Chapter-2

# AN INVENTORY MODEL WITH MARKOV DEPENDENT DEMAND QUANTITIES*

## 2.1 Introduction

In this chapter we deal with a continuous review (s,S) inventory model in which it is assumed that the quantity demanded by each arrival depends on the quantity demanded by the previous arrival and the maximum quantity demanded is $a \leqslant s$. Specifically, the quantities demanded by the successive arrivals form a Markov chain. Some work have been done earlier in which the assumption of independence on the quantities demanded is relaxed. Karlin and Fabens (1959), Iglehart and Karlin (1962) consider the case of a discrete-time inventory model where the demands are assumed to arise from a Markov process. They assume that at the beginning of each period the system is in one of N states labelled 1,2,...,N which are observed by the inventory manager before he orders. If the demand process is in state j in a period, a demand distribution, $\varphi_j$ will be operative in the period. The demand process changes state according to known transition probabilities with the transition from period to period governed by a Markov process.

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* Appeared in Cahiers du C.E.R.O., Vol.33, 1991.

When the demands at each arrival epochs are dependent the structure of the (s,S) optimal policy is not changed; but the main difference is that the choice of the quantity replenished at an order placing epoch will depend upon the demands in the cycle just completed. So the demand process changes the state of the inventory level according to a set of known transition probabilities with the transition at each demand epoch governed by a Markov chain defined over a state space $\{1,2,...,a\}$.

Section 2.2 deals with the description of the model. The various notations used in the sequel are also explained in that section. Section 2.3 discusses the analysis of the model. Limiting distribution of the system is investigated in Section 2.4. The model discussed here can be suitably applied in situations like bonus demands in major companies on recurring basis. The aim of the management is to minimise the total cost by distribution of optimum amount to the satisfaction of both the employees and the employer. An optimisation problem associated with the model is discussed in Section 2.5. A numerical illustration is done in the last section.

## 2.2. Description of the model

An (s,S) inventory model with the maximum capacity of the warehouse being fixed as S units with zero lead time is is considered. It is assumed that each arrival demands a random number (integer valued) of items; but the maximum quantity that can be demanded is restricted to a with $a \leqslant s$. The basic assumption of our model is that the quantity demanded by each arrival depends on the quantity demanded by the previous arrival so that the quantities demanded by the successive arrivals form a Markov chain defined over the state space $\{1,2,\ldots,a\}$. The interarrival times of demands are independent and identically distributed random variables following distribution function $G(.)$ and probability density function $g(.)$ with mean $\mu$ (assumed finite). Replenishment is assumed to be instantaneous and such that whenever the inventory drops to s or below for the first time after each replenishment an order is placed to bring the stock level back to S. To avoid perpetual shortage it is assumed that $S > 2s$. The following notations are used in the sequel:

$I(t)$ – On-hand inventory level at time t.

$*$ denotes convolution. For example $(F*G)(t) = \int_{-\infty}^{\infty} F(t)dG(t-u)$

$g^{*k}(.)$ – k-fold convolution of $g(.)$ with itself.

E denotes the set $\{1,2,\ldots,a\}$

$E_1 = \{s+1,\ 2+2,\ldots,\ S-1,\ S\}$

$N^0 = \{0,1,2,\ldots\}$

$P_i(n,t)$ = Probability that $I(t)=n$ given that the initial reordering level is i.

$[x]$ denotes the largest integer less than or equal to x.

$$\delta_{[i]} = \begin{cases} 0 & \text{if } i \text{ is a positive integer} \\ 1 & \text{otherwise} \end{cases}$$

$$\sigma_{\{[\frac{S-n}{a}],0\}} = \begin{cases} 1 & \text{if } [\frac{S-n}{a}] = 0 \\ 0 & \text{if } [\frac{S-n}{a}] > 0 \end{cases}$$

2.3 <u>Analysis of the model</u>

Let $0 = T_o < T_1 < T_2 < \ldots$ be the successive demand epochs and $X_o, X_1, X_2, \ldots$ be the quantities demanded by the successive arrivals at these epochs. Then by our assumption $\{X_n, n \in N^o\}$ constitutes a Markov chain defined on the state space E with the initial probability

$$p_i = Pr\ (X_o=i),\ i \in E.$$

Let us assume without loss of generality that $p_i = 1$ and $p_j = 0$ for $j \neq i$, $j \in E$.

