($n = 3$ and $n = 9$) of $p - 1$. It can be seen that if $n = 3$, then $\text{Pr}(G) = \frac{3}{19}$ and $\frac{|G|}{|Z(G)|} = 57$, whereas $\text{Pr}(G) < \frac{11}{75}$ if $n = 9$.

If $|G'| = 25$, then, by Lemma 2.2.3, we must have $G' \cong C_5 \times C_5$ and $|G| \equiv 3 \mod(6)$. So, using Lemma 2.3.2(b) and the second part of Result 1.5.13, we have $\text{Pr}(G) = \frac{11}{75}$ and $\frac{|G|}{|Z(G)|} = 75$.

**Case 3.** $G' \not\subseteq Z(G)$ and $G' \cap Z(G) \neq \{1\}$. In this case, $|G'| = 9, 21$ or $25$ ($|G'| \neq 15$, by Lemma 2.3.1(a)).

If $|G'| = 9$, then $|G' \cap Z(G)| = 3$. Using Theorem 2.2.6, we see that $\frac{17}{81}$ is the only value of $\text{Pr}(G)$, in the interval $[\frac{11}{75}, 1]$, and it occurs when $C_G(G')$ is abelian. Also, in that case, we have $\frac{|G|}{|Z(G)|} = 27$.

If $|G'| = 21$, then, by Lemma 2.3.1(b), $\frac{|G|}{|C_G(G')|} = |G' \cap Z(G)| = 3$. Using Result 1.5.13 together with Lemma 2.2.2 and Lemma 2.3.1(b), we see that $\frac{20}{189}$ is the only value of $\text{Pr}(G)$, in the interval $[\frac{11}{75}, 1]$, and it occurs when $C_G(G')$ is abelian. Also, in that case $\frac{|C_G(G')|}{|Z(G)|} = 21$ (using Result 1.2.13), and hence $\frac{|G|}{|Z(G)|} = 63$.

Finally, if $|G'| = 25$, then $|G' \cap Z(G)| = 5$. But, using Theorem 2.2.6, we see that $\text{Pr}(G) < \frac{11}{75}$.

Thus, in view of Remark 2.2.1, the theorem is completely proved. \qed
We conclude the section with the following remark.

**Remark 2.3.5.** In [44, Section IV], Rusin classifies all finite groups $G$ with $\Pr(G) > \frac{11}{32}$. However, there are a few points that are worth noting.

(a) In Case 2, he surprisingly misses out one situation, namely,

$$\Pr(G) = \frac{5}{14}, \ G' \cong C_7, \ G' \cap Z(G) = \{1\}, \ \frac{G}{Z(G)} \cong D_{14},$$

where $D_{14}$ is the dihedral group of order 14. Accordingly, this situation does not appear in Table 1.5.3.

(b) In Case 3, he claims to have been able to show that if $|G'| = 4$ and $|G' \cap Z(G)| = 2$, then

$$\Pr(G) = \frac{1}{4} \left( 1 + \frac{1}{2^{2t}} + \frac{1}{2} \cdot \frac{1}{2^{2s}} \right),$$

where $2^{2s} = |C_G(G') : Z(C_G(G'))|, \ 2^{2t} = |\frac{G}{G' \cap Z(G)} : Z(\frac{G}{G' \cap Z(G)})|$ and $s + 1 \geq t \geq 1$. But, as noted in Theorem 2.2.6, we always have $t = 1$, and in that case, the value of $\Pr(G)$ obtained by him coincides with the one given by Theorem 2.2.6(a) for $p = 2$.

