Chapter-3

SOME INVENTORY MODELS WITH MARKOV DEPENDENCE*

3.1 Introduction

In the previous chapter, the assumption of Markov dependence was made on the quantity demanded by successive arrivals. In this chapter the dependence structure is introduced in the (s,S) inventory models in two different ways. In Model I, the successive quantities replenished are dependent- dependence being on the just previous replenished quantity only, whereas in Model II the reorder levels vary according to a Markov chain. Both models deal with zero lead time. Model I considers the case of bulk demands and Model II that of unit demand.

Ever since the book by Arrow, Karlin and Scarf appeared (1958), many researchers have formulated discrete or continuous review inventory problems through (s,S) policy. Sahin (1983) examines an (s,S) inventory model with bulk demands and random lead time. She obtains the binomial moments of the time dependent and limiting distribution of the inventory deficit. This is also analysed by Ramanarayanan and Jacob (1987) where they examine only the time dependent behaviour of the system.

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* Model II discussed in this chapter appeared in Opsearch, Vol.27, No.1, 1990.

An (s,S) policy with the quantity demanded not exceeding what is available in the stock is examined by Krishnamoorthy and Manoharan (1990). They have derived the system size distribution in the steady state.

In this chapter we consider two Models. In Model I the inventory level is not necessarily brought back to its maximum at a replenishment epoch; instead the successive replenished quantities are assumed to form a Markov chain defined over a state space to be specified. Such a situation arises in the case of financing companies which give loans for building constructions, purchase of vehicles etc. where a fresh loan quantity depends upon the previous loan amount which have been already availed.

Ramanarayanan and Jacob (1986) discuss the case of an (s,S) inventory model with unit demand, random lead time and varying ordering levels. The method suggested by them is not computationally tractable and further, passage to the limit is extremely difficult. Krishnamoorthy and Manoharan (1991) discuss the same model and obtain the correlation between the number of demands during a lead time and the length of the next inventory dry period. Model II is on an inventory policy with Markov dependent reordering levels.

Section 3.2 deals with the description of Model I. System size probability distribution at arbitrary time point and steady state behaviour are obtained. Illustration by a numerical example and cost function over a cycle are examined in the same section.

Section 3.3 is concerned with the description and analysis of Model II. System size probabilities and the limiting distribution are obtained. An optimal decision rule is also discussed. Further a numerical example is also given.

The following notations are used in this chapter:

$I(t)$ - Inventory level at time t ($t \geq 0$)

$*$ - Convolution. For example $(F*G)(t) = \int_{-\infty}^{\infty} F(t)\, dG(t-u)$

$f^{*n}(.)$ - n-fold convolution of $f(.)$ with itself.

p_k stands for the probability that k units are demanded by an arrival, $k = a, a+1, \ldots, b-1, b$.

a and b represent the minimum and maximum number of items that will be demanded by an arriving customer. We assume that $0 < a \leq b$ and $0 \leq s-b+1 \leq s$.

$E = \{c, c+1, c+2, \ldots, S-s\}$; $c \geq b$.

$A = \{s-b+1, s-b+2, \ldots, s-1, s\}$

$H = \{s+1, s+2, \ldots, S-1, S\}$

$\bar{E} = \{0,1,2,\ldots, s\}$

$\bar{A} = \{1,2,\ldots, S\}$

$N^0 = \{0,1,2,\ldots,\}$

$[x]$ - The largest integer less than or equal to x.

$P_{(i,I)}(n,t)$ - Probability that $I(t)=n$ given that the initial quantity replenished is i units and the initial ordering level is I.

$P_i(n,t)$ - Probability that $I(t)=n$ given that the initial ordering level is i.

3.2. Model I

This model considers an inventory policy where the quantity demanded by an arriving customer lie between a and b with a and b positive integers and $a \leq b$, $s-b+1 \geq 0$. The demand quantities are independent and identically distributed random variables having the discrete distribution p_k, $k=a,a+1,\ldots,b-1,b$. We assume that the time between demands are independent and identically distributed random variables, independent of demand magnitudes, with distribution function $G(.)$ which is absolutely continuous and $g(t)dt = dG(t)$ with first moment μ_1 (assumed finite). Lead time is zero and shortage is not permitted. The maximum capacity of the warehouse is fixed to be S units.

Due to demands that take place the inventory position decreases and as soon as the level falls to A due to a demand for the first time after each replenishment, an order is placed to bring the inventory to its maximum ie. if at the time of ordering the onhand inventory is i, $i \in A$, then the quantity ordered is (S-i) units. Replenishment is instantaneous with the assumption that the successive quantities replenished form a Markov chain defined on the state space E. Let the one-step transition probability matrix associated with this Markov chain be

$$\mathbb{P}_1 = ((\, q_{ij} \,)),\ i,j \in E \qquad (1)$$

<u>Analysis</u>

Suppose $0 = T_o < T_1 < T_2 < \ldots < T_n < \ldots$ are the successive time epochs at which the ordering level falls to A for the first time after the previous replenishments. Specifically let $Y_o, Y_1, Y_2, \ldots$ be the ordering levels and $X_o, X_1, X_2, \ldots$ be the quantities replenished at these epochs. Then by our assumption $\{X_n,\ n=0,1,2,\ldots\}$ forms a Markov chain defined over the state space E with the one-step transition probabilities q_{ij} as defined in (1).

