Chapter-4

BULK DEMAND INVENTORY SYSTEM WITH RANDOM LEAD TIME AND SERVER VACATION

4.1 Introduction

Apart from the previous chapters which deal with zero lead time inventory problems, the present chapter investigates an (s,S) inventory model with random lead time. One more factor that plays a role here is the server vacation which is initiated as soon as the inventory becomes dry. This type of model fits into a number of real situations corresponding to the seller's market.

Random lead time inventory problems as treated as a stochastic process is analogous to a queueing problem (see chapters 15-17 of Studies in Mathematical Theory of Inventory and Production by Arrow, Karlin and Scarf (1958)) with random arrivals (deliveries) and departures (demands). Scarf (1960) treats a dynamic inventory model with random lead times, but under the restriction that a new order may be placed only at a time when there are no outstanding orders. Detailed analysis of continuous review (s,S) inventory systems have been carried out by several other authors and results relating to the probability distribution of the inventory level and the optimal choice of the levels s and S have been discussed. Sahin (1979) treats an inventory model where the demand quantities follow a continuous distribution with lead time remaining a constant.

Daniel and Ramanarayanan (1988) is the first to introduce server vacation to inventory models. They assume that the quantity demanded by each arriving customer is exactly one. Madhusoodanan (1989) considers a model similar to that of our present model in which he extends the technique of Ramanarayanan and Jacob (1987) to the situation where the server goes for vacation. This method has a drawback, namely that it uses the matrix of transition time densities and its convolutions to arrive at the expression for the probability distribution of the inventory level. Here we give an expression for the system size probabilities using a simple technique.

Section 4.2 introduces the model and explains notations and the assumptions of this chapter. Section 4.3 shows how the model is analysed by embedding a Markov renewal process in the random process representing inventory level. The system size probabilities and also the reliability of the system at arbitrary time point are obtained in Sections 4.4 and 4.5 respectively

4.2. Description

We consider a continuous review (s,S) inventory system with quantity demanded by each arriving customer following a discrete distribution on the set $E = \{1,2,....,a\}$ with a as the maximum quantity that can be demanded. The interarrival times of demands are independent and identically distributed random variables having distribution function G(.) which is absolutely continuous with density g(.). The maximum capacity of the store is fixed as S units. Due to demands that take place over time, the level of the inventory falls and when the level reaches s or below for the first time an order is placed for replenishment. If the ordering level is i, then the ordering quantity is S-i. The lead times are assumed to be independent and identically distributed random variables with distribution function F(.) and density function f(.). These are independent of the demand process and the ordering level. If order materialisation does not take place when the inventory level falls to zero, the server goes on vacation for a random duration having distribution function H(.) having density function h(.). On return if the server finds that the order has not materialised he again goes for vacation of random duration which is independent of and having the same distribution as the previous one. This process continues until on return he finds the order having realised. The demands that emanate during a dry period will not be met and therefore will be deemed to be lost. The vacation durationsare also assumed to be independent of the demand process and lead times.

The notations used in this chapter are explained below.

$[x]$ denotes the largest integer less than or equal to x.

$G(.)$ and $g(.)$ respectively, represent the distribution function and density function of interarrival time of demands.

$F(.)$ and $f(.)$ stand, respectively, for the lead time distribution and density.

$H(.)$ and $h(.)$ are the distribution function and density function of vacation times.

p_i = Probability that i units are demanded by an arriving customer $(i=1,2,...,a)$

$*$ denotes convolution

$$\phi(s) = \sum_{i=1}^{a} p_i s^i$$

$p_i^{(b)}$ = Probability of b consecutive demands consuming i units. This is the coefficient of s^i in $[\phi(s)]^{*b}$

$\hat{p}_i$, for $i=1,2,....,a$ denotes the probability of at least (of course not more than a) i units being demanded by a customer.

$\hat{p}_i^{(n)}$ stands for the probability of at least i units being demanded by n customers.

$m(.) = \sum_{n=o}^{\infty} h^{*n}(.)$ ie. the renewal density of vacation.

$A(.)$ = The renewal density of lost demands (during a dry period).

