Chapter-7

FINITE CAPACITY $G/E_k/1$ AND $M/G^{a,b}/1$ QUEUEING SYSTEMS

7.1 Introduction

This chapter deals with two models of finite capacity, single server queueing systems. Model-I is about $G/E_k/1$ queueing system in which the interarrival times of customers are assumed to follow a general distribution. Each arrival to the system induces an additional k phases and the service times in each phase follow exponential distribution. Model-II investigates a queueing system with general bulk service rule with batch sizes varying from a to b. The system capacity is assumed to be finite in both the Models.

There are many situations in real-life where we encounter such types of queueing models as described above. The transportation process involving buses, trains, aeroplanes etc. all deal with such type of queueing systems.

Jacob, Krishnamoorthy and Madhusoodanan (1988) obtain the time dependent solution to $M/G^{a,b}/1$ model whereas the same model with vacation is studied by Jacob and Madhusoodanan (1987). Their approach is based on matrix convolution method. Here we make use of a much more simplified tool- Markov Renewal theoretic approach.

Section 7.2 describes the models and notations used in the chapter. Analysis, time-dependent system size probabilities and the steady state distributions of the $G/E_k/1$ queueing system are carried out in Section 7.3. Analysis of $M/G^{a,b}/1$ queueing model, its system size probabilities, steady state behaviour and virtual waiting time distribution are derived in Section 7.4.

7.2. Description of the Models

Model I- The $G/E_k/1$ system

The interarrival times of customers to the single server queueing system are assumed to be independent and identically distributed random variables following distribution function G(.) with density g(.). The system is of finite capacity b. The service pattern follows Erlang distribution of order $k(k \geq 2)$. The queue discipline is FCFS. Here the arrival of each customer induces additional k phases into the system provided there are atmost (b-1) units in the system. Service in each phase is exponentially distributed with same rate μ. Arrivals taking place when the system is full are lost. The system size at any time point is the number of phases in the system at that instant. The service mechanism is such that the server becomes idle and stops serving only when there is no unit in the system.

Model II- The $M/G^{a,b}/1$ Queue

We assume that the model under consideration has a waiting room (W.R) and a service station (S.R) each of capacity b. Customers arrive one by one according to a homogeneous Poisson process of rate λ and join the W.R subject to the capacity restriction. All arrivals taking place when the W.R. is full are lost to the system. Service times are independent and identically distributed random variables following distribution function H(.) and density function h(.). Services are in batches with at least a customers and a maximum of b in each batch. When the service of a batch is completed the server scans the W.R. and transfers all those in the W.R. provided there are at least a customers, to the S.S. On the other hand if the server finds less than a customers waiting for service, he goes for vacation for a random length of time following distribution function F(.) and density function f(.). On return, if the system size is still less than a, he takes another vacation immediately which is independent of and identically distributed as the previous one. This process continues until on return he finds at least a units waiting for service.

<u>Notations</u>

$*$ denotes convolution. $g^{*n}(x)$ denotes the n-fold convolution of $g(x)$ with itself.

N denotes $\{1,2,3,4,.....\}$

N^o denotes the set $\{0,1,2,...\}$

E is the set $\{0,1,2,..., k,k+1,...,bk\}$

$(Q(i,j,t))$ for $i,j \in E$ is a $(bk+1)x(bk+1)$ matrix whose $(i,j)^{th}$ entry is $Q(i,j,t)$

$P_i(j,t)$ is the probability that at time t there are j phases in the system given that the system has started with i phases initially.

Let $\Lambda_n(x) = \dfrac{e^{-\lambda x}(\lambda x)^n}{n!}$, $n=0,1,2,...,b-1$ and

$$\bar{\Lambda}_b(x) = \sum_{n \geq b} \frac{e^{-\lambda x}(\lambda x)^n}{n!}$$

γ denotes the Gamma density where

$$\gamma_{\lambda,k}(x) = (e^{-\lambda x}\lambda^k x^{k-1})/(k-1)!, \quad k=1,2,3,...$$

E_1 is the set $\{0,1,2,...,a-1\}$

E_2 is the set $\{0,1,2,...,b\}$

$\hat{g}(\alpha)$ - the Laplace transform of $g(t)$.

