Chapter-8

TRANSIENT SOLUTION TO $M^X/G^Y/1/b$ QUEUE WITH VACATION*

8.1. Introduction

This chapter discusses a vacation queueing model in which the system can be operated only when it is full, but only a random number of units are taken in a batch for service. Unlike the previous chapters, matrix convolution technique is adopted here to arrive at the time-dependent system size probabilities.

Chaudhary (1979) obtains the limiting probabilities of queue lengths at random and departure epochs in the case of an $M^X/G/1$ queueing system. The transient and stationary behaviour of the M/G/1/k queue, with a fixed maximum number of customers, k, in the system at any time is studied by Cohen (1969). Bagchi and Templeton (1973) make use of Cohen's method to generalise his results to $M^X/G^Y/1/k$ queue.

Section 8.2 deals with the description of the model together with the notations and preliminaries used in this chapter. Transition time densities and expressions for renewal density are given in Section 8.3. Time dependent system size probabilities and distribution of the busy period

*Appeared in Calcutta Statistical Association Bulletin, Vol.39, March and June 1990, Nos. 153-154, pp.21-29.

are derived in Sections 8.4 and 8.5 respectively. Virtual waiting time distribution is derived in the last section.

8.2 Description of the model

A single server queueing system with the arrival pattern following a compound Poisson process is considered. The random variable X represents the number of customers arriving in a batch for service with the distribution of X defined as

$$\Pr\{X=i\} = p_i, \quad i=1,2,\ldots,b.$$

Let $$\phi_1(s) = \sum_{i=1}^{b} p_i s^i$$

Again customers in batch arrive at Poisson rate λ and joins the queue in the waiting room (W.R). The system is of finite capacity b and arrivals occuring when the W.R. is full are lost to the system. The service commences only when the waiting room (W.R) is full (b) and then a random number of units Y are taken in a batch to the service station (S.S)for service. The service pattern follows a general distribution $G_Y(.)$ with density $g_Y(.)$ where Y is the batch size taken for service with the distribution of Y given by $\Pr\{Y=r\}= q_r$, $r=1,2,\ldots,b$ with $\phi_2(s) = \sum_{k=1}^{b} q_k s^k$. On completion of the service of a batch, if the waiting room (W.R) contains less than b customers, the server immediately takes a vacation for a random duration having general distribution H(.) with density function h(.). The vacation policy is of the exhaustive type- every time the server returns after vacation, if the system size is less than b the server again goes on vacation whose duration has the same distribution H(.). The random variables X and Y are assumed to be independent. Further the sizes of the arriving batches are independent, so are the sizes of the batches taken for service.

Notations and preliminaries

p_j^{*i} - the coefficient of s^j in $[\phi_1(s)]^i$

N(t) represents the number of arrival instants upto time t so that

$$Pr\{X_1+X_2+\ldots+X_{N(t)}=j\} = \sum_{i=1}^{j} p_j^{*i}(e^{-\lambda t} (\lambda t)^i/i!)$$

Denote Pr $\{X_1+X_2+\ldots+X_{N(t)}=j\}$ as $\Lambda_j(t)$

Let $\bar{\Lambda}_k(t)$ represents the probability that at least k arrivals occur upto time t.

Let $f_{ij}(x)$ be the probability that a batch of size i taken for service at time zero completes the service in (x,x+dx] and j units arrive in (o,x] such that $j+(b-i)\geq b$. Hence

$$f_{ij}(x) = \sum_{k=1}^{j} p_j^{*k} \frac{e^{-\lambda x}(\lambda x)^k}{k!} g_i(x)\, q_i$$

$$= \Lambda_j(x)\, g_i(x)\, q_i, \quad j=i,i+1,\ldots,b;\ i=1,2,\ldots,b$$

$P_{ji}(t)$ represents the probability that at time t there are i units in the W.R and j units in the S.S.

$$i = b-j,\ b-j+1,\ldots,b;\ j=1,2,\ldots,b$$

8.3 Transition time densities

For $j<i$ and $i=1,2,\ldots,b$, define $f_{ij}(x)=0$

Write

$$\mathbb{F}(x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) & \cdot\cdot & \cdot\cdot & f_{1b}(x) \\ 0 & f_{22}(x) & \cdot\cdot & \cdot\cdot & f_{2b}(x) \\ 0 & 0 & f_{33}(x) & \cdot\cdot & f_{3b}(x) \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ 0 & 0 & 0 & 0 & f_{bb}(x) \end{bmatrix}$$

Introduce

$$\mathbb{H}(x) = \begin{bmatrix} f_{10}(x) & 0 & 0 & 0 & \cdots & 0 \\ f_{20}(x) & f_{21}(x) & 0 & 0 & \cdots & 0 \\ f_{30}(x) & f_{31}(x) & f_{32}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ f_{bo}(x) & f_{b1}(x) & f_{b2}(x) & . & \cdots & f_{bb-1}(x) \end{bmatrix}$$

where $f_{ij}(x) = 0$ for $j \geqslant i$; $i=1,2,\ldots,b$.