$I(t)$ = Inventory level (onhand inventory) at time t, $t \geq 0$.

B is the set of points $\{s-a+1, s-a+2, \ldots, s-1, s\}$

Assumptions

We assume that the maximum quantity demanded by an arriving customer is a with $1<a<s$. Also it is assumed that $S > 2s$. These assumptions are made to avoid perpetual shortage. Nevertheless, they are not explicitly used. Even when quantity demanded exceeds what is available, the customer goes off with the available number of items.

4.3. Analysis

Suppose $0 = T_o < T_1 < T_2 < \ldots < T_n \ldots$ be the successive time points at which orders are placed for replenishment and $X_o, X_1, X_2, \ldots, X_n, \ldots$ be the corresponding inventory levels (ordering levels), $X(T_n) = X_n$. Assume that $X(o) = i$, for $i = s-a+1, s-a+2, \ldots, s-1, s$ and hence an order is placed at the instant of commencement of inventory. Then we have

<u>Theorem</u>

$(X,T) = \{(X_n,T_n),\ n=0,1,2,\ldots\}$ forms a Markov Renewal process (MRP) with semi Markov kernel

$$Q(i,j,t) = \Pr\{X_{n+1}=j;\ T_{n+1}-T_n \leqslant t \mid X_n=i\}\ , i,j \in B \text{ and } t \geqslant 0$$

Proof follows easily from the definition of MRP.

To get $Q(i,j,t)$ we proceed as follows:

The event $\{X_{n+1}=j;\ T_{n+1}-T_n \leqslant t \mid X_n=i\}$ can occur in two mutually exclusive ways (and these are exhaustive):

(i) Before an order materialisation the inventory level drops to zero due to demands and so there is a dry period and hence the server goes on vacation.

(ii) No dry period between order placement and its materialisation.

Hence

$$Q(i,j,t) = Q_1(i,j,t) + Q_2(i,j,t)$$

where,

$Q_1(i,j,t)$ represents the transition probability from i to j in time less than or equal to t with order placed when level is at i, not materialising before the system emptying and $Q_2(i,j,t)$ that without any dry period between transition from i to j, that is, in this case the order which is placed when level reaches $i (i \leqslant s)$ for the first time after the previous replenishment, materializes before the system becomes empty and then due to a number of demands the next replenishment order is placed when the inventory level reaches $j (j \leqslant s)$ for the first time after replenishment.

We have,

$$Q_1(i,j,t) = \int_{z=o}^{t} \int_{u=z}^{t} \int_{w=u}^{x} \int_{x=w}^{t} \int_{y=u}^{t} \Big(\sum_{b=1}^{i-1} \sum_{r=\max\{1,[\frac{b}{a}]\}}^{b} g^{*r}(z) p_b^{(r)} g(u-z) \hat{p}_{i-b}$$

$$m(w-u)\; h(x-w)\; \frac{(F(x)-F(u))}{(1-F(w))}\; A(y-u)$$

$$\sum_{n=\max\{1,[\frac{S-i-j}{a}]\}}^{S-i-j} \frac{G^{*n}(t-y)-G^{*(n+1)}(t-y)}{(1-G(x-y))}\; p_{S-i-j}^{(n)}$$

$$dy\; dx\; dw\; du\; dz \qquad (1)$$

$$Q_2(i,j,t) = \int_{u=o}^{t} \int_{v=u}^{t} \sum_{b=o}^{i-1} \sum_{r=[\frac{b}{a}]}^{b} g^{*r}(u)$$

$$p_b^{(r)}\; f(v) \sum_{n=\max\{1,[\frac{S-b-j}{a}]\}}^{S-b-j} \frac{G^{*n}(t-u)-G^{*(n+1)}(t-u)}{(1-G(v-u))}$$

$$p_{S-b-j}^{(n)}\; dv\; du \qquad (2)$$

where we define $p_o^{(o)}$ as 1 and $p_b^{(o)} = 0$ for $b > 0$.