That is $\hat{g}(\alpha) = \int_0^\infty e^{-\alpha t} g(t)dt$

$(Q_1(i,j,t))$ is a square matrix of order a whose $(i,j)^{th}$ entry is $Q_1(i,j,t)$.

$P_{(o,\ell)}((i,j),t)$ is the probability that i units are undergoing service and j units are in the W.R at time t under the condition that at time zero the server went on vacation as there were only ℓ ($\leqslant a-1$) units in the system, with either $i=0$ or $a \leqslant i \leqslant b$; $j=0,1,2,\ldots,b$.

7.3. Model I: $G/E_k/1$ Queue

Let Y_t denotes the number of phases in the system at time t. Then the process $\{Y_t, t \geqslant 0\}$ is a semi-regenerative process with state space E. Identify $0=T_o, T_1, T_2, \ldots$ as the initial, first, second, ... arrival instants. Let X_n denote the number of phases present in the system just prior to the n^{th} arrival. Then the process $\{(X,T)\} = \{(X_n, T_n), n \in N^o\}$ is the embedded Markov renewal process.

Define

$$Q(i,j,t) = Pr\{X_{n+1}=j; T_{n+1}-T_n \leqslant t \mid X_n = i\},\quad i,j \in E, t \geqslant 0$$

Then (i) for $0 \leqslant i \leqslant (b-1)k$; $j=0,1,2,\ldots,i,\ldots.bk$

$$Q(i,j,t) = \int_0^t \{(e^{-\mu u}(\mu u)^{(i+k-j)}) | (i+k-j)!\} \, g(u)du$$

(ii) for $(b-1)k < i \leqslant bk,\ j \leqslant i$

$$Q(i,j,t) = \int_o^t \{(e^{-\mu u}(\mu u)^{i-j}/(\ i-j\ )!\}\, g(u)du$$

The semi-Markov kernel over E is

$$\mathbb{Q} = \{Q(i,j,t),\ i,j \in E,\ t \geqslant 0\}$$

Let $Q(t) = (Q(i,j,t)),\ i,j \in E$

For all $n \in N$ define

$$Q^n(i,j,t) = \Pr\{X_n=j;\ T_n \leqslant t \mid X_o=i\},\quad i,j \in E,\ t \geqslant 0$$

with $\quad Q^o(i,j,t) = I(i,j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

For all $t \geqslant 0$, $n \in N$ we have the recursive relation,

$$Q^{n+1}(i,m,t) = \sum_{j \in E} \int_o^t Q(i,j,du)\ Q^n(j,m,t-u)$$

The Markov renewal function is given by

$$R(i,j,t) = \sum_{n=o}^{\infty} Q^{*n}(i,j,t),\ i,j \in E,\ t \geqslant 0$$

with the Markov renewal kernel $\mathbb{R}$.

Let $\mathbb{R}(t) = (R(i,j,t))$ for $i,j \in E,\ t \geqslant 0$

Since E is finite, the Markov renewal kernel can be computed by taking Laplace transforms.

System size probabilities

Assume that initially there are a units (ak phases) in the system. Then $P_{ak}(i,t)$ satisfies the Markov renewal equations so that

$$P_{ak}(i,t) = \sum_{m \in E} \int_0^t R(ak,m,du)\, K_m(i,t-u)$$

where $K_m(i,t) \;= \Pr\{Y_t=i,\ T_1>t \mid X_o=m\}$ and is obtained as follows:

(i) for $0 \leqslant m \leqslant bk$, $i=0$

$$K_m(i,t) = \{(e^{-\mu t}(\mu t)^m)/m!\}\,[1-G(t)]$$

(ii) for $0 \leqslant m \leqslant (b-1)k$, $1 \leqslant i \leqslant m+k$

$$K_m(i,t) = \frac{e^{-\mu t}(\mu t)^{m+k-i}}{(m+k-i)!}\,[1-G(t)]$$

(iii) finally for $(b-1)k < m \leqslant bk$, $1 \leqslant i \leqslant m$

$$K_m(i,t) = \frac{e^{-\mu t}(\mu t)^{m-i}}{(m-i)!}\,[1-G(t)]$$

Steady state distribution

To obtain the limiting distribution of the system size, first we verify whether (X,T) is irreducible, recurrent and aperiodic. For this we assume that the expected number of phases L completed during an interarrival time is greater than k. With this assumption (X,T) becomes irreducible and recurrent.

That is $E(L) = \int_{o}^{\infty} \sum_{\ell \in E} \ell \frac{e^{-\mu u}(\mu u)^{\ell}}{\ell !} g(u)du > k$

Also the sojourn time in state 0 has an exponentially distributed component and so Q(o,j,t) is not a step function. Hence all states are aperiodic in (X,T). So (X,T) is an irreducible aperiodic and recurrent process. Further the state space E is finite.

Let π be an invariant measure for X which gives the stationary distribution of the process (X,T). This π is obtained from the solution of the set of linear equations $\pi\mathbb{P}=\pi$ subject to the condition $\pi e=1$ where π is a (bk+1) component row vector and e = (1,1,...,1) which is a (bk+1)-component column vector.

Further $\mathbb{P} = ((p_{i,j}))$, $i,j \in E$ is a matrix of order (bk+1) x (bk+1) where

$$p_{i,j} = \Pr\{X_{n+1}=j \mid X_n=i\} = Q(i,j,\infty)$$

$$\pi \mathbb{P} = \pi \text{ implies that}$$

$$\sum_{i \in E} \pi(i)\, p_{i,j} = \pi(j), \; j \in E$$

The expected sojourn time in state j is

$$\tau(j) = \int_0^\infty [1 - \sum_{i \in E} Q(i,j,t)]dt$$

Let $\tau = \{\tau(0), \tau(1), \ldots, \tau(bk)\}$

We compute the limiting probabilities as

$$\lim_{t \to \infty} P_{ak}(i,t) = \underline{P}(i) = \frac{\sum_{j=0}^{i} \pi(j)\, n(j,i)}{\sum_{j \in E} \pi(j)\, \tau(j)}$$

where $n(j,i) = \int_0^\infty K_j(i,t)dt$

The limiting distribution of thesystem size probabilities are also obtained by the method of Laplace transformations as follows:

We have already obtained

$$P_{ak}(i,t) = \sum_{m \in E} \int_0^t R(ak,m,du)\, K_m(i,t-u) \qquad \text{(A)}$$

Let

$$\hat{P}_\alpha(ak,i) = \int_0^\infty e^{-\alpha t} P_{ak}(i,t)dt$$

$$\hat{R}_\alpha(ak,m) = \int_0^\infty e^{-\alpha t} R(ak,m,t)dt$$

and

$$\hat{K}_\alpha(m,i) = \int_0^\infty e^{-\alpha t} K_m(i,t)dt$$

Taking Laplace transform of both sides of (A) and applying Tauberian theorem (Widder (1948)) we have

$$\lim_{t \to \infty} P_{ak}(i,t) = \lim_{\alpha \to 0} \hat{P}_\alpha(ak,i)$$

$$= \lim_{\alpha \to 0} \left(\sum_{m \in E} \hat{R}_\alpha(ak,m) \hat{K}_\alpha(m,i) \right)$$

7.4. Model-II- $M/G^{a,b}/1$ Queue

Let g(u)du be the probability that the service of a batch which has started at time zero is completed in the interval (u,u+du] and during this service time at least a arrivals have taken place. Thus

$$g(u)du = \sum_{j=a}^{\infty} \Lambda_j(u) \; h(u)du \tag{1}$$

where h(.) is the service time density function. Let X(t) be the number of units underoing service and Y(t) be the number of units waiting for service at time t.