Let $\underline{f}_i(x) = (0,0,0,\ldots,0,f_{ii}(x), f_{ii+1}(x),\ldots,f_{ib}(x))$, for $i=1,2,\ldots,b$. Then $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ is a b-component column vector. Taking the ℓ^{th} co-ordinate of the above vector and naming it as $K_i^{\ell}(x)$ we see that the probability that the system starting with the service of a batch of size $i(1 \leqslant i \leqslant b)$ units initially, continues to work uninterruptedly and finally the service of a batch of size ℓ has been completed in $(x,x+dx]$ and ℓ arrivals have occured during this (last) service time to make the system size full again.

Now $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{H})(x)$ is a b-component row vector.

Let $F_i(x) = (\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{H})(x)$. This stands for the probability that a busy period that has started with the service of a batch of size i units initially and after serving n more batches, has ended in (x,x+dx] with atmost (b-1) units waiting for service. Let the ℓ^{th} coordinate of the vector $F_i(x)$ be denoted as $F_i^{\ell}(x)$. Hence $F_i^{\ell}(x)$ represents the probability that the system starting with the service of a batch of size $i(1\leqslant i\leqslant b)$ units, the busy period ends in (x,x+dx] with ℓ units waiting for service, $\ell=0,1,2,\ldots,b-1$.

<u>Renewal density</u>

The time points at which a busy period is initiated after each vacation are regeneration points. Let $T_i, i=1,2,\ldots$ represent the successive vacation completion points.

Let a busy period be initiated with the service of a batch of size i $(1\leqslant i\leqslant b)$ units and z be the time epoch at which the busy period has ended with atmost b-1 units in the W.R. Assume that m vacations complete in (z,v), the last being completed in (v,v+dv) at which the system size is again less than b and so the server goes on vacation which is completed in (u,u+du). The service now starts as there are b units waiting. Hence

$$A(u) = \Pr\{u < T_n \leqslant u+du\}$$

$$= \int_o^u \sum_{i=1}^{b} \sum_{\ell=o}^{b-1} F_i^{\ell}(z) \int_z^u \Big(\sum_{m=o}^{\infty} h^{*m}(v-z)\Big) \sum_{j=o}^{b-1-\ell} \wedge_j(v-z)h(u-v)$$

$$\bar{\wedge}_{b-(\ell+j)}(u-v)dv\ dz$$

Therefore the renewal density is given by

$$M(u) = \Pr\{u < T_1+T_2+ \ldots T_n \leqslant u+du\}$$

$$= \sum_{n=1}^{\infty} A^{*n}(u)$$

8.4 System size probabilities

$P_{oi}(t)$ is the probability that there is no unit in the S.S and i units are there in the W.R, i=0,1,2,...,b-1,b.

(i) For i=1,2,...,b-1

$$P_{oi}(t) = \int_o^t \Big[\sum_{a=1}^{b} \sum_{\ell=o}^{i} F_a^{\ell}(u) \sum_{m=o}^{\infty} h^{*m}(t-u)\wedge_{i-\ell}(t-u) +$$

$$M(u) \int_u^t \sum_{a=1}^{b} \sum_{\ell=o}^{i} F_a^{\ell}(z-u) \sum_{m=o}^{\infty} h^{*m}(t-z)\wedge_{i-\ell}(t-z)dz\Big]du \quad (1)$$

(ii) For i=b.