The right hand side of (1) is arrived at as follows:

The inventory level drops to $i \in B$ (B being visited for the first time after the previous replenishment). We take this as the time origin. Then r demands take place until time z (the r^{th} being at z) which together take away atmost i-1 units and the next demand that take place at time u makes the inventory dry whereupon the server goes on vacation. There are a number of demands lost, the last one taking place at time y (this is represented by A(y-u)). The server returns after each vacation to find the inventory dry and hence goes back for a fresh vacation (this is represented by m(w-u)). The last vacation is completed at time x since the replenishment takes place in (w,x). Now the inventory level is (S-i). Exactly n demands take place bringing down the inventory to $j \in B$ (this is the first visit to B after the previous replenishment). Hence an order is placed for (S-j) units. A similar argument yields the right hand side of (2) except that in this case there is no dry period.

Now define $R(i,j,t) = \sum_{n=o}^{\infty} Q^{*n}(i,j,t)$ which is the expected number of visits to state j in (o,t] starting initially at i, $i,j \in B$, $t \geq 0$ where

$$Q^{o}(i,j,t) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

4.4 System size probabilities

The stock level $\{I(t), t \geq 0\}$ is a discrete valued stochastic process defined on the state space $\{0,1,2,\ldots,S\}$. Define

$$X(t) = X_n, \quad \text{for} \quad T_n \leq t < T_{n+1}$$

Let $Z(t) = (I(t), X(t))$. Then clearly $\{Z(t), t \geq 0\}$ is a semi-regenerative process with state space $\{0,1,2,\ldots,S\}$ x $\{s-a+1, s-a+2,\ldots,s-1,s\}$ and (X,T) is the Markov renewal process embedded in it. Further assume that (X,T) is irreducible, recurrent and aperiodic.

Let $P((n,j),t) = Pr\{Z(t)=(n,j)\}$, for $n=0,1,2,\ldots,S$ and $j=s-a+1,\ldots,s-1,s$

Then

$$P((n,j),t) = Pr\{Z(t)=(n,j), T_1 > t\} + Pr\{Z(t)=(n,j), T_1 \leq t\}$$

$$= K((n,j),t) + \sum_{i \in B} \int_0^t Q(i,j,du)\, P((n,j),t-u) \qquad (3)$$

where for every $(n,j) \in \{0,1,2,\ldots,S\}$ x $\{s-a+1, \ldots, s-1,s\}$ the mapping $t \longrightarrow P((n,j),t)$ is Borel measurable and bounded over finite intervals. Further $K((n,j),t)$ is directly Riemann integrable for every $(n,j) \in \{0,1,2,\ldots,S\}$ x $\{s-a+1,\ldots,s-1,s\}$. Therefore the Markov renewal equation (3) has one and only one solution given by

$$P((n,j),t) = \sum_{i \in B} \int_0^t R(i,j,du)\, K((n,j),t-u), \quad n=0,1,2,\ldots,S; \quad j=s-a+1,\ldots,s \text{ and } t \geq 0.$$

From this we can compute $P((n,j),t)$ for different values of n and j.

$K((n,j),t)$ for different values of n and j are computed as follows:

Case (i): When n=S, $K((S,j),t) = (1-G(t))\, F(t), \; j \in B$

Case (ii): For $S-j+1 \leq n < S$

$$K((n,j),t) = \int_{v=0}^{t} \int_{z=0}^{v} \sum_{c=0}^{j-1} \sum_{r=\max\{0,[\frac{c}{a}]\}}^{b} (g^{*r}(z)\, p_c^{(r)})\, f(v)$$

$$\sum_{b=\max\{1,[\frac{S-c-n}{a}]\}}^{S-c-n} \frac{G^{*b}(t-z)-G^{*(b+1)}(t-z)}{1-G(v-z)}\; p_{S-c-n}^{(b)}\, dv\, dz$$

It is to be noted that in this case, there can be no dry period and this is taken care of by the probability of atmost $c(\leq j-1)$ units being sold off until replenishment takes place.