Define $Z(t) = \{X(t), Y(t)\}$. Then the stochastic process $\{Z(t), t \geq 0\}$ is a semi-regenerative process defined over the state space $E_1 \times E_2$.

The time epochs at which the server goes for vacation after a service with less than a units waiting for service are the busy period termination epochs. Let $0=T_o, T_1, T_2, \ldots$ be the time epochs of successive busy period terminations and Y_n, $n=0,1,2,\ldots$ be the number of units in the system (ie. in the W.R.) at time T_n+. Then the process $\{(Y,T)\} = \{(Y_n, T_n), n \in N^o\}$ is a time homogeneous Markov renewal process defined over the set E_1. The kernal of the semi-Markov process is

$$\{Q_1(i,j,t),\ i,j \in E_1,\ t \geq 0\}$$

where the $Q_1(i,j,t)$ are given by

$$Q_1(i,j,t) = \Pr\{Y_{n+1}=j; T_{n+1}-T_n \leq t \mid Y_n = i\},\quad i,j \in E_1,\ t \geq 0$$

$$= \int_o^t \int_u^t \int_v^t \sum_{n=o}^{\infty} f^{*n}(u)\ f(v-u) \sum_{r=o}^{a-1-i} \Lambda_r(u)\, \bar{\Lambda}_{a-(i+r)}(v-u)$$

$$\sum_{m=1}^{\infty} g^{*(m-1)}(w-v)\ h(t-w)\ \Lambda_j(t-w)\,dw\,dv\,du \quad (2)$$

The Markov Renewal function is given by

$$R_1(i,j,t) = \sum_{n=o}^{\infty} Q_1^{*n}(i,j,t),\quad i,j \in E_1,\ t \geq 0.$$

System size probabilities

We assume that initially, ie. at time $T_0=0$, the server just enters a vacation period after completing a busy period so that the initial state of the process is

$$Z(o) = \{X(o), Y(o)\} = (o,\ell) \text{ for some } \ell \in E_1$$

Let

$$P_{(o,\ell)}((i,j),t) = \Pr\{Z(t)=(i,j)|Z(o)=(o,\ell)\} \; , \ell \in E_1.$$

Then

$$P_{(o,\ell)}((i,j),t) = \Pr\{Z(t)=(i,j),T_1>t|Z(o)=(o,\ell)\} + \Pr\{Z(t)=(i,j),T_1\leqslant t|Z(o)=(o,\ell)\}$$

Now

$$\Pr\{Z(t)=(i,j),\ T_1\leqslant t|Z(o)=(o,\ell)\} = \sum_{k\in E_1}\int_o^t Q_1(\ell,k,du)\; P_{(o,k)}((i,j),t-u)$$

and

$$\Pr\{Z(t)=(i,j);T_1>t|Z(o)=(o,\ell)\} = K_{(o,\ell)}((i,j),t)$$

Then $P_{(o,\ell)}((i,j),t)$ satisfies the Markov renewal equation. Hence the solutions are given by

$$P_{(o,\ell)}((i,j),t) = \sum_{k\in E_1}\int_o^t R_1(\ell,k,du)\; K_{(o,k)}((i,j),t-u)$$

where

(i) for $a<i\leqslant b$; $j=0,1,2,\ldots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \int_u^t \sum_{n=o}^{\infty} f^{*n}(u)f(v-u) \sum_{r=o}^{a-1-k} \wedge_r(u)\bar{\wedge}_{a-(k+r)}(v-u)$$

$$\int_v^t \sum_{m=1}^{\infty} g^{*m-1}(w-v) \int_w^t h(x-w)\wedge_i(x-w)\ [1-H(t-x)]$$

$$\wedge_j(t-x)dx\ dw\ dv\ du$$

(ii) for $i=a$; $j=0,1,2,\ldots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \int_u^t \sum_{n=o}^{\infty} f^{*n}(u)f(v-u) \sum_{r=o}^{a-1-k} \wedge_r(u)$$

