Chapter-5

FINITE CAPACITY QUEUEING SYSTEMS WITH STATE DEPENDENT NUMBER OF STAGES OF SERVICE

5.1. Introduction

This chapter introduces a class of finite capacity single server queueing models in which the server offers a random number of stages of service say k, k=1,2,...,m to each unit depending upon the system size at the onset of its service. First we examine Model I in which the arrival pattern follows Poisson process of rate λ. The distribution of service times in all stages are independent and identically distributed (i.i.d) random variables following distribution function G(.) and probability density function g(.) with mean μ_1(assumed to be finite). Thus the system under consideration generalises the truncated (truncated at m) bilateral Phase-type distribution (see Shanthikumar 1985) in which arrival takes place according a Poisson process.

In Model II, the interarrival times follow a general distribution F(.) with probability density function f(.) and mean a (finite). Service times in each stage are i.i.d. random variables following exponential distribution with parameter μ(fini

Consider a service facility manned by a single server and providing m different types of services. The services are given one by one by the same server with each unit demanding m stages of service. The system is of finite capacity b so that customers arriving when the system is full are lost to the system. However, each unit at the commencement of its service is offered a random number of stages of service (at least equal to one and atmost m) depending upon the number of units in the system at that epoch. Specifically, we assume that the number k of stages of service offered when the number of units in the system at the commencement of service is j has probability q_{jk}, $j=1,2,\ldots,b$, $k=1,2,\ldots,m$. We encounter such a situation as modelled in this chapter in workshops where machines of identical nature are brought for overhauling.

In the queueing literature one can find that the service characteristics change dynamically to accommodate variations in the queue size. Hillier et. al. (1964), Gupta (1967) and Rosenshine (1967) have examined queueing systems in which the service rates are an instantaneous functions of the system state Shantikumar (1979) discusses a class of queueing models in which the service time of a customer at a single server facility is dependent on the queue size at the onset of its service. Bertsimas (1990) analyses the $C_k/C_m/s$ system where C_k is the class of Coxian probability density functions of order k.

Whereas he does not assume the state-dependent number of stages of service we do make this assumption and further assume that the service time in each stage has a general distribution in Model I. Further service times in all stages are i.i.d random variables independent of the system size.

The following are the notations that are used in the sequel.

Let $\Lambda_j(x)$ denote the probability that j arrivals take place in an interval of duration x.

When interarrival times follow exponential distribution with parameter λ, we have

$$\Lambda_j(x) = e^{-\lambda x} (\lambda x)^j / j!, \quad j = 0,1,2,\ldots,b-1$$

and let $\bar{\Lambda}_b(x)$ be defined as

$$\bar{\Lambda}_b(x) = \sum_{j \geq b} e^{-\lambda x} (\lambda x)^j / j!$$

E denotes $\{(o,o,o)\} \cup \{(i,j,k): i \in (1,2,\ldots,b);$
$j \in (1,2,\ldots,m);$
$k \in (0,1,2,\ldots,j-1)\}$

X(t) - the system size at time t

Y(t) - the number of stages of service offered at the onset of service to the unit undergoing service at time t.

Z(t) - the number of stages of service completed by the unit undergoing service at time t.

The system is in (i,j,k) means X(t)=i, Y(t)=j, Z(t)=k, $(i,j,k) \in E$ when X(t)=0, then Y(t)=0 and Z(t)=0.

* - convolution

N - the set of natural numbers

N^o - $\{0\} \cup N$

$\gamma_{\mu,r}(u)$ - is the gamma density with parameters μ and r, $\mu > 0$ and r = 1,2,...

Analysis, system size probabilities and the limiting distribution of Model I are provided in Section 5.2. Section 5.3 deals with the analysis, system size probabilities and the limiting distribution of Model II. The last section illustrates computational problem associated with the models.

5.2. Model I

Analysis

Let $0 = T_o < T_1 < T_2 < \ldots < T_n \ldots$ be the successive stage service completion epochs, $X_o, X_1, X_2, \ldots, X_n, \ldots$ be the number of units in the system just after $T_o, T_1, T_2, \ldots, T_n, \ldots$. Let $Y_o, Y_1, Y_2, \ldots$ and $Z_o, Z_1, Z_2, \ldots$ respectively denote the number of stages of service offered and the number of stages of service completed by the unit undergoing service at time T_n, $n=0,1,2,\ldots$ Then the process $\{(X,Y,Z),T\} = \{(X_n,Y_n,Z_n),T_n; n \in N^o\}$ is easily seen to constitute a Markov renewal process defined over the state space E.