Case (iii): For $s+1 \leq n \leq S-j$

$$K((n,j),t) = \int_{z=o}^{y} \int_{v=z}^{y} \int_{x=v}^{t} \int_{w=v}^{x} \int_{y=v}^{x} \sum_{c=1}^{j-1} \sum_{r=\max\{1,[\frac{c}{a}]\}}^{c} g^{*r}(z)$$

$$p_c^{(r)} g(v-z)\ \hat{p}_{j-c}\ m(w-v)\ h(x-w)\ \frac{F(x)}{1-F(w)}$$

$$A(y) \sum_{b=\max\{1,[\frac{S-j-n}{a}]\}}^{S-j-n} \frac{G^{*b}(t-y)-G^{*(b+1)}(t-y)}{1-G(x-y)}\ p_{S-j-n}^{(b)}$$

$$dy\ dw\ dx\ dv\ dz$$

$$+ \int_{z=o}^{t} \int_{v=o}^{t} \sum_{c=o}^{j-1} \sum_{r=\max\{o,[\frac{c}{a}]\}}^{c} g^{*r}(z)\ p_c^{(r)} f(v)$$

$$\frac{(G^{*b}(t-z)-G^{*(b+1)}(t-z))}{1-G(v-z)}\ p_{S-c-n}^{(b)}\ dv\ dz$$

In the above the first term on the right hand side represents the situation where there is a dry period and the second term is for the case with no dry period. The way in which they are arrived at is on the same lines as that used for arriving at (1) and (2) of Section 4.3.

<u>Case (iv)</u>: For n satisfying the condition $n=j=s-a+1, s-a+2, \ldots, s-1, s$.

$$K((j,j),t) = (1-G(t))(1-F(t))$$

<u>Case (v)</u>: For $0<n<j$

$$K((n,j),t) = (1-F(t)) \sum_{b=\max\{1,[\frac{j-n}{a}]\}}^{j-n} (G^{*b}(t)-G^{*(b+1)}(t))\, p_{j-n}^{(b)}$$

and finally,

<u>Case (vi)</u>: For $n=0$, we get

$$K((o,j),t) = (1-F(t)) \sum_{b=\max\{1,[\frac{j}{a}]\}}^{\infty} (G^{*b}(t)-G^{*(b+1)}(t))\, p_j^{(b)}$$

In the expression for $P((o,j),t)$ we allow b to take arbitrary large values. This only means that demands take place even when inventory is dry and hence they are not met.

4.5. System Reliability

System reliability at time t denoted by $R_1(t)$, is defined as the probability that the system is working at time t. In this case $R_1(t)$ is the probability that the server is available and hence inventory level is larger than zero.

$$R_1(t) = \Pr\{Z(t) \neq (o,j)\}$$

The event $\{Z(t) \neq (o,j)\}$ can happen in three mutually exclusive ways as follows:

(i) Last order is placed in (u,u+du]; but no replenishment until t and no dry period until time t.

(ii) Last order is placed in (u,u+du]; replenishment takes place before inventory level becomes zero

(iii) Last order is placed in (u,u+du]; replenishment takes place during a dry period; the server returns after vacation before time t.

Therefore

$$R_1(t) = \sum_{i \in B} \int_o^t R(i,j,du)\Big\{(1-F(t-u)) \int_{v=u}^{t} \sum_{b=o}^{j-1} \sum_{r=[\frac{b}{a}]}^{b} g^{*r}(v-u)\,p_b^{(r)} (1-G(t-v))\,dv\Big]$$

$$+\Big\{\int_{v=u}^{t} \sum_{b=o}^{j-1} \sum_{r=[\frac{b}{a}]}^{b} g^{*r}(v-u)\; p_b^{(r)}\; f(v-u)dv\Big\}$$

$$+\Big\{\int_{v=u}^{t} \int_{z=v}^{t} \int_{w=z}^{x} \int_{x=w}^{t} \int_{y=z}^{x} \sum_{b=1}^{j-1} \sum_{r=\max\{1,[\frac{b}{a}]\}}^{b} g^{*}(v-u)p_b^{(r)} g(z-v)$$

$$\hat{p}_{j-b}\; m(w-z)\; h(x-w)\; A(y-z)\; \left(\frac{F(x)-F(u)}{1-F(w)} \right)dx\; dy\; dw\; dz\; dv\Big\}\Big\}$$