$$\{\wedge_{a-(k+r)}(v-u)[1-H(t-v)]\wedge_j(t-v)$$

$$+\bar{\wedge}_{a-(k+r)}(v-u) \int_v^t \int_w^t \sum_{m=1}^{\infty} g^{*(m-1)}(w-v)\ h(y-w)$$

$$\wedge_a(y-w)\ [1-H(t-y)]\wedge_j(t-y)dy\ dw\}\ dv\ du$$

(iii) for $i=0$, $j=k,k+1,\ldots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \sum_{n=o}^{\infty} f^{*n}(u) \sum_{m=o}^{a-1-k} \wedge_m(u)[1-F(t-u)]\bar{\wedge}_{j-(k+m)}(t-u)du$$

and 0 otherwise

Virtual waiting time distribution

Let W(t) be the virtual waiting time of a customer in the queue.

Let $\Pr\{W(t) \leqslant x \mid Z(t) = (i,j),\ Z(o) = (o,\ell)\} = B\{W(t) \leqslant x\}$

Then

(i) for $a \leqslant i \leqslant b;\ a-1 \leqslant j \leqslant b-1$

$$B\{W(t) \leqslant x\} = \int_o^t \int_u^t \sum_{k \in E_1} R_1(\ell,k,du) \sum_{n=o}^{\infty} f^{*n}(v-u) \int_v^t f(w-v) \sum_{r=o}^{a-1-k} \Lambda_r($$

$$\Lambda_{i-(r+k)}(w-v)\, \Lambda_j(t-w)\,[1-H(t-w)] \int_t^{t+x} h(\tau)\,d\tau\ dw\ dv$$

(ii) for $a \leqslant i \leqslant b;\ o \leqslant j < a-1$

$$B\{W(t) \leqslant x\} = \int_o^t \int_u^t \sum_{k \in E_1} R_1(\ell,k,du) \sum_{n=o}^{\infty} f^{*n}(v-u)$$

$$\int_v^t f(w-v) \sum_{r=o}^{a-1-k} \Lambda_r(v-u)\Lambda_{i-(r+k)}(w-v)$$

$$\Lambda_j(t-w)\ [1-H(t-w)] \int_t^{t+x} h(\tau)\bar{\Lambda}_{a-(j+1)}(\tau-t)\,d\tau\ dw\ dv$$

(iii) for $i=0,\ 0 \leqslant j < b-1$

$$B\{W(t) \leqslant x\} = \int_o^t \sum_{k \in E_1} R_1(\ell,k,du) \int_t^{t+x} f(v-u)\Lambda_{j-k}(t-u)\bar{\Lambda}_{a-(j+1)}(v-t$$

(iv) for $a \leq i \leq b$; $j=b$

$$B\{W(t)\leq x\} = \int_o^t \sum_{k\in E_1} R_1(\ell,k,du) \int_u^t \sum_{n=o}^{\infty} f^{*n}(v-u)$$

$$\int_v^t f(w-v) \sum_{r=o}^{a-1-k} \Lambda_r(v-u) \Lambda_{i-(r+k)}(w-v)$$

$$\bar{\Lambda}_b(t-w) \int_t^{t+x} h(\tau-w) \int_\tau^{t+x} h(z-\tau)\,\bar{\Lambda}_{a-1}(z-\tau)dz\,d\tau\,dw\,dv$$

finally for $i=0$, $j=b$

$$B\{W(t)\leq x\} = \int_o^t \sum_{k\in E_1} R_1(\ell,k,du) \int_t^{t+x} f(v-u)\bar{\Lambda}_{b-k}(t-u)$$

$$\int_v^{t+x} h(w-v)\,\bar{\Lambda}_{a-1}(w-v)dw\,dv$$

Next consider the case in which the server does not go for vacation.