$$P_{ob}(t) = \int_o^t \int_u^t [\sum_{a=1}^{b} \sum_{\ell=o}^{b-1} F_a^{\ell}(u) \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{j=o}^{b-1-\ell} \wedge_j(v-u)$$

$$\bar{\wedge}_{b-(\ell+j)}(t-v)[1-H(t-v)]$$

$$+ M(u)(\sum_{a=1}^{b} \sum_{\ell=o}^{b-1} F_a^{\ell}(v-u) \int_v^t \sum_{m=o}^{\infty} h^{*m}(z-v) \sum_{j=o}^{b-1-\ell} \wedge_j(z-v)\bar{\wedge}_{b-(\ell+j)}(t-z)$$

$$(1-H(t-z))dz]dv\ du \qquad (2)$$

(iii) For j=1,2,...,b; i=0,1,2,...,b

$$P_{ji}(t) = (1-G_j(t))q_j \wedge_{i-(b-j)}(t)+ \int_o^t(M(u)+ \sum_{a=1}^{b} \sum_{\ell=1}^{b} K_a^{\ell}(u))$$

$$(1-G_j(t-u))q_j\wedge_{i-(b-j)}(t-u)\ du \qquad (3)$$

and (iv) for j=i=o

$$P_{oo}(t) = G_b(t)\ q_b \wedge_o(t)+ \int_o^t \int_u^t (M(u)+ \sum_{a=1}^{b} \sum_{\ell=1}^{b} K_a^{\ell}(u))$$

$$\{g_b(v-u)\ q_b \wedge_o(t-u) \sum_{m=o}^{\infty} h^{*m}(t-v)\}\ dv\ du \qquad (4)$$

8.5. Busy Period Distribution

For the present model, define the busy period of the system as the time duration for which the system continuously remains non-empty. To obtain the so-defined busy period distribution we proceed as follows:

Delete the first column of $\mathbb{H}(.)$ and represent the corresponding $((b \times (b-1)))$ matrix by $\mathbb{H}_1(.)$.

$$\mathbb{H}_1(x) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ f_{21}(x) & 0 & 0 & \cdots & 0 \\ f_{31}(x) & f_{32}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ f_{b1}(x) & f_{b2}(x) & . & \cdots & f_{bb-1}(x) \end{bmatrix}$$

Hence $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{H}_1)(x)$ is a (b-1) component row vector. Denote this by $F_i^{(1)}(x)$ whose ℓ^{th} coordinate we write as $F_i^{(1)\ell}(x)$. Then $F_i^{(1)\ell}(x)$ represents the probability that the system starting with the service of a batch of size i units $(1 \leq i \leq b)$ initially, the busy period ends in $(x, x+dx]$ with ℓ units waiting for service, $\ell = 1,2,\ldots,b-1$.

Define $\quad {}_oA(u) = \Pr\{u < T_n \leq u+du\}$

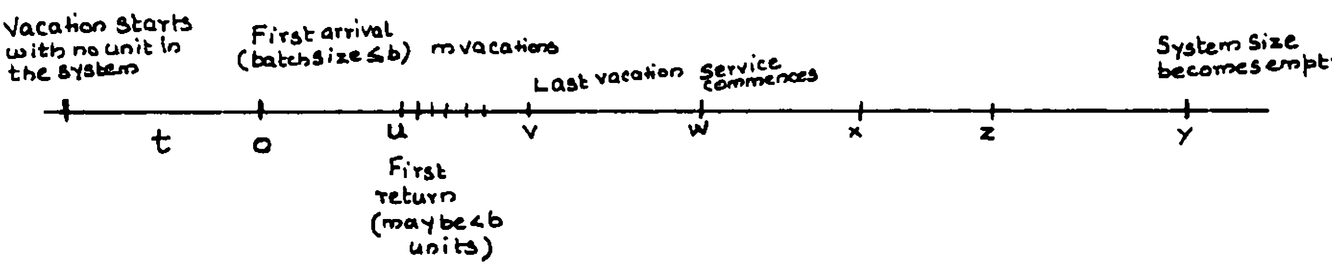
Then
$$ {}_oA(u) = \int_o^u \int_z^u \sum_{i=1}^{b} \sum_{\ell=1}^{b-1} F_i^{(1)\ell}(z) \sum_{m=o}^{\infty} h^{*m}(v-z)h(u-v) \sum_{j=o}^{b-1-\ell} \wedge_j(v-z)\,\overline{\wedge}_{b-(\ell+j)}(u-v)\, dv\, dz $$