(c) While summarizing all the possibilities for $\Pr(G)$, Rusin writes that if $C_2 \times C_2 \cong G' \subseteq Z(G)$, then $\frac{G}{Z(G)} \cong C_2^3$ or $C_2^4$, no matter whether $\Pr(G) = \frac{7}{16}$ or $\frac{35}{64}$ (compare with Table 1.5.3). However, arguing in the same manner as we have done in a similar situation in the proof of Theorem 2.3.3, namely, $C_3 \times C_3 \cong G' \subseteq Z(G)$, it can be seen that $\frac{G}{Z(G)} \cong C_2^3$ or $C_2^4$ according as $\Pr(G) = \frac{7}{16}$ or $\frac{35}{64}$. 

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(d) He also writes that if \( \Pr(G) = \frac{3}{8} \), \( G' \cong C_6 \) and \( G' \cap Z(G) \cong C_2 \), then 
\[
\frac{G}{Z(G)} \cong C_2 \times S_3 \text{ or } T,
\]
where \( T \) is the non-abelian group of order 12 besides \( A_4 \) and \( C_2 \times S_3 \) (compare with Table 1.5.3). However, it is well-known that \( T \not\cong \frac{G}{Z(G)} \) for any \( G \).

2.4 A lower bound for \( \Pr(G') \)

Given a finite group \( G \) with \( |\text{cd}(G)| = 2 \), M. R. Pournaki and R. Sobhani [40, Corollary 2.3] have proved that

\[
\Pr(G) \geq \frac{1}{|G'|} \left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right)
\]

with equality if and only if \( G \) is of central type.

In this section, we obtain this lower bound for \( \Pr(G) \) without imposing any additional restriction on \( G \). We also derive certain necessary and sufficient conditions, in terms of standard group-theoretic concepts, for the attainment of this bound. As a consequence we obtain some characterizations for finite nilpotent groups whose commutator subgroups have prime order. In particular, we prove certain results, concerning finite groups having \( p \) as the smallest prime dividing their orders, whose proofs are either not available or not easily accessible in the literature.

**Theorem 2.4.1.** If \( G \) is a finite group, then

\[
\Pr(G) \geq \frac{1}{|G'|} \left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right).
\]

In particular, \( \Pr(G) > \frac{1}{|G'|} \) if \( G \) is non-abelian.
Proof. We prove this theorem in two different ways — one using the class equation (1.1.1) and the other using the degree equation (1.2.1).

Class equation method: Let $x_1, x_2, \ldots, x_t$ constitute the complete set of representatives of the conjugacy classes in $G$ consisting of non-central elements. Clearly, $t = k(G) - |Z(G)|$. So, by the class equation, we have

$$|G| = |Z(G)| + \sum_{i=1}^{t} |C_{G}(x_i)| \leq |Z(G)| + |G'|(k(G) - |Z(G)|),$$

using (1.1.1). Thus,

$$\Pr(G) = \frac{k(G)}{|G|} \geq \frac{1}{|G'|} \left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right).$$

Degree equation method: Note that $\chi(1)^2 \leq |G : Z(G)|$ for all $\chi \in \text{Irr}(G)$. Also, the number of linear characters in $\text{Irr}(G)$ is given by $|G : G'|$. So, by the degree equation,

$$|G| = |G : G'| + \sum_{\chi \in \text{Irr}(G), \chi(1) \neq 1} \chi(1)^2 \leq |G : G'| + \sum_{\chi \in \text{Irr}(G), \chi(1) \neq 1} |G : Z(G)|$$

$$= |G : G'| + |G : Z(G)|(k(G) - |G : G'|).$$

Thus, once again, we have the desired inequality. \qed

Remark 2.4.2. If $G$ is a finite group, then, using Result 1.5.7 with $H = Z(G)$, we also have

$$\Pr(G) \geq \frac{1}{|G : Z(G)|^2}.$$
However, it can be readily checked that
\[
\frac{1}{|G : Z(G)|^2} \leq \frac{1}{|G'| \left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right)}
\]
with equality if and only if \( G \) is abelian.

The following theorem while providing us with several equivalent conditions, that are necessary as well as sufficient for equality to hold in Theorem 2.4.1, relates itself to the subject matter of [17] and [36], dealing respectively with the groups of central type that are nilpotent and the \( CN \)-groups, that is, the groups in which the centralizer of every element is normal. It is also very much in line with Remark 1.1.2.