We assume that the Markov chain $\{X_n, n \in N^o\}$ to be irreducible and aperiodic with the one-step transition probability matrix

$$\mathbb{P} = ((\ p_{i,j}\ )),\ i,j \in E \text{ where}$$

$$p_{i,j} = \Pr\left\{ X_{n+1}=j \mid X_n=i \right\}$$

Let $Y_o, Y_1, Y_2, \ldots$ be the stock levels just after meeting the demands at $T_o, T_1, T_2, \ldots$ . Then

$$Y_n = \begin{cases} Y_{n-1}-X_n & \text{if } Y_{n-1}-X_n > s \\ S & \text{if } Y_n-X_n \leqslant s \end{cases}$$

From the description of $X_n$ and $Y_n$, $n = 0,1,2,\ldots$ we easily see that the two dimensional stochastic process $\left\{(X_n,Y_n), n \in N^o\right\}$ constitutes a Markov chain defined over the state space $E \times E_1$. The corresponding one-step transition probabilities associated with the Markov chain $\left\{(X_n,Y_n),\ n \in N^o\right\}$ can be generated from the given one-step transition probabilities associated with the demand process.

<u>Theorem 1</u>

The stochastic process $\left\{(X,Y),T\right\} = \left\{(X_n,Y_n),T_n;\ n \in N^o\right\}$ is a Markov renewal process defined over the state space $E \times E_1$ with the corresponding semi-Markov kernel given by

$$\left\{ Q\left\{(i,I),\ (j,J),t\right\},\ i,j \in E;\ I,J \in E_1,\ t \geqslant 0 \right\}$$

where

$$Q\{(i,I),(j,J),t\} = Pr\{(X_{n+1}=j,\ Y_{n+1}=J);\ T_{n+1}-T_n \leqslant t \mid (X_n=i,\ Y_n=I)\}$$

$$= \int_0^t p_{i,j}\, g(u)du$$

$$= p_{i,j}\, G(t)$$

Proof:

The interarrival times of demands are positive, independent and identically distributed random variables. Hence the demand epochs constitute a renewal process. By our basic assumption that the successive quantities demanded forms a Markov chain, the demand magnitude at $T_{n+1}$ depends only on the demand magnitude at $T_n$ and not on $T_r$, $r = 0,1,2,\ldots,n-1$. Further the demand magnitudes are independent of the stock levels. Hence considering two successive demand epochs say $T_n$ and $T_{n+1}$

$$Pr\{(X_{n+1}=j,\ Y_{n+1}=J);\ T_{n+1}-T_n \leqslant t \mid (X_o,Y_o),(X_1,Y_1),\ldots,(X_n=i,\ Y_n=I);\ T_o,T_1,\ldots,T_n\}$$

$$= Pr\{(X_{n+1}=j,\ Y_{n+1}=J);\ T_{n+1}-T_n \leqslant t \mid (X_n=i, Y_n= I)\}$$

Since $T_{n+1}-T_n$, $n=0,1,2,\ldots$, are i.i.d random variables with probability density function $g(.)$ and $\{X_n, n \in N^o\}$ is a Markov chain which is independent of $\{Y_n, n \in N^o\}$,

$$Pr\{(X_{n+1}=j, Y_{n+1}=J); T_{n+1}-T_n \leq t | (X_n=i, Y_n=I)\}$$

$$= \int_0^t p_{i,j}\, g(u)\, du$$

$$= p_{i,j}\, G(t)$$

$$= Q\{(i,I), (j,J),t\}$$ which proves the theorem.

As soon as the stock level falls to an element in $\{s-a+1, s-a+2,\ldots, s-1,s\}$ for the first time after each replenishment, next order for replenishment is placed so as to bring the inventory level back to S. Initially due to a demand of magnitude $i(i \in E)$ we assume that the inventory level falls to s or below so that $X_o = i$ and $Y_o = S$. Looking at the successive time epochs $0 = T_o^{(1)}, T_1^{(1)}, T_2^{(1)}, \ldots$ at which the inventory level is brought to S, let $F\{(i,S), (j,S),t\}$ denote the probability that two consecutive replenishments take place in an amount of time $\leq t$ such that the initial demand is for a quantity i and the next demand that leads to replenishment is for a quantity j; $i,j \in E$. Then

$$F\{(i,S),(j,S),t\} = \sum_{n=[\frac{S-s}{a}]+\delta_{[\frac{S-s}{a}]}}^{S-s} Q^{*n}\{(i,S),(j,S),t\}; \quad i,j \in E;\ t \geq 0$$

Now define the function $R\{(.),(.),t\}$ by

$$R\{(i,S),(j,S),t\} = \sum_{m=o}^{\infty} F^{*m}\{(i,S),(j,S),t\}$$

with

$$F^{o}\{(i,S),(j,S),t\} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}, \quad i,j \in E,\ t \geq 0$$

$F^{*m}\{(i,S),(j,S),t\}$ is obtained from the recursive relation

$$F^{*(m+1)}\{(i,S),(j,S),t\} = \sum_{k \in E} \int_{o}^{t} F\{(i,S),(k,S),du\} F^{*m}\{(k,S),(j,S),t-u\}, \quad i,j \in E;\ t \geq 0.$$

Since $I(t)$ denotes the onhand inventory level at time t, $I(t) = Y_n$ for $T_n \leq t < T_{n+1}$, so the process $\{I(t), t \geq 0\}$ is a semi-Markov process defined on the state space $E_1$.