Initially at time $T_o = 0$, due to a demand, let $Y_o=s$ so that a replenishment by a quantity say $X_o = S-s$ occurs at the instant of commencement of inventory. (One can as well proceed with the assumption $X_o=i$ with probability q_i, $i \in E$).

Identify $0 = T_o = T_{o,o}, T_{o,1}, \ldots, T_{o,r_o} = T_1 = T_{1,o}, T_{1,1}, T_{1,2}, \ldots, T_{1,r_1} = T_2 = T_{2,o}, \ldots$ as the successive demand epochs. Then $\{T_{n,i} - T_{n,i-1},\ i=1,2,\ldots,r_n\ ;\ n \in N^o\}$ is a sequence of positive, independent and identically distributed random variables and so forms a renewal process. Introduce yet another sequence of random variables $\{Z_{n,i},\ i=1,2,\ldots,r_n;\ n \in N^o\}$ where $Z_{n,i}$ represents the inventory level just after meeting the demand at $T_{n,i}$. The process $\{(X,Y,Z)\} = \{(X_n, Y_n, Z_{n,i}),\ i = 1,2,\ldots, r_n;\ n \in N^o\}$ turns out to be a three dimensional Markov chain. Then we have

Theorem-1

The stochastic process $\{(X,Y,Z),T\} = \{(X_n, Y_n, Z_{n,i}), T_{n,i};\ i = 1,2,\ldots, r_n;\ n \in N^o\}$ is a Markov Renewal Process defined over the state space E x A x H with the semi-Markov kernel defined by $((Q\{(\ell,L,k)(j,J,m),t\}))$ where

(i) between two consecutive demand epochs both of which are not replenishment epochs

$$Q\{(j,J,k),(j,J,m),t\} = Pr\{X_n=j, Y_n=J, Z_{n,i+1}=m;\ T_{n,i+1}-T_{n,i} \leq t \mid X_n=j, Y_n=J, Z_{n,i}=k\}$$

$$= \int_0^t p_{k-m}\, g(u)du,\quad j \in E;\ J \in A;\ m,k \in H;\ t \geq 0$$

and

(ii) between two successive demand epochs in which the current demand epoch happens to be a replenishment epoch as well

$$Q\{(\ell,L,k)(j,J,m),t\} = \Pr\{X_{n+1}=j,\ Y_{n+1}=J,\ Z_{n+1}=m;\ T_{n+1}-T_{n,r_n-1}\leqslant t \mid X_n=\ell,\ Y_n=L, Z_{n,r_n-1}=k\}$$

$$= \int_0^t p_{k-J}\, q_{\ell j}\, g(u)du,\quad \ell,j\in E;\ L,J\in A;\ k,m\in H;\quad n = 0,1,2,\ldots$$

<u>Proof</u>

The interarrival times of demands $T_{n,i} - T_{n,i-1}$, $i = 1,2,\ldots,r_n$; $n \in N^o$ are assumed to be independent and identically distributed random variables following distribution function G(.) and density function g(.). Further the demand quantities are independent and also does not depend upon the length of the time elapsed between demands. Hence considering time epochs like $T^+_{n,i-1}$ and $T^+_{n,i}$, ($T^+_{n,i}$ represents the time epoch just after meeting a demand) $i = 1,2,\ldots,r_n-1$, $n \in N^o$

$$\Pr\{X_n=j,\ Y_n=J,\ Z_{n,i}=m,\ T_{n,i}-T_{n,i-1}\leqslant t \mid X_o,X_1,\ldots,X_n=j;\ Y_o,Y_1,Y_2,\ldots,Y_n=J;\ Z_{n,1},Z_{n,2},\ldots,Z_{n,i-1}=k;\ T_{n,1},T_{n,2},\ldots,T_{n,i-1}\}$$

$$= \Pr\{(X_n=j,\ Y_n=J,\ Z_{n,i}=m);\ T_{n,i}-T_{n,i-1}\leqslant t \mid (X_n=j,Y_n=J,Z_{n,i-1}=k)\}$$

$$= Q\{(j,J,k),\ (j,J,m),t\},\quad j\in E;\ J\in A,\ m,k\in H;\ t \geqslant 0.$$

Here exactly one demand occurs and the demand is for a quantity (k-m) so that

$$Q\{(j,J,k),\ (j,J,m),t\} = \int_0^t p_{k-m}\, g(u)du$$

For case (ii) we consider demand epochs like $T^+_{n,r_{n-1}}$ and T^+_{n+1}. Due to a demand at $T_{n,r_n} = T_{n+1}$, the stock level drops to $J \in A$ so that the demand is for a quantity k-J where k is the inventory level prior to the demand at T_{n+1}. The replenishment at T_{n+1} is by a quantity j and the just previous replenished quantity is ℓ where $j, \ell \in E$. Then by the assumptions of our models,

$$\Pr\{(X_{n+1}=j,\ Y_{n+1}=J,\ Z_{n+1}=m;\ T_{n+1}-T_{n,r_n-1} \leqslant t \mid X_o, X_1, \ldots, X_n = \ell;$$
$$Y_o, Y_1, \ldots, Y_n = L\ ;\ Z_{n,1},\ Z_{n,2}, \ldots, Z_{n,r_n-1} = k; T_{n,1}, T_{n,2}, \ldots T_{n,r_n-1}$$
$$= \Pr\{X_{n+1}=j,\ Y_{n+1}=J,\ Z_{n+1}=m;\ T_{n+1}-T_{n,r_n-1} \leqslant t \mid X_n = \ ,Y_n=L, Z_{n,r_n-1}=k\}$$
$$= Q\{(\ell,L,k),\ (j,J,m),t\}$$
$$= \int_0^t p_{k-J}\ q_{\ell j}\ g(u)du$$

Hence the theorem.