**Theorem 2.4.3.** For a finite non-abelian group \( G \), the statements given below are equivalent.

(a) \( \text{Pr}(G) = \frac{1}{|G'| \left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right)} \).

(b) \( \text{cd}(G) = \{1, |G : Z(G)|^{1/2}\} \), which means that \( G \) is of central type with \( |\text{cd}(G)| = 2 \).

(c) \( |C_G(x)| = |G'| \) for all \( x \in G - Z(G) \).

(d) \( C_G(x) = G'x \) for all \( x \in G - Z(G) \); in particular, \( G \) is a nilpotent group of class 2.

(e) \( C_G(x) \leq G \) and \( G' \cong \frac{G}{C_G(x)} \) for all \( x \in G - Z(G) \); in particular, \( G \) is a \( CN \)-group.

(f) \( G' = \{[y, x] : y \in G\} \) for all \( x \in G - Z(G) \); in particular, every element of \( G' \) is a commutator.
Proof. The equivalence of (a) and (b) follows from the degree equation method used for proving Theorem 2.4.1; on the other hand, the class equation method gives the equivalence of (a) and (c).

The equivalence of (c) and (d) follows from (1.1.d). For the particular case in (d), note that if $G' \not\subseteq Z(G)$, then there exists a commutator $g \in G - Z(G)$, and so, we have $C_G(g) = G'g = G'$, which is impossible.

If (d) holds, then, for each $x \in G - Z(G)$, the map $f : G \rightarrow G'$, given by $f(y) = yxy^{-1}x^{-1}$, defines a surjective homomorphism with kernel $C_G(x)$. Thus (e) holds. On the other hand, (c) follows immediately from (e).

Finally, it is easy to see that (d) and (f) are equivalent. \hfill \square

Theorem 2.4.1 and Theorem 2.4.3 not only allow us to obtain some characterizations for finite nilpotent groups of class 2 whose commutator subgroups have prime order, but also enable us to re-establish certain facts (essentially due to K. S. Joseph [26]) concerning the smallest prime divisors of the orders of finite groups.

Proposition 2.4.4. Let $G$ be a finite group and $p$ be the smallest prime divisor of $|G|$.

(a) If $p \neq 2$, then $\Pr(G) \neq \frac{1}{p}$.

(b) When $G$ is non-abelian, $\Pr(G) > \frac{1}{p}$ if and only if $|G'| = p$ and $G' \subseteq Z(G)$.

Proof. (a) If $\Pr(G) = \frac{1}{p}$, then Result 1.5.2 gives $|G'| \leq p + 1$. In addition, if $p \neq 2$, we have $|G'| = p$. But then, by Theorem 2.4.1, we have $\Pr(G) > \frac{1}{p}$, which is a contradiction. This proves part (a).
(b) If $\Pr(G) > \frac{1}{p}$, Result 1.5.2 gives $|G'| < p + 1$, whence $|G'| = p$. Now, by Lemma 2.1.1(a), we have

$$\left| \frac{G}{C_G(G')} \right| \leq |\text{Aut}(G')| = p - 1.$$  

Hence, we must have $C_G(G') = G$; equivalently, $G' \subseteq Z(G)$. This proves part (b), noting that its converse follows from Theorem 2.4.1. \[
\square
\]

As an immediate consequence, we have

**Corollary 2.4.5.** If $G$ is a finite group with $\Pr(G) = \frac{1}{3}$, then $|G|$ is even.

From Proposition 2.4.4, we also have, as an immediate corollary, the following improvement to Result 1.5.6.

**Corollary 2.4.6.** Let $G$ be a finite group and $p \neq 2$ be the smallest prime divisor of $|G|$. If $G$ is non-abelian with $G' \cap Z(G) = \{1\}$, then $\Pr(G) < \frac{1}{p}$.