Defining $P_i(n,t)$ as $Pr\{I(t)=n \mid X_o=i\}$ with $n \in E_1$ and $i \in E$ we see that $P_i(n,t)$ satisfies the Markov renewal equations (Cinlar 1975a). Thus

(i) for n = S,

$$P_i(S,t) = \Pr\{I(t)=S;\ T_1>t|X_o=i\} + \Pr\{I(t)=S;T_1\leq t|X_o=i\}$$

$$= K_i(S,t) + \sum_{j\in E}\int_o^t F\{(i,S),(j,S),du\}\, P_j(S,t-u)$$

where

$$K_i(S,t) = \Pr\{I(t)=S;\ T_1>t|X_o=i\} = 1-G(t), \quad \text{and}$$

(ii) for n = s+1, s+2,..., S-1

$$P_i(n,t) = K_i^{(1)}(n,t) + \sum_{j\in E}\int_o^t Q\{(i,S),(j,S-j),du\}P_j(n,t-u)$$

where

$$K_i^{(1)}(n,t) = \Pr\{I(t)=n;\ T_1^{(1)} > t|X_o = i\}$$

$$= \int_o^t \sum_{m=[\frac{S-n}{a}]+\sigma\{[\frac{S-n}{a}],0\}}^{S-n} \sum_{j\in E} Q^{*m}\{(i,S),(j,n),du\}$$

$$[1-G(t-u)]$$

The solutions are given by

$$P_i(S,t) = \sum_{j\in E}\int_o^t R\{(i,S),(j,S),du\}\, K_j(S,t-u)$$

and for n = s+1, s+2, ..., S-1

$$P_i(n,t) = \sum_{j\in E}\int_o^t R\{(i,S),(j,S),du\}\, K_j^{(1)}(n,t-u).$$

## 2.4 Limiting distribution

To compute the steady state probabilities it is necessary that at each demand epoch, not only the quantities demanded but also the corresponding inventory levels after meeting the demands are to be known. From the given probabilities governing the demand process, the transition probability matrix $(( p^{(1)}_{(i,I),(\ell,L)} ))$ corresponding to the two dimensional Markov chain $\{(X_n,Y_n),\ n \in N^o\}$ can be derived where

$$p^{(1)}_{(i,I),(\ell,L)} = \Pr\{(X_{n+1}=\ell,\ Y_{n+1}=L) \mid (X_n=i, Y_n=I)\}$$

$$i,\ell \in E,\ I,L \in E_1$$

The state space of this Markov chain is $\{(i_1,I_1) \mid i_1=1,2,\dots,a;\ I_1 = s+1,s+2,\dots,S-1\} \cup \{(1,S),(2,S),\dots,(a,S)\}$ with $i_1+I_1 \leqslant S$. $(( p^{(1)}_{(i,I),(\ell,L)} ))$ is computed and is given in Table-1.

We have assumed $\{X_n,\ n \in N^o\}$ to be irreducible and aperiodic and hence is the Markov chain $\{(X_n,Y_n), n \in N^o\}$. Let $\pi$ be the stationary probability vector of the Markov chain $\{(X_n,Y_n), n \in N^o\}$.

That is $\pi = \{\pi(1,s+1),\ \pi(2,s+1), \dots \pi(a,s+1),\ \pi(1,s+2),\ \pi(2,s+2),\ \dots,\ \pi(1,S-1),\ \pi(1,S),\ \pi(2,S),\ \dots,\ \pi(a,S)\}$