Further let

$$_{o}M(u) = Pr\{u < T_1+T_2+\ldots+T_n \leq u+du\}$$

Then $$_{o}M(u) = \sum_{n=1}^{\infty} {}_{o}A^{*u}(u)$$

Now we derive the distribution of the busy period



Suppose at time O the first arrival has occured (with a batch size $\leq$ b units) during the vacation where we assume that the vacation has started with no unit in the system. The distribution of the above defined busy period is given by B(y) where

$$B(y) = Pr\{y < Y \leq y+dy\}$$

$$= \int_{(t)} \int_{o}^{y} \int_{u}^{y} \int_{v}^{y} \sum_{j+\ell_1+\ell_2 \leq b-1} \Big\{ (p_j \lambda e^{-\lambda t} (h(t+u) * \sum_{m=o}^{\infty} h^{*m}(v-u) h(w-v))$$

$$(\sum_{k=1}^{\ell_1} p_{\ell_1}^{*k} \frac{e^{-\lambda u}(\lambda u)^k}{k!} (\sum_{r=1}^{\ell_2} e^{-\lambda(v-u)} \frac{(\lambda(v-u))^r}{r!} p_{\ell_2}^{*r})$$

$$\left(\sum_{\ell_3=b-(j+\ell_1+\ell_2)}^{b} \sum_{n=1}^{\ell_3} \left(e^{-\lambda(w-v)} \frac{(\lambda(w-v))^n}{n!} p_{\ell_3}^{*n} \right) \right.$$

$$\left(g_b(y-w) q_b \wedge_o(y-w) + \int_w^y \int_x^y [\, {}_oM(x-w) \left(\sum_{i=1}^{b} \sum_{\ell=1}^{b} K_i^{\ell}(z-x) \right) f_{bo}(y-z)\right.$$

$$\left. dz\, dx]) \right\} dw\, dv\, du\, dt \qquad (5)$$

8.6 Virtual waiting time distribution

By virtual waiting time at time t in the queue, we mean the amount of time an arrival has to wait in the queue before it being taken for service if it were to arrive at time t (Takac's (1962)).

Let the virtual waiting time at time t be W_t. Expression (5) is an upper bound for W_t. We get sharper bounds in (o,b-1). In this case,

$P\{W_t \leqslant x\}$ is the probability that vacation is completed at or prior to t+x and a batch of size b is taken for service + probability that a batch of size (b-1) is taken for service after completing the vacation and during its service time which ends at or prior to t+x at least (b-1) units have arrived + Probability that a batch of size (b-2) is taken for service and .... + ... We illustrate this by restricting b=2. The different possibilities for the state of the system at time t in this case are $\{(0,0), (0,1), (0,2), (1,1), (1,2), (2,0), (2,1), (2,2)\}$

(i) For (0,0):

$$P\{W_t \leqslant x\} = \int_0^t g_2(u)\; q_2\Lambda_o(t) \int_t^{t+x} h(v-u) \qquad \bar{\Lambda}_1(v-t)dv\,du +$$

$$\int_0^t \sum_{a=1}^{2}\sum_{\ell=1}^{2} K_a^{\ell}(u) \int_u^t g_2(w-u)q_2\Lambda_o(t-u)$$

$$\int_t^{t+x} h(v-w) \qquad \bar{\Lambda}_1(v-t)dv\,dw\,du + \int_0^t\int_0^t M(u)g_2(w-u)q_2\Lambda_o(t-u)$$

$$\int_t^{t+x} h(v-w) \qquad \bar{\Lambda}_1(v-t)dv\,dw\,du \tag{1}$$

(ii) For (0,1):

$$P\{W_t \leqslant x\} = \int_0^t (g_2(u)q_2\Lambda_1(t)+g_1(u)q_1\Lambda_o(t)) \int_t^{t+x} h(v-u)\sum_{i=0}^{\infty}\Lambda_i(v-t)$$

$$\{q_2 + \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) + \int_s^{t+x} f_{10}(s-v)h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)\}$$

$$dz\,ds\,dv\,du + \int_0^t \Big(\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)\Big)$$

$$\int_u^t (g_2(w-u)q_2\Lambda_1(t-u)+g_1(w-u)q_1\Lambda_o(t-u)) +$$

$$\int_t^{t+x} h(v-w) \sum_{i=o}^{\infty} \wedge_i(v-t) \{q_2 + \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v) h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)\} \, dz \, ds \, dv \, dw \, du +$$

$$\int_o^t M(u) \int_u^t (g_2(w-u) q_2 \wedge_1(t-u) + g_1(w-u) q_1 \wedge_o(t-u))$$

$$\int_t^{t+x} h(v-w) \sum_{i=o}^{\infty} \wedge_i(v-t) \{q_2 + \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v) \, h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)\} \, dz \, ds \, dv \, dw \, du$$