Finally, in this section we have the following result.

**Proposition 2.4.7.** Let $G$ be a finite group and $p$ be a prime. Then the following statements are equivalent.

(a) $|G'| = p$ and $G' \subseteq Z(G)$.

(b) $G$ is of central type with $|\text{cd}(G)| = 2$ and $|G'| = p$.

(c) $G$ is a direct product of a $p$-group $P$ and an abelian group $A$ such that $|P'| = p$ and $\gcd(p, |A|) = 1$.

(d) $G$ is isoclinic to an extra-special $p$-group; consequently, $|G : Z(G)| = p^{2k}$ for some positive integer $k$.

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In particular, if \( G \) is non-abelian and \( p \) is the smallest prime divisor of \( |G| \), then the above statements are also equivalent to the condition \( \Pr(G) > \frac{1}{p} \).

Proof. The equivalence of (a) and (b) follows from Theorem 2.4.3(b)(d) and Result 1.2.7.

In view of Result 1.1.3, (c) follows from (a), considering \( P \) to be the Sylow \( p \)-subgroup of \( G \) and \( A \), the product of the other Sylow subgroups (if any) of \( G \).

If (c) holds, then \( G \) is isoclinic to the \( p \)-group \( P \). Since \( |P'| = p \) and \( P' \cap Z(P) \neq \{1\} \), we have \( P' \subseteq Z(P) \). Therefore, from Result 1.1.8, it follows that \( P \) is isoclinic to a finite group \( H \) such that \( Z(H) = H' \cong P' \), which in view of Result 1.1.3, makes \( H \) into an extra-special \( p \)-group. Thus, (d) follows from the fact that isoclinism is transitive (using Result 1.1.6 for the second part).

If (d) holds, then it follows that \( |G'| = p \) and \( G/Z(G) \) is abelian. Hence, we have (a).

The particular case follows from Proposition 2.4.4(b).

\( \square \)

Remark 2.4.8. In view of Result 1.2.8, the expression \( |\text{cd}(G)| = 2 \) in the statement (b) of Proposition 2.4.7 is superfluous.

2.5 Some additional observations

Consider the symmetric group \( S_n \) of degree \( n \geq 3 \) and the corresponding alternating group \( A_n \). It is well-known that

\[
\Pr(S_n) = \frac{P(n)}{n!},
\]

(2.5.a)
where \( P(n) \) denotes the number of partitions of \( n \). This, in fact, follows from a well-known result that \( k(S_n) = P(n) \), where \( k(S_n) \) is the number of conjugacy classes of \( S_n \).

In [7], J. Dénes, P. Erdős and P. Turán have derived that \( k(A_n) = \frac{1}{2}(P(n) + 3Q(n)) \), where \( Q(n) \) is the number of partitions of \( n \) having distinct odd parts. Therefore, it follows that

\[
\Pr(A_n) = \Pr(S_n) + \frac{3Q(n)}{n!}.
\]

Thus, \( \Pr(A_n) > \Pr(S_n) \).

Consider the dihedral group \( D_{2n} \), \( n \geq 1 \), and the quaternion group \( Q_{2n+1} \), \( n \geq 2 \). In [31], P. Lescot has deduced that

\[
\Pr(D_{2n}) = \begin{cases} 
\frac{n+3}{4n} & \text{if } n \text{ is odd} \\
\frac{n+6}{4n} & \text{if } n \text{ is even}
\end{cases}
\]

and

\[
\Pr(Q_{2n+1}) = \frac{2^{n-1} + 3}{2^{n+1}}.
\]

Clearly, \( \Pr(D_{2n}) \to \frac{1}{4} \) and \( \Pr(Q_{2n+1}) \to \frac{1}{4} \) as \( n \to \infty \). In this regard, Lescot enquired whether there is any other natural family of groups with the same property. It may be mentioned here that the following groups also have the same property:

\[
M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1} \rangle,
\]

\[
Q_{4m} = \langle a, b : a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1} \rangle
\]

and

\[
SD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{-1+2^{n-2}} \rangle.
\]