| | (1,s+1) | (2,s+1) | ... | (a,s+1) | ... | (a,s+2) | ... | (a,S-a) | ... | (1,S-1) | (1,S) | (2,S) | ... | (a,S) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (1,s+1) | 0 | 0 | | .. | | .. | | .. | | 0 | $p_{11}$ | $p_{12}$ | ... | $p_{1a}$ |
| (2,s+1) | 0 | 0 | | .. | | .. | | .. | | 0 | $p_{21}$ | $p_{22}$ | ... | $p_{2a}$ |
| ⋮ | | | | | | | | | | | | | | |
| (a,s+1) | 0 | 0 | | .. | | .. | | .. | | 0 | $p_{a1}$ | $p_{a2}$ | ... | $p_{aa}$ |
| (1,s+2) | $p_{11}$ | 0 | 0 | .. | | .. | | .. | | 0 | 0 | $p_{12}$ | ... | $p_{1a}$ |
| (2,s+2) | $p_{21}$ | 0 | 0 | .. | | .. | | .. | | 0 | 0 | $p_{22}$ | ... | $p_{2a}$ |
| ⋮ | | | | | | | | | | | | | | |
| (a,s+2) | $p_{a1}$ | 0 | 0 | .. | | .. | | .. | | 0 | 0 | $p_{a2}$ | ... | $p_{aa}$ |
| ⋮ | | | | | | | | | | | | | | |
| (a,S-a) | | | | | | | | | | | | | | |
| ⋮ | | | | | | | | | | | | | | |
| (1,S-1) | | | | | | | | | | | | | | |
| (1,S) | 0 | 0 | | .. | | .. | | $p_{1a}$ | ... | $p_{11}$ | 0 | .. | ... | 0 |
| (2,S) | 0 | 0 | | .. | | .. | | $p_{2a}$ | ... | $p_{21}$ | 0 | .. | ... | 0 |
| ⋮ | | | | | | | | | | | | | | |
| (a,S) | 0 | 0 | | .. | | .. | | $p_{aa}$ | ... | $p_{a1}$ | .. | .. | ... | 0 |

Table-1: The Transition Probability matrix $((p_{((i,I),(\ell,L))}))$

These satisfy the relation

$$\pi(\ell,L) = \sum_{i=1}^{a} \sum_{I=s+1}^{S} \pi(i,I)\ p^{(1)}_{((i,I),(\ell,L))}$$

with

$$\sum_{i\in E} \sum_{I\in E_1} \pi(i,I) = 1$$

The uniqueness of $\pi$ follows from Bhat (1984). For, we have assumed $a \leqslant s$ and so the state space E of the Markov chain $\{X_n,\ n \in N^o\}$ is finite. Hence the state space of the Markov renewal process $\{(X,Y),T\}$ has only a finite number of elements in it. Further $\{(X_n,Y_n),\ n \in N^o\}$ is irreducible and aperiodic since $\{X_n, n \in N^o\}$ is irreducible and aperiodic. Hence the invariant measure $\pi$ is unique.

Since the interarrival times of demands are i.i.d. random variables, the mean sojourn in any state is equal to the mean of the interarrival time distribution of the demands. So the mean sojourn time in state $(\ell,L)$, $\ell \in E$, $L \in E_1$ is

$$m(\ell,L) = \int_0^\infty (1-G(t))dt = \mu \quad \text{(assumed finite)}$$

Following Cinlar (1975a) the limiting probabilities are obtained as given below.

(i) for n = S,

$$\lim_{t \to \infty} P_i(S,t) = \frac{\sum_{j=1}^{a} \pi(j,S)\ m(j,S)}{\sum_{\ell=1}^{s} \sum_{L=s+1}^{S} \pi(\ell,L)\ m(\ell,L)}$$

$$= \sum_{j=1}^{a} \pi(j,S)$$

(ii) for n = S-a+1, S-a+2,...,S-2,S-1,

$$\lim_{t \to \infty} P_i(n,t) = \sum_{j=1}^{S-n} \pi(j,n)$$

(iii) for n = s+1, s+2,..., S-a-1, S-a

$$\lim_{t \to \infty} P_i(n,t) = \sum_{j=1}^{a} \pi(j,n)$$

We note from the above that the limiting probabilities are independent of the initial state i, as is expected from the theory of Markov chains. Let $\lim_{t \to \infty} P_i(n,t) = \underline{P}(n)$. The following theorem easily follows from the above discussion.

<u>Theorem 2</u>

If the demand quantities are independent and identically distributed random variables on the set E, then the limiting stationary distribution is discrete uniform.

## 2.5. Optimisation problem

For any inventory model the decision variables are to be so chosen that the objective function associated with that model attains the minimum value at these values of the decision variables. Here the objective function associated with our model is the total expected cost (for any cycle) per unit time under steady state. The decision variables are s and S for a given fixed value of a.