(iii) <u>For (0,2)</u>

$$P\{W_t \leqslant x\} = \int_o^t [g_2(u) q_2 \wedge_2(t) + g_1(u) q_1 \wedge_1(t)] \int_t^{t+x} h(v-u) \sum_{i=o}^{\infty} \wedge_i(v-t)$$

$$\int_v^{t+x} [\sum_{j=1}^{\infty} f_{2j}(s-v) + \sum_{j=o}^{\infty} f_{1j}(s-v) [\int_s^{t+x} q_2 + \sum_{j=o}^{\infty} f_{1j}(z-s)] +$$

$$\int_s^{t+x} f_{20}(s-v) \, h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s) dz \, ds \, dv \, du +$$

$$\int_o^t [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)] \int_u^t [g_2(w-u) q_2 \wedge_2(t-u) +$$

$$g_1(w-u) \, q_1 \wedge_1(t-u)] \int_t^{t+x} h(v-w) \sum_{i=o}^{\infty} \wedge_i(v-t)$$

$$\{\int_v^{t+x} [\sum_{j=1}^{\infty} f_{2j}(s-v) + \sum_{j=o}^{\infty} f_{1j}(s-v) \, [\int_s^{t+x} q_2 + \int_s^{t+x} \sum_{j=o}^{\infty} f_{ij}(z-s)]$$

$$\int_s^{t+x} f_{20}(s-v) \, h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)]\} \, dz \, ds \, dv \, dw \, du$$

$$+ \int_o^t M(u) \int_u^t [g_2(w-u)q_2 \wedge_2(t-u)+g_1(w-u)q_1 \wedge_1(t-u)]$$

$$\int_t^{t+x} h(v-w) \sum_{i=o}^{\infty} \wedge_i(v-t)\{ \int_v^{t+x} [\sum_{j=1}^{\infty} f_{2j}(s-v)+ \sum_{j=o}^{\infty} f_{1j}(s-v)$$

$$[\int_s^{t+x} (q_2+ \sum_{j=o}^{\infty} f_{1j}(z-s))] + \int_s^{t+x} f_{20}(s-v) h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)]\}$$

$$dz\ ds\ dv\ dw\ du$$

<u>For (1,1)</u>

$$P \{W_t \leq x\} = \wedge_o(t) \int_t^{t+x} g_1(u) q_1[q_2+ \int_u^{t+x} \sum_{j=1}^{\infty} f_{1j}(v-u)$$

$$+ \int_v^{t+x} f_{10}(v-u) h(z-v) \sum_{i=1}^{\infty} \wedge_i(z-v) dz\ dv\ du$$

$$+ \int_o^t [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)] \wedge_o(t-u) \int_t^{t+x} g_1(v-u)$$

$$q_1 \{ q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) + \int_t^{t+x} f_{10}(s-v)h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)\}$$

$$dz\ ds\ dv\ du$$

$$+ \int_o^t M(u) \wedge_o(t-u) \int_t^{t+x} g_1(v-u) q_1 \{ q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v)$$

$$+ \int_s^{t+x} f_{10}(s-v) h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)\} dz\ ds\ dv\ du$$

For (1,2)

$$P\{W_t \leqslant x\} = \Lambda_1(t) \int_t^{t+x} g_1(u)\; q_1 \int_u^{t+x} \sum_{j=1}^{\infty} f_{2j}(v-u) +$$

$$[\sum_{j=o}^{\infty} f_{1j}(v-u)\; [q_2 + \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v)\; h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)]] +$$

$$\int_v^{t+x} f_{20}(v-u)\; h(z-v) \sum_{i=1}^{\infty} \Lambda_i(z-v) dz\; ds\; dv\; du +$$

$$\int_o^t [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)]\, \Lambda_1(t-u) \int_t^{t+x} g_1(v-u)\; q_1 \int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(w-v) +$$

$$[\sum_{j=o}^{\infty} f_{1j}(w-v)(q_2 + \int_w^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-w) +$$

$$\int_s^{t+x} f_{10}(s-w)\; h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)] +$$

$$\int_w^{t+x} f_{20}(w-v)\; h(z-w) \sum_{i=1}^{\infty} \Lambda_i(z-w) dz\; ds\; dw\; dv\; du +$$

$$\int_o^t M(u)\; \Lambda_1(t-u) \int_t^{t+x} g_1(v-w)\; q_1 \int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(w-v)$$

$$\int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(w-v) + [\sum_{j=o}^{\infty} f_{ij}(w-v)(q_2 + \int_w^{t+x} \sum_{j=1}^{\infty} f_{ij}(s-w) +$$

$$\int_s^{t+x} f_{10}(s-w)\; h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)] +$$

$$\int_{w}^{t+x} f_{20}(w-v)\ h(z-w) \sum_{i=1}^{\infty} \wedge_i(z-w) dz\ ds\ dw\ dv\ du$$