The next step is to obtain an expression for the Markov renewal function. To this end we proceed as follows.

Consider two successive replenishment epochs T_n and T_{n+1}. Define $F\{(\ell,L,\ell+L),(j,J,j+J),t\}$ as the probability that $T_{n+1}-T_n \leqslant t$ and the replenished quantity and ordering level at T_{n+1} are, respectively, j,J conditional on ℓ and L as the replenished quantity and ordering level respectively at T_n, $\ell, j \in E$ and $L,J \in A$. Thus

$$F\{(\ell,L,\ell+L),(j,J,j+J),t\} = \{\Pr\ X_{n+1}=j,\ Y_{n+1}=J,\ Z_{n+1}=j+J;\ T_{n+1}-T_n \leqslant t \mid X_n=\ell,\ Y_n=L,\ Z_n=\ell+L\}$$

$$\ell, j \in E;\ L,J \in A;\ t \geqslant 0.$$

$$= \sum_{m=[\frac{\ell+L}{b}]}^{\ell+L-(s+1)+1} Q^{*m}\{(\ell,L,\ell+L),(j,J,j+J),t\}$$

$$= \int_0^t \sum_{m=[\frac{\ell+L}{b}]}^{\ell+L-(s+1)+1} g^{*m}(u)\ q_{\ell j} \sum_{\substack{i_1+i_2+\dots+i_m=\ell+L-J \\ i_1+i_2+\dots+i_{m-1}<\ell+L-s}} p_{i_1} p_{i_2} \dots p_{i_m}\, du$$

Define

$$R\{(\ell,L,\ell+L),(j,J,j+J),t\} = \sum_{n=0}^{\infty} F^{*n}\{(\ell,L,\ell+L),(j,J,j+J),t\}$$

with

$$F^{0}\{(\ell,L,\ell+L),(j,J,j+J),t\} = \begin{cases} 1 & \text{for } (\ell,L,\ell+L)=(j,J,j+J) \\ 0 & \text{otherwise} \end{cases}$$

and $F^{*n}\{(\ell,L,\ell+L),\ (j,J,j+J),t\}$ is obtained from the recursive relation

$$F^{*(n+1)}\{(\ell,L,\ell+L),(j,J,j+J),t\} = \sum_{i\in E}\ \sum_{I\in A}\ \int_0^t F\{(\ell,L,\ell+L),(i,I,i+I),du\}$$

$$F^{*n}\{(i,I,i+I),(j,J,j+J),t-u\},$$

$$\ell,j \in E,\ L,J \in A;\ t \geqslant 0.$$

Since $I(t)$ denotes the inventory level at time t, $I(t) = Z_{n,i}$ for $T_{n,i} \leqslant t < T_{n,i+1}$, $i=1,2,\ldots,r_n$; $n \in N^o$ and so $\{I(t), t \geqslant 0\}$ is a semi-Markov process defined over H.

Let $P_{(S-s,s)}(n,t) = \Pr\{I(t)=n|X_o=S-s,\ Y_o=s\}$ for $n\in H$, $t\geqslant 0$.

Then $P_{(S-s,s)}(n,t)$ satisfies the Markov renewal equations (Cinlar 1975a).

Thus we have

(i) for $n=S$

$$P_{(S-s,s)}(S,t) = K_{(S-s,s)}(S,t) + \sum_{i\in E}\ \sum_{I\in A}\ \int_0^t F\{(S-s,s,S),$$

$$(i,I,i+I),du\}$$

$$P_{(i,I)}(S,t-u)$$

where

$$K_{(S-s,s)}(S,t) = \Pr\{I(t)=S,\ T_{o,1}>t|X_o=S-s,\ Y_o=s\}$$

$$= 1-G(t)$$

(ii) for n = S-1, S-2, ..., s+1

$$P_{(S-s,s)}(n,t) = K^{(1)}_{(S-s,s)}(n,t) + \sum_{i\in E}\sum_{I\in A}\int_0^t F\{(S-s,s,S),(i,I,i+I),du\}\, P_{(i,I)}(n,t-u)$$

where

$$K^{(1)}_{(S-s,s)}(n,t) = \int_0^t \sum_{m=[\frac{S-n}{b}]+1}^{S-n} Q^{*m}\{(S-s,s,S),(S-s,s,n),du\}[1-G(t-u)]$$

Hence the solutions are given by

$$P_{(S-s,s)}(S,t) = \int_0^t R\{(S-s,s,S),(S-s,s,S),du\}\, K_{(S-s,s)}(S,t-u)$$

and for n = S-1, S-2,..., s+1

$$P_{(S-s,s)}(n,t) = \sum_{\substack{j\in E \\ j+J \geqslant n}}\sum_{J\in A}\int_0^t R\{(S-s,s,S),(j,J,j+J),du\}\, K^{(1)}_{(j,J)}(n,t-u)$$

where

$$K^{(1)}_{(j,J)}(n,t) = \int_0^t \sum_{m=[\frac{j+J-n}{b}]+1}^{j+J-n} Q^{*m}\{(j,J,j+J),(j,J,n),du\}\,[1-G(t-u)]$$

Steady state analysis

Let $\lim_{t \longrightarrow \infty} P_{(S-s,s)}(n,t) = \underline{P}(n)$ for $n \in H$

To obtain the limiting probabilities of the system size we proceed in the following manner.