Consider the epochs of successive stage completion (not service completion) of a unit. Define

$$Q\{(i_1,j,k_1),(i_2,j,k_2=k_1+1),t\}$$

$$= \Pr\{(X_{n+1}=i_2,\ Y_{n+1}=j,\ Z_{n+1}=k_2),\ T_{n+1}-T_n \leqslant t \mid (X_n=i_1,\ Y_n=j,\ Z_n=k_1)\}$$

$$= \begin{cases} \int_0^t \Lambda_{i_2-i_1}(u)\, g(u)du, & \text{for } i_1=1,2,\ldots,b-1,\ i_2=i_1+r_1 \\ & \text{where } r_1=0,1,2,\ldots,b-i_1;\ j=1,2,\ldots,m \text{ and } k_1=0,1,2,\ldots,j-2 \\ \int_0^t \bar{\Lambda}_{b-i_1}(u)\, g(u)du, & \text{for } i_1=1,2,\ldots,b;\ i_2=b \\ & j=1,2,\ldots,m,\ k_1=0,1,2,\ldots,j-2 \end{cases}$$

In the case when the service commencement epoch of a stage is such that it is the last stage for the unit undergoing service, then

$$Q\{(i_1,j_1,j_1-1),(i_2,j_2,0),t\} = \Pr\{(X_{n+1}=i_2,\ Y_{n+1}=j_2,\ Z_{n+1}=0\ \ T_{n+1}-T_n \leq t \mid (X_n=i_1, Y_n=j_1, Z_n=j_1-1)\}$$

$$= \begin{cases} \int_0^t \Lambda_{i_2-i_1+1}(u)\, q_{i_2j_2}\, du, & \text{for } i_1=1,2,\dots,b-1,\ i_2=i_1+r_1-1 \\ \quad \text{where } r_1=0,1,2,\dots,b-i_1 & \\ \int_0^t \bar{\Lambda}_{b-i_1+1}(u)\, g(u)\, q_{i_2j_2} du, & \text{for } i_1=1,2,\dots,b;\ i_2=b,\ j_1,j_2=1,2,\dots,m \end{cases}$$

with the provision that if $i_1=1$ and $i_2=0$, then

$$Q\{(1,j_1,j_1-1),(0,0,0),t\} = \int_0^t e^{-\lambda u}\, g(u)du$$

For all $n \in N$ define

$$Q^n\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \Pr[(X_n,Y_n,Z_n)=(i_2,j_2,k_2); T_n \leq t\ \ (X_0,Y_0,Z_0)=(i_1,j_1,k_1)\}$$

Then we have the recursive relation

$$Q^{n+1}\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \sum_{i=i_1-1}^{b} \sum_{j=o}^{m} \sum_{k=o,k_1+1}$$

$$\int_o^t Q\{(i_1,j_1,k_1),(i,j,k),du\} Q^n\{(i,j,k),(i_2,j_2,k_2),t-u\}$$

Finally define the Markov renewal function

$$R\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \sum_{n=o}^{\infty} Q^{*n}\{(i_1,j_1,k_1)(i_2,j_2,k_2),t\}, t\geq 0$$

and $(i_1,j_1,k_1),(i_2,j_2,k_2) \in E$

with $Q^o\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \begin{cases} 1 \text{ for } (i_1,j_1,k_1)=(i_2,j_2,k_2) \\ 0 \text{ for } (i_1,j_1,k_1)\neq(i_2,j_2,k_2) \end{cases}$

System size probabilities

Without loss of generality we may assume that at time $T_o=0$ a stage service completion has just taken place so that the state of the system is $(X_o,Y_o,Z_o) = (i_o,j_o,k_o)$ (assumed fixed). Consider the three dimensional process $L(t)=\{X(t),Y(t),Z(t)\}$. Then the process $\{L(t),\ t\geq 0\}$ is the associated semi-regenerative process with the Markov renewal process $\{(X,Y,Z),T\}$ embedded in it.

Define

$$P_{(i_o,j_o,k_o)}((i,j,k),t) = Pr\{L(t)=(i,j,k)|L(o)=(i_o,j_o,k_o)\}$$

Then $$P_{(i_o,j_o,k_o)}((i,j,k),t) = Pr\{L(t)=(i,j,k);T_1>t|L(o)=(i_o,j_o,k_o)\}$$
$$Pr\{L(t)=(i,j,k);T_1\leq t|L(o)=(i_o,j_o,k_o)\}$$

Let $Pr\{L(t) = (i,j,k);\ T_1 > t|\ L(o) = (i_o,j_o,k_o)\}$ represents $K_{(i_o,j_o,k_o)}((i,j,k),t)$. Then

$$K_{(i_o,j_o,k_o)}((i,j,k),t) = K_{(i_o,j_o,k_o)}((i,j_o,k_o),t)$$

$$= \begin{cases} \Lambda_{i-i_o}(t)\ [1-G(t)], & i \neq b \\ \bar{\Lambda}_{b-i_o}(t)\ [1-G(t)], & i=b \end{cases}$$

Now $K_{(.)}(.,t)$ is bounded over finite intervals and directly Riemann integrable. So $P_{(i_o,j_o,k_o)}((i,j,k),t)$ satisfies Markov renewal equation (Cinlar 1975 a).