For (2,0)

$$P\{W_t \leqslant x\} = \wedge_o(t) \int_{t}^{t+x} g_2(u)\ q_2\ [\wedge_o(u-t) \int_{u}^{t+x} h(v-u) \sum_{i=1}^{\infty} \wedge_i(v-u)]$$

$$+ \sum_{i=1}^{\infty} \wedge_i(u-t)\ dv\ du\ +$$

$$\int_{o}^{t} [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)] \wedge_o(t-u) \int_{t}^{t+x} g_2(v-u) q_2 \Big\{ [\wedge_o(v-t)$$

$$\int_{v}^{t+x} h(z-v) \sum_{i=1}^{\infty} \wedge_i(z-v)] + \sum_{i=1}^{\infty} \wedge_i(v-t) \Big\}\ dz\ dv\ du\ +$$

$$\int_{o}^{t} M(u) \wedge_o(t-u) \int_{t}^{t+x} g_2(v-u)\ q_2 \Big\{ [\wedge_o(v-t) \int_{v}^{t+x} h(z-v)$$

$$\sum_{i=1}^{\infty} \wedge_i(z-v)] + \sum_{i=1}^{\infty} \wedge_i(v-t) \Big\}\ dz\ dv\ du$$

For (2,1)

$$P\{W_t \leqslant x\} = \wedge_1(t) \int_{t}^{t+x} g_2(u)\ q_2 [q_2 + \int_{u}^{t+x} \sum_{j=1}^{\infty} f_{1j}(v-u)\ +$$

$$\int_{v}^{t+x} f_{10}(v-u)\ h(z-v) \sum_{i=1}^{\infty} \wedge_i(z-v)]\ dz\ dv\ du$$

$$\int_{o}^{t} [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)] \wedge_1(t-u) \int_{t}^{t+x} g_2(v-u) q_2 [q_2 + \int_{v}^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_{s}^{t+x} f_{10}(s-v) h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)] dz\ ds\ dv\ du\ +$$

$$\int_o^t M(u)\ \wedge_1(t-u) \int_t^{t+x} g_2(v-u)\ q_2[q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{ij}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v)\ h(z-s)\ \sum_{i=1}^{\infty} \wedge_i(z-s)]dz\ ds\ dv\ du$$

<u>For (2,2)</u>:

$$P\{W_t \leqslant x\} = \wedge_2(t) \int_t^{t+x} g_2(u)\ g_2\Big\{ \int_u^{t+x} \sum_{j=1}^{\infty} f_{2j}(v-u) +$$

$$\sum_{j=o}^{\infty} f_{1j}(v-u)\ [q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v)\ h(z-s)\ \sum_{i=1}^{\infty} \wedge_i(z-s)]\Big\}\ dz\ ds\ dv\ du +$$

$$\int_o^t (\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)) \wedge_2(t-u) \int_t^{t+x} g_2(v-u)\ q_2\Big\{ \int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(s-v)+$$

$$\sum_{j=o}^{\infty} f_{1j}(s-v)[q_2+ \int_s^{t+x} \sum_{j=1}^{\infty} f_{1j}(w-s)+ \int_w^{t+x} f_{10}(w-s)h(z-w)$$

$$\sum_{i=1}^{\infty} \wedge_i(z-w)]\Big\}dz\ dw\ ds\ dv\ du +$$

$$\int_o^t M(u) \wedge_2(t-u) \int_t^{t+x} g_2(v-u)\ q_2+\Big\{ \int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(s-v) +$$

$$\sum_{j=o}^{\infty} f_{1j}(s-v)\ [q_2+ \int_s^{t+x} \sum_{j=1}^{\infty} f_{1j}(w-s)+ \int_w^{t+x} f_{10}(w-s)\ h(z-w)$$

$$\sum_{i=1}^{\infty} \wedge_i(z-w)]\Big\}\ dz\ dw\ ds\ dv\ du.$$