We have seen that the three dimensional process $\{(X_n, Y_n, Z_{n,i}),\ i=1,2,\ldots,r_n;\ n \in N^o\}$ forms a Markov chain with state space

$$\{(i,j,k) \mid i=c,c+1,\ldots,M;\ j=s-b+1,\ s-b+2,\ldots,s-1,s;\ k=s+1,\ s+2,\ldots,j+i-1,\ j+i\}$$

where $k \leqslant S$. From the given one-step transition probabilities associated with the replenished quantities, the one-step transition probability matrix $\mathbb{P}_2 = ((\ p^{(1)}_{(i,j,k),(i',j',k')}\ ))$ associated with the three dimensional Markov chain can be obtained. The stationary distributions are then computed as

$$\pi(i',j',k') = \sum_{i \in E} \sum_{j \in A} \sum_{k \in H} \pi(i,j,k)\ p^{(1)}_{(i,j,k),(i',j',k')},$$

$$i' \in E,\ j' \in A;\ \text{and}\ k' \in H.$$

Since the transition to any state takes place at a demand epoch, the mean sojourn time in any state is given by

$$m(i,j,k) = \int_0^{\infty} (1-G(t)dt = \mu_1$$

which is the mean interarrival time between demands. The limiting probabilities are now computed as follows:

$$\underline{P}(S) = \frac{\pi(S-s,s,S)\ m(S-s,s,S)}{\sum_{i\in E}\sum_{j\in A}\sum_{k\in H}\pi(i,j,k)\ m(i,j,k)}$$

$$= \pi(S-s,\ s,S)$$

Similarly

$$\underline{P}(S-1) = \sum_{j=s-1}^{s}(\pi(S-s,j,S-1)) + \pi(S-s-1,\ s,S-1)$$

$$\underline{P}(S-2) = \sum_{j=s-2}^{s}(\pi(S-s,j,S-2)) + \sum_{j=s-1}^{s}(\pi(S-s-1,j,S-2)) + \pi(S-s-2,s,S-2)$$

$$\vdots$$

$$\underline{P}(s+1) = \sum_{i=c}^{S-s}\sum_{j=S-b+1}^{s}\pi(i,j,s+1)$$

Thus

$$\underline{P}(n) = \sum_{i=c}^{S-s}\sum_{j=s-b+1}^{s}\pi(i,j,n),\ n=s+1,s+2,\ldots,s-b+1+c$$

and

$$\underline{P}(S-m) = \sum_{j=s-m}^{s}(\pi(S-s,j,S-m)) + \sum_{j=s-(m-1)}^{s}(\pi(S-s-1,j,S-m)) + \ldots$$

$$+ \sum_{j=s-1}^{s}(\pi(S-s-(m-1),j,S-m)) + \pi(S-s-m,s,S-m),$$

$$m=0,1,2,\ldots,S-(s-b+1)-(c+1).$$

Cost function over a cycle

For the inventory model under consideration, a cycle is the length of duration between two successive replenishment epochs. A typical plot of the stock level is shown in Fig.1.