Hence the solutions are given by

(i) for $(i,j,k) \neq (o,o,o)$,

$$P_{(i_o,j_o,k_o)}((i,j,k),t) = \int_o^t \sum_{i_1 \leq i} R\{(i_o,j_o,k_o),(i_1,j,k),du\} \; K_{(i_1,j,k)}((i,j,k),t-u)$$

(ii) Probability that the system size is zero at time t is

$$P_{(i_o,j_o,k_o)}((o,o,o),t) = \int_o^t R\{(i_o,j_o,k_o),(o,o,o),du\} \Lambda_o(t-u)$$

Limiting Behaviour

We start with a given set of q_{ij}'s, $i=1,2,\ldots,b-1$; $j=1,2,\ldots,m$. From the given q_{ij}'s, the one-step transition probabilities corresponding to the Markov chain $\{(X,Y,Z)\} = \{(X_n,Y_n,Z_n),\ n \in N^o\}$ is evaluated as follows.

Define

$$p_{((i_1,j_1,k_1),(i_2,j_2,k_2))} = Pr\{(X_{n+1}, Y_{n+1}, Z_{n+1})=(i_2,j_2,k_2) \mid (X_n,Y_n,Z_n) = (i_1,j_1,k_1)\}$$

For $i_1=0$ and $i_2=0$

$$p_{((i_1,j_1,k_1),(i_2,j_2,k_2)} = \int_o^\infty \lambda e^{-\lambda u} \int_u^\infty q_{11}\, g(v-u)\, \Lambda_o(v-u)dv\, du$$

For $i_1=o$, $i_2=1$, $j_2=1,2,\ldots,m$ and $k_2=0$

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$$p_{((i_1,j_1,k_1),(i_2,j_2,k_2))} = \int_0^{\infty} \lambda e^{-\lambda u} \int_u^{\infty} q_{11}\; g(v-u)\Lambda_1(v-u) q_{1j_2} dv\, du$$

Likewise the various transition probabilities can be computed for different values of $(i_1,j_1,k_1),(i_2,j_2,k_2) \in E$. From these the stationary probabilities $\pi(i,j,k),(i,j,k) \in E$ associated with the Markov chain $\{(X,Y,Z)\}$ are computed from the solution of the equations

$$\pi(i,j,k) = \sum_{i_1=o}^{b} \sum_{j_1=o}^{m} \sum_{k_1=o}^{j_1-1} \pi(i_1,j_1,k_1)\, p_{((i_1,j_1,k_1),(i,j,k))}$$

$$\text{with} \sum_{i=o}^{b} \sum_{j=o}^{m} \sum_{k=o}^{j-1} \pi(i,j,k)=1$$

The mean sojourn time in any state (i,j,k) is $m(i,j,k)$ and

$$\begin{aligned} m(i,j,k) &= E[T_{n+1}-T_n \leqslant t \mid (X_n=i,Y_n=j,Z_n=k)] \\ &= \mu_1 \text{ (assumed finite)} \end{aligned}$$

Define

$\int_0^{\infty} K_{(i_1,j,k)}((i,j,k),t)dt$ as $n((i_1,j,k),(i,j,k))$

Let $\lim_{t \to \infty} P_{(i_o,j_o,k_o)}((i,j,k),t) = \underline{P}(i,j,k)$

Then $\underline{P}(i,j,k)$ is obtained as

$$\underline{P}(i,j,k)=\frac{\sum\limits_{i_1\leqslant i}\pi(i_1,j,k)\ n((i_1,j,k),(i,j,k)}{\sum\limits_{i_1=o}^{b}\ \sum\limits_{j_1=o}^{m}\ \sum\limits_{k_1=o}^{j_1-1}\pi(i_1,j_1,k_1)\ m(i_1,j_1,k_1)}\qquad \text{(Cinlar 1975b)}$$

Hence for $i\neq 0$,

$$\underline{P}(i,j,k) = (\mu_1)^{-1}\sum_{i_1\leqslant i}\pi(i_1,j,k)\ n((i_1,j,k)(i,j,k))$$

and for $i=0$

$$\underline{P}(o,o,o) = (\mu_1)^{-1}\ \pi(o,o,o)\int_o^\infty \wedge_o(t)dt$$

5.3. Model-II

Analysis

Let $0 = T_o,T_1,T_2,\ldots$ be the successive arrival instants; $X_o,X_1,X_2,\ \ldots$ be the system size just prior to $T_o,T_1,T_2,\ldots$ and Y_i,Z_i, $i=0,1,2,\ldots$ are as defined in Model I. Then the process $\{(X,Y,Z),T\} = \{(X_n,Y_n,Z_n);T_n\ n \in N^o\}$ constitute a Markov renewal process defined over the state space E with

$$Q_1\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = Pr\{(X_{n+1}=i_2,Y_{n+1}=j_2,Z_{n+1}=k_2);$$
$$T_{n+1}-T_n\leqslant t\,|\,(X_n=i_1,Y_n=j_1,Z_n=k_1)\}$$

$$= \int_o^t