The expected inventory level E(I) at any instant of time is

$$E(I) = \sum_{n=s+1}^{S} n\, P(n)$$

$$= \sum_{j=1}^{a} \left\{ \sum_{n=s+1}^{S-a} n\, \pi(j,n) + S\, \pi(j,S) \right\} + \sum_{j=1}^{S-n} \sum_{n=S-a+1}^{S-1} n\, \pi(j,n)$$

We shall call the time elapsed between two successive demands that result in the replenishment of the inventory as the length of a cycle. Suppose in the steady state the quantity replenished at a demand epoch is M and Z denotes the length of the cycle just completed. The joint density function of M and Z be denoted by $f_j(m,z)$. Then

$$f_j(m,z) = P_j\{M=m,\ z \leqslant Z < z+dz\}$$

where j is the quantity demanded by the last arrival in the previous cycle.

$$f_j(m,z) = \sum_{k=[\frac{S-m}{a}]+\delta_{[\frac{S-m}{a}]}}^{S-s} \Big\{ P_j[(M=m,\ z\leqslant Z<z+dz) \Big| \text{k arrivals demanded totally } m(>(S-s)) \text{ units of which the first } (k-1) \text{ arrivals demanded less than } (S-s) \text{ units}]$$

$$\times \Pr(\text{k arrivals demanded totally } m(>(S-s)) \text{ units of which the first } (k-1) \text{ arrivals demanded less than } (S-s) \text{ units}) \Big\}$$

$$= \sum_{k=[\frac{S-m}{a}]+\delta_{[\frac{S-m}{a}]}}^{S-s} \sum_{\substack{i_1,i_2,\ldots,i_k \\ i_1+i_2+\ldots+i_k=m \\ i_1+i_2+\ldots+i_{k-1}<S-s}} p_{j,i_1}\, p_{i_1,i_2} \cdots p_{i_{k-1},i_k}\, g^{*k}(z)$$

Hence the expected quantity replenished per unit time is

$$E_j\left(\frac{M}{Z}\right) = \int_o^\infty \sum_{m=S-s}^{S-s+a-1} \frac{m}{z} f_j(m,z)dz$$

The probability density function of the duration of a cycle is

$$\sum_{m=S-s}^{S-s+a-1} f_j(m,z)$$

Therefore, the expected length of a cycle is

$$E_j(Z) = \int_o^\infty z \sum_{m=S-s}^{S-s+a-1} f_j(m,z)dz$$

Let K be the fixed ordering cost, c be the variable procurement cost and h be the holding cost per unit per unit time. So the total expected cost per unit time is $\bar{F}_j(s,S)$ and is given by

$$\bar{F}_j(s,S) = \frac{K}{E_j(Z)} + cE_j\left(\frac{M}{Z}\right) + h\,E(I)$$

where

$E_j(Z)$, $E_j(\frac{M}{Z})$ and $E(I)$ are as already defined. Thus for given K,c,h, one-step transition probability matrix of the demand process and interarrival time distribution, the optimal value of the pair (s,S) can be computed.

## 2.6. Numerical illustration

Let the one-step transition probability matrix associated with the demand process be

$$\mathbb{P} = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$

and let the interarrival times of demands follow exponential distribution with mean $\lambda = 0.5$.

The stationary probability vector $\pi$ is computed and $E(I)$ obtained. Then $E_j(Z)$, $E_j(\frac{M}{Z})$ and $\bar{F}_j(s,s)$ for j=1 and K=10, c=1, h=1 are computed and tabulated as follows.

| (s,S) | $\pi$ | E(I) | $E_1(\frac{M}{Z})$ | $E_1(Z)$ | $\bar{F}_1(s,S)$ |
|---|---|---|---|---|---|
| (2,5) | {.02, .41, .08, .17, .32} | 3.82 | 1.81 | 2.06 | 10.49 |
| (2,6) | {.12, .11, .04, .26, .15, .08, .24} | 4.56 | 2.04 | 1.84 | 12.02 |
| (2,7) | {.09, .05, .09, .11, .04, .13, .15, .06, .28} | 5.34 | 1.23 | 0.95 | 17.10 |
| (3,7) | {.106, .1, .043, .233, .143, .071, .304} | 5.68 | 2.86 | 2.97 | 11.90 |
| (3,8) | {.05, .16, .11, .06, .02, .25, .08, .08, .19} | 6.03 | 1.11 | 1.21 | 19.25 |
| (3,9) | {.06, .09, .05, .12, .07, .07, .03, .16, .10, .05, .20} | 6.68 | 0.76 | 0.49 | 27.92 |

From the above table, we see that corresponding to the pairs (2,5), (2,6) and (2,7), the total expected cost per unit time is minimum for the pair (2,5) and corresponding to (3,7), (3,8) and (3,9), the optimal pair is (3,7) and corresponding to (2,7) and (3,7), the optimal pair is (3,7) for the given set of values of K, c, h and $\lambda$ .