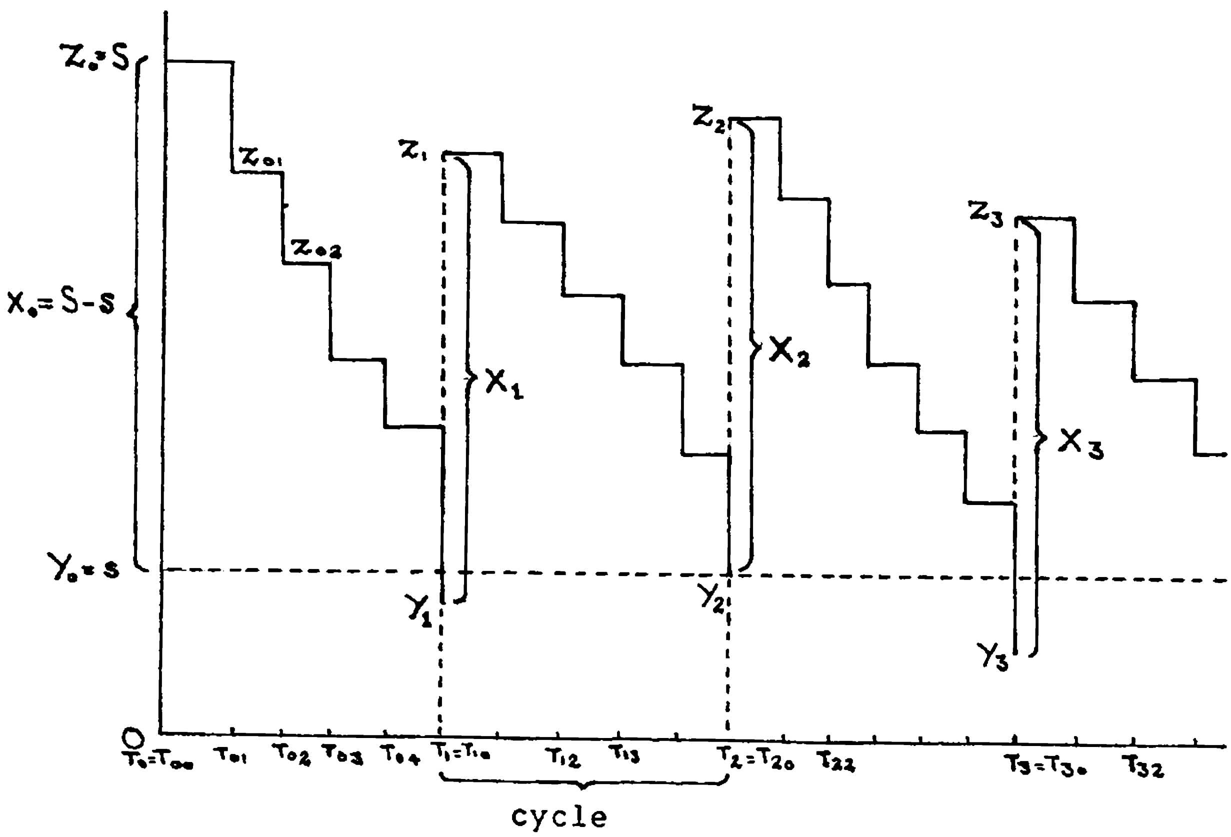

Fig.1.

The total cost over a particular cycle assuming j as the replenishing quantity under steady state is computed as follows: The objective function is the total expected cost per unit time over a cycle under steady state which is so chosen that it attains a minimum value corresponding to the quantity replenished.

Considering two successive replenishment epochs, let Y denotes the ordering level at the current replenishment epoch, Z be the length of the cycle just completed. Then the conditional density function of Y and Z given that X is the reordering level and j is the quantity replenished at the beginning of the cycle is denoted by

$$f_{j,x}(y,z)$$

Hence

$$f_{j,x}(y,z)dz = \sum_{k=[\frac{x+j}{b}]}^{x+j-(s+1)+1} P_{j,x}\left\{Y=y, z\leqslant Z<z+dz \;\middle|\; \begin{array}{l}\text{k demands occurred,}\\ \text{totally consuming}\\ \text{(x+j-y) items and}\\ \text{(k-1) demands}\\ \text{consumed less than}\\ \text{(x+j-s) items.}\end{array}\right\}$$

$$\times \Pr\left\{\begin{array}{l}\text{k demands occurred totally consuming (x+j-y) items}\\ \text{and (k-1) demands consumed less than (x+j-s) items}\end{array}\right\}c$$

$$= \sum_{k=[\frac{x+j}{b}]}^{x+j-(s+1)+1} g^{*k}(z) \sum_{\substack{i_1, i_2, \ldots, i_k \\ i_1+\ldots+i_k=(x+j-y) \\ i_1+\ldots+i_{k-1}<(x+j-s)}} p_{i_1} p_{i_2} \ldots p_{i_k} \, dz$$

Hence the conditional expected value of a cycle =

$$E_{j,x}(Z) = \int_0^\infty z\left(\sum_{y=s-b+1}^{s} f_{j,x}(y,z)\right)dz$$

The conditional expected inventory level over a cycle =

$$E_{j,x}(I) = \sum_{n=s+1}^{x+j} n \, Pr\{I=n \mid j,x\}$$

$$= \sum_{n=s+1}^{x+j} n \, \pi(j,x,n)$$

Hence the conditional expected total cost over a cycle per unit time is

$$\bar{F}_{j,x}(c) = \left\{ \sum_{y=s-b+1}^{s} \frac{K+\gamma j}{E_{j,x}(z)} \, \pi(j,y,y+j) \right\} + h \sum_{n=s+1}^{x+j} n \, \pi(j,x,n)$$

where K is the fixed cost of ordering, γ is the procurement cost per unit and h is the holding cost per unit per unit time.

Clearly the above function is convex.

Numerical example

Let the one-step transition probability matrix associated with the given Markov chain constituted by the successive quantities replenished be given by

$$\mathbb{P}_1 = ((\, q_{ij} \,)) = \begin{bmatrix} 1/2 & 1/2 \\ 1/5 & 4/5 \end{bmatrix}$$

with the maximum capacity of the warehouse being S=4 and let s=1. Let a=1 and b=2 with $p_1 = 1/3$ and $p_2 = 2/3$. Assume that the interarrival times of demands follow exponential distribution with parameter $\lambda = 0.5$.

The set E $= \{2,3\}$ so that $q_{22} = 1/2$; $q_{23} = 1/2$; $q_{32} = 1/5$ and $q_{33} = 4/5$.