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This is because certain tedious computations yield

\[
\Pr(M_{2mn}) = \begin{cases} 
\frac{m+3}{4m} & \text{if } m \text{ is odd} \\
\frac{m+6}{4m} & \text{if } m \text{ is even}, 
\end{cases}
\]

\[
\Pr(Q_{4m}) = \frac{m+3}{4m}
\]

and

\[\Pr(SD_{2^n}) = \frac{2^{n-2} + 3}{2^n}.
\]

In [16, Corollary 1.2], I. V. Erovenko and B. Sury have shown, in particular, that for every integer \(k > 1\) there exists a family \(\{G_n\}\) of finite groups such that \(\Pr(G_n) \to \frac{1}{k^2}\) as \(n \to \infty\). In this connection, we have the following observation.

**Proposition 2.5.1.** For every integer \(k > 1\) there exists a family \(\{G_n\}\) of finite groups such that \(\Pr(G_n) \to \frac{1}{k}\) as \(n \to \infty\).

**Proof.** In view of Result 1.5.4, it is enough to show the existence of a family of finite groups \(\{G_n\}\) such that \(\Pr(G_n) \to \frac{1}{p}\) as \(n \to \infty\). Put \(G_n = ES(n, p)\), an extra-special \(p\)-group of order \(p^{2n+1}\). Then by Result 1.5.9, we have

\[
\Pr(G_n) = \frac{1}{p} + \frac{p-1}{p^{2n+1}} \to \frac{1}{p} \quad \text{as } n \to \infty.
\]

This completes the proof. \(\square\)

We conclude this section with the following proposition which says that the reciprocal of every positive integer can be realized as the commutativity degree of some finite group.

**Proposition 2.5.2.** For every positive integer \(n\) there exists a finite group \(G\) such that \(\Pr(G) = \frac{1}{n}\).
Proof. We shall prove the proposition by induction on \( n \). If \( n = 1 \), we may take \( G \) to be any abelian group. If \( n = 2 \), we may take, in view of (2.5.a), \( G = S_3 \). So, assume that \( n \geq 3 \) and that the proposition is true for all positive integers \( k < n \).

**Case 1.** \( n \equiv 0 \text{ or } 2 \pmod{4} \).

In this case, \( n = 2^\alpha m \), where \( \alpha, m \) are positive integers and \( m \) is odd. Clearly \( m < n \). So, by induction hypothesis there exists a finite group \( G \) such that \( \Pr(G) = \frac{1}{m} \). Hence, using (2.5.a) and Result 1.5.4, we have

\[
\Pr(G \times (S_3)^\alpha) = \Pr(G) \cdot (\Pr(S_3))^\alpha = \frac{1}{m} \cdot \frac{1}{2^\alpha} = \frac{1}{n}.
\]

**Case 2.** \( n \equiv 1 \pmod{4} \).

In this case, \( \frac{n+3}{4} \) is a positive integer and \( \frac{n+3}{4} < n \). So, by induction hypothesis, there exists a finite group \( G \) such that \( \Pr(G) = \frac{4}{n+3} \). Hence,

\[
\Pr(D_{2n} \times G) = \frac{n+3}{4n} \cdot \frac{4}{n+3} = \frac{1}{n}.
\]

**Case 3.** \( n \equiv 3 \pmod{4} \).

In this case, \( \frac{n+1}{4} \) is a positive integer and \( \frac{n+1}{4} < n \). So, by induction hypothesis, there exists a finite group \( G \) such that \( \Pr(G) = \frac{4}{n+1} \). Hence,

\[
\Pr(D_{6n} \times G) = \frac{3n+3}{12n} \cdot \frac{4}{n+1} = \frac{1}{n}.
\]

This completes the proof. \( \square \)