As in the previous case, the semi-regenerative process is $\{Z(t), t\geq 0\}$ defined over the state space $E_1 \times E_2$. Let $0 = T_o, T_1, T_2, \ldots$ be the successive time points at which the server becomes idle with less than a customers waiting for service and Y_n be the number of customers waiting for service at time T_n so that $Y_n = Y(T_n+)$. Here also

$$\{(Y,T)\} = \{(Y_n,T_n),\ n \in N^o\}$$

is the time homogeneous Markov renewal process defined over E_1 with

$$Q_2(i,j,t)= \int_o^t \gamma_{\lambda,a-i}(u) \int_u^t \sum_{m=1}^{\infty} g^{*(m-1)}(v-u)\ h(t-v)\wedge_j(t-v)dv\ du$$

and

$$R_2(i,j,t)= \sum_{n=o}^{\infty} Q_2^{*n}(i,j,t) \qquad (3)$$

The system size probabilities in this case are given by

$$P_{(o,\ell)}((i,j),t) = \sum_{k\in E_1} \int_o^t R_2(\ell,k,du)\ K_{(o,k)}(i,j),t-u)$$

where $K_{(o,k)}((i,j),t)$ is as defined earlier and the expressions are given by

(i) for $a<i\leq b$, $j=0,1,2,\dots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \int_u^t [\gamma_{\lambda,a-k}(u) \sum_{m=1}^{\infty} g^{*m-1}(v-u)$$
$$\int_v^t h(w-v)\wedge_i(w-v)[1-H(t-w)]\wedge_j(t-w)]dw\ dv\ du$$

(ii) for $i=a$, $j=0,1,2,\dots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \gamma_{\lambda,a-k}(u)\{[1-H(t-u)]\wedge_j(t-u) +$$
$$\int_u^t \int_v^t \sum_{m=1}^{\infty} g^{*(m-1)}g(v-u)h(w-v)$$
$$\wedge_a(w-v)[1-H(t-w)]\wedge_j(t-w)dw\ dv\}\ du$$

finally (iii) for $i=0$, $j \leq a-1$

$$K_{(o,k)}((i,j),t) = \wedge_{j-k}(t)$$

The virtual waiting time distribution in this case conditioned on the system size are given below:

(i) for $a \leq i \leq b$; $a-1 \leq j \leq b-1$

$$B\{W(t) \leq x\} = \int_o^t \int_u^t \int_v^t \sum_{k=o}^{a-1} R_2(\ell,k,du)\ \gamma_{\lambda,a-k}(v-u) \sum_{m=1}^{\infty} g^{*m-1}(w-v)$$

$$\int_w^t h(y-w)\wedge_i(y-w)[1-H(t-w)] \int_t^{t+x} h(\tau)\wedge_j(t-y)\ d\tau\ dy\ dw\ dv$$

(ii) for $a \leq i \leq b$; $0 \leq j < a-1$

$$B\{W(t) \leq x\} = \int_o^t \int_u^t \sum_{k=o}^{a-1} R_2(\ell,k,du)\ \gamma_{\lambda,a-k}(v-u) \int_v^t \sum_{m=1}^{\infty} g^{*(m-1)}(w-v)$$

$$\int_w^t h(y-w)\wedge_i(y-w)[1-H(t-w)]$$

$$\int_t^{t+x} h(\tau)\wedge_j(t-y)\ \gamma_{\lambda,a-1-j}(\tau-t)d\tau\ dy\ dw\ dv$$

finally for $i=0$; $o \leq j < a-1$

$$B\ \{W(t) \leq x\} = \int_o^t \int_u^t \sum_{k \in E_1} R_2(\ell,k,du)\ \gamma_{\lambda,a-k}(v-u) \int_v^t \sum_{m=1}^{\infty} g^{*(m-1)}(w-v)$$

$$\int_w^t h(y-w) \sum_{r=o}^{j} \wedge_r(y-w)\wedge_{j-r}(t-y) \int_t^{t+x} \gamma_{\lambda,a-(j+1)}(z)dz\ dy\ dw\ dv$$