From $\mathbb{P}_1$ we compute $\mathbb{P}_2$ defined over the state space $\{(2,0,2), (2,1,3), (2,1,2), (3,0,3), (3,0,2), (3,1,4), (3,1,3), (3,1,2)\}$ as follows:

$$p^{(1)}_{(2,0,2),(2,0,2)} = p_2\, q_{22}$$

$$p^{(1)}_{(2,0,2),(2,1,3)} = p_1\, q_{22}$$

$$\left.\begin{array}{l} p^{(1)}_{(2,0,2),(2,1,2)} \\ p^{(1)}_{(2,0,2),(3,0,2)} \\ p^{(1)}_{(2,0,2),(3,1,3)} \\ p^{(1)}_{(2,0,2),(3,1,2)} \end{array}\right\} = 0$$

$$p^{(1)}_{(2,0,2),(3,0,3)} = p_2\ q_{23}$$

$$p^{(1)}_{(2,0,2),(3,1,4)} = p_1\ q_{23}$$

$$p^{(1)}_{(2,1,3)(2,0,2)}$$

$$\left.\begin{array}{l} p^{(1)}_{(2,1,3),(3,0,3)} \\ p^{(1)}_{(2,1,3),(3,0,2)} \\ p^{(1)}_{(2,1,3),(3,1,3)} \\ p^{(1)}_{(2,1,3),(3,1,2)} \end{array}\right\} = 0$$

$$p^{(1)}_{(2,1,3),(2,1,3)} = p_2\ q_{22}$$

$$p^{(1)}_{(2,1,3),(2,1,2)} = p_1$$

$$p^{(1)}_{(2,1,3),(3,1,4)} = p_2\ q_{23}$$

The other transition probabilities can be obtained in a similar way. Hence

$$\mathbb{P}_2 = \begin{bmatrix} 1/3 & 1/6 & 0 & 1/3 & 0 & 1/6 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 1/3 & 1/6 & 0 & 1/3 & 0 & 1/6 & 0 & 0 \\ 0 & 2/15 & 0 & 0 & 1/3 & 8/15 & 0 & 0 \\ 2/15 & 1/15 & 0 & 8/15 & 0 & 4/15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 2/15 & 0 & 0 & 0 & 8/15 & 0 & 1/3 \\ 2/15 & 1/15 & 0 & 8/15 & 0 & 4/15 & 0 & 0 \end{bmatrix}$$

From $\mathbb{P}_2$, the stationary probability vector π and then $\underline{P}(n)$ are computed. Finally $\overline{F}_{j,x}(c)$ for x=1 and j=2,3 are calculated and tabulated as follows.

	π	n	$\underline{P}(n)$	j	$E_{j,1}(z)$	$E_{j,1}(I)$	$\overline{F}_{j,1}(c)$
K=10, γ=1, h=1, S=4, s=1, a=1, b=2, p_1=1/3, p_2=2/3, λ =0.5	{0.11, 0.5, 0.17, 0.12, 0.04, 0.03, 0.01, 0.02}	2	0.34	2	2.67	1.84	4.58
		3	0.63	3	2.93	0.2	0.87
		4	0.03				

For the given inventory problem, the conditional expected total cost per unit time over a cycle is minimum corresponding to the replenishing quantity j=3.

3.3. Model-II

Description

Model II deals with a continuous review single commodity inventory problem where we assume that each demand is exactly for one unit. The interarrival times of demands are independent and identically distributed random variables following distribution function G(.) with **density** function g(.) and having finite first moment a . The maximum capacity of the warehouse is fixed as S units. Lead time is zero and no shortage is permitted. Further we assume that the reorder levels vary according to a Markov chain with state space $\{0,1,2,\dots,s\}$, $s \leqslant S-1$. The quantity replenis ed is always equal to $M=S-s$. In the present analysis we identify a two dimensional Markov chain in the underlying process, thereby gaining more information about the process.

Analysis

The assumption of our model is that the reordering levels are governed by a Markov chain. Denoting $X_o, X_1, X_2, \dots$ as the initial, first, second, ... reordering levels, $\{X_n, n=0,1,2,\dots\}$ forms a Markov chain defined over $\{0,1,2,\dots,s\}$ with initial probability

$Pr\ (X_o=s) = 1$ and

$Pr\ (X_o=i) = 0, \quad i=0,1,2,\ldots,s-1$

The one-step transition probability matrix $\mathbb{P}_3$ is given by

$$\mathbb{P}_3 = ((\ p_{ij}\ )) \text{ where}$$

$$p_{ij} = \begin{cases} Pr\{X_{n+1}=j|X_n=i\}, & i,j = 0,1,2,\ldots,s \\ 0 & \text{for } i,j > s \end{cases}$$

Let $0 = T_o < T_1 < T_2 < \ldots$ be the successive demand epochs and $Y_o, Y_1, Y_2, \ldots$ be the corresponding inventory levels after meeting the demands at $T_o, T_1, T_2, \ldots$. Then $I(T_n+) = Y_n$. The process $\{I(t),\ t \geqslant 0\}$ is a semi-Markov process defined on $\{1,2,\ldots,S\}$. The next procedure is to get the embedded Markov Renewal Process. For that we should have the information regarding the most recent reordering level ie. considering the pair (X_n, Y_n), if X_n denotes the last reordering level just prior to T_n, then the process $\{(X,Y),T\} = \{(X_n,Y_n),T_n;\ n \in N^o\}$ forms the associated embedded Markov Renewal Process defined over the state space $\bar{E} \times \bar{A}$. The semi-Markov kernel is given by $((\ Q_i(j,k,t)))$ where

$$Q_i(j,k,t) = Pr\{Y_{n+1}= k;\ T_{n+1}-T_n \leqslant t | Y_n=j,\ X_n=i\}$$

$$i = 0,1,2,\ldots,s$$
$$j,k = 1,2,\ldots,S,\ t \geqslant 0,$$

where X_n stands for the reordering level just prior to T_n. The maximum value that j can take is i+M where $i+M \leqslant s+M$ so that $Q_i(j,k,t)$, i=0,1,2,...,s; j,k = 1,2,...,S are given by

$$Q_i(j,k,t) = \begin{cases} 1-G(t), & j=i+M;\ k=j \\ G(t), & j=k+1;\ k>s \\ G(t)(1-p_{ik}), & j=s+1,\ k=s \\ G(t)p_{ik}, & j=s+1,\ k=s \\ G(t)\ 1-(p_{ij}+p_{ik}), & j=s,s-1,\ldots,1,\ k=j-1 \\ G(t)\{1-(p_{is}+p_{is-1}+\ldots+p_{ij})\}\,p_{ij-1}, & j=s,s-1,\ldots,1;\ k=j-1+M \\ G(t)\{1-(p_{is}+p_{is-1}+\ldots+p_{ij+1})\}\,p_{ij}; & j=s,s-1,\ldots,1;\ k=j+M-1 \end{cases}$$

To obtain the Markov Renewal function, we proceed as follows. Initially at the commencement of inventory the stock level is s and an order is placed; a replenishment occurs instantaneously so that Y_o=S and X_o=s with p(s) = 1.

Define F(i,j,t) as the probability that an order is placed when the level is i and the next order is placed when the level is j, i,j=0,1,2,...,s and the time duration in these is less than or equal to t.

ie. $F(i,j,t) = Q_i^{*i+M-j}(i+M,j+M,t),\ j \leqslant i+M \leqslant s+M.$

The Markov Renewal function is given by

$$R(s,j,t) = \sum_{m=o}^{\infty} \{ \sum_{i=o}^{s} F^{*m}(i,j,t) * F(s,j,t),$$

$$j=0,1,2,...,s, \quad t \geqslant 0.$$

where

$$F^{o}(i,j,t) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$F^{*m}(i,j,t)$ is obtained from the recursive relation

$$F^{*(m+1)}(i,j,t) = \sum_{k=o}^{s} \int_{o}^{t} F(i,k,du)\ F^{*m}(k,j,t-u) \quad \text{(Cinlar 1975a)}$$

Define $P_s(n,t) = \Pr\{I(t)=n | X_o=s\}$, $n=S,S-1,...,M+1, M, M-1,...$
$s+1, s,s-1,...,1.$

$P_s(n,t)$ satisfies the Markov renewal equation so that

$$P_s(n,t) = 1-G(t) + \int_o^t Q_s(S,S-1,du)\ P_s(n,t-u)$$

Hence the solution is given by

$$P_s(S,t) = \int_o^t R(s,s,du)\ (1-G(t-u))$$

Similarly for n = S-1,

$$P_s(S-1,t) = \int_o^t \sum_{j=s-1}^{s} R(s,j,du) \int_u^t Q_j^{*(M+j-(S-1))}(M+j,S-1,v-u)\,[1-G(t-v)]\,dv$$

$\vdots$

for n = M+1,

$$P_s(M+1,t) = \int_o^t \sum_{j=1}^{s} R(s,j,du) \int_u^t Q_j^{*(M+j-(M+1))}(M+j,\ M+1,\ v-u)\,[1-G(t-v)]\,dv$$

In general

$$P_s(M+i,t) = \int_o^t \sum_{j=1}^{s} R(s,j,du) \int_u^t Q_j^{*(M+j-(M+i))}(M+j,M+i,v-u)\,[1-G(t-v)]dv\ ,\ i=1,2,\ldots,s.$$

and

$$P_s(n,t) = \int_o^t \sum_{j=o}^{s} R(s,j,du) \int_u^t Q_j^{*(M+j-n)}(M+j,n,v-u)\,[1-G(t-v)]dv\ ,\ n=1,2,\ldots,M$$

Limiting distribution

Let $\mathbb{P}_4$ denotes the one-step transition probability matrix corresponding to the Markov chain $\{(X_n,Y_n),n=0,1,2,\ldots\}$ with

$$\mathbb{P}_4 = ((\ p^{(1)}_{(i,j),(\ell,k)}\ )),\ i,\ell = 0,1,2,\ldots,s;\quad j,k = 1,2,\ldots,S \ \text{ with}$$

$$p^{(1)}_{(i,j),(\ell,k)} = Pr\{(X_{n+1}=\ell,\ Y_{n+1}=k)\ |\ (X_n=i,\ Y_n=j)\}$$

$\mathbb{P}_4$ is computed from $\mathbb{P}_3$ in the following way

$$p^{(1)}_{(i,j),(\ell,k)} = \begin{cases} 1 & , i=0,1,2,\ldots,s;\ j=1,;\ \ell=0,\ k=M \\ 1 & , i=0,1,2,\ldots,s;\ j=s+2,\ s+3,\ldots,s+M; \\ & \quad \ell=i,\ k=j-1 \\ \dfrac{p_{i\ell}}{\sum\limits_{r=o}^{s-1} p_{or}} & , i=0,1,2,\ldots,s;\ j=2,3,\ldots,s;\ \ell=j-1; \\ & \quad k=M+j-1 \\ 1-\dfrac{p_{i\ell}}{\sum\limits_{r=o}^{s-1} p_{or}} & , i=0,1,2,\ldots,s;\ j=2,3,\ldots,s;\ \ell=i; \\ & \quad k=j-1 \\ p_{is} & , i=0,1,2,\ldots,s;\ j=s+1;\ell=s;\ k=M+s \\ 1-p_{is} & , i=0,1,2,\ldots,s;\ j=s+1,\ell=i;\ k=s \end{cases}$$

The stationary distributions are given by

$$\pi = \{\pi(0,1),\ \pi(0,2),\ldots,\ \pi(0,M),\ \pi(1,1),\ \ldots,\ \pi(1,M+1),\ldots, \pi(s,1)\ \pi(s,2),\ \ldots\ \pi(s,S)\}$$

where

$$\pi(\ell,k) = \sum_{i=o}^{s} \sum_{j=1}^{S} \pi(i,j)\ p^{(1)}_{(i,j),(\ell,k)},\ \ell = 0,1,2,\ldots,s,\ k=1,2,\ldots,S$$

The mean sojourn time in a state (ℓ,k) where ℓ is the last reordering level and k is the stock level after meeting a demand is

$$m(\ell,k) = \int_0^\infty (1-Q_\ell(k,k-1,t))\,dt, \quad \ell = 0,1,2,\ldots,s; \quad k= 1,2,\ldots, S.$$

$$= \int_0^\infty (1-G(t))dt$$

$$= a$$

The limiting probabilities are given by

$$\lim_{t \to \infty} \Pr\,(I(t)=S|X_o=s) = \lim_{t \to \infty} P_s(S,t) = \underline{P}(S)\,(\text{say})$$

$$= \frac{\pi(s,S)\; m(s,S)}{\sum_{\ell=0}^{s} \sum_{k=1}^{S} \pi(\ell,k)\; m(\ell,k)}$$

$$= \pi(s,S).$$

Similarly $\lim_{t \to \infty} P_s(S-1,t) = \underline{P}(S-1)$

$$= \sum_{j=s-1}^{s} \pi(j,S-1)$$

In general

$$\underline{P}(n) = \sum_{j=o}^{s} \pi(j,n), \quad n=1,2,\ldots,M$$

and

$$\underline{P}(M+i) = \sum_{j=i}^{s} \pi(j,M+i), \quad i=1,2,\ldots,s.$$

Optimisation

The objective function is the total expected cost per unit time in the steady state. The decision variable, M should be so chosen that the objective function is minimum for that value of M.

The expected time elapsed between two successive demands = $\int_{0}^{\infty} (1-G(t)) = a < \infty$. Under steady state, assuming j, j = 0,1,2,...,s as the reordering level, the expected time elapsed between two successive orders is (j+M-j)a = Ma. Therefore the expected number of orders per unit time = $\frac{1}{Ma}$. The expected inventory level at any instant of time is given by

$$E(I) = \sum_{n=1}^{S} n\,\underline{P}(n)$$

Therefore the total expected cost per unit time in the steady state is

$$\bar{F}(M) = \sum_{\ell=o}^{s} \sum_{k=1}^{S} \left\{\frac{K+cM}{Ma}\right\} \pi(\ell,k) + h\,E(I)$$

$$= \left\{\frac{K+cM}{Ma}\right\} \sum_{\ell=o}^{s} \sum_{k=1}^{S} \pi(\ell,k) + h \sum_{n=1}^{S} n\underline{P}(n)$$

where K is the fixed order cost, c is the variable procurement cost per unit, h is the holding cost per unit per unit time.

The optimal value of M is that value of M for which $\bar{F}(M)$ is minimum. It is readily verified that $\bar{F}(M)$ is a convex function in M. The optimal value of M is obtained from the two relations

$$\bar{F}(M) \leqslant \bar{F}(M+1)$$
$$\bar{F}(M) \leqslant \bar{F}(M-1)$$

Illustrations

1) Let $\mathbb{P}_3 = \begin{bmatrix} .7 & .2 & .1 \\ .1 & .7 & .2 \\ .2 & .1 & .7 \end{bmatrix}$

with s=2 and a = 0.5

Keeping s fixed, M is allowed to vary and the conditional probabilities in each case are obtained. For K=50, c=1, h=1, we have the following table.

	Value of M	$\bar{F}(M)$
K=50, c=1, h=1, a=0.5, s=2	2	56.83
	3	38.26
	4	30.78
	5	25.94
	6	23.12
	8	19.59
	10	18.106
	11	17.62
	12	17.99
	13	24.128

The optimal value of M is 11 for the given range of M.

2) For Markov chain $\{X_n, n=0,1,2,...\}$ with state space $\{0,1,2\}$ let

$$\mathbb{P}_3 = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$$

with a = 0.5.

Proceeding on the same line as in Problem 1 with K=10, c=1 and h=1, we obtained the following table.

K=10, c=1, h=1, s=2	Value of M	$\bar{F}(M)$
	3	13.80
	4	10.77
	5	9.34
	6	9.71

The optimal value of M is seen to be 5 for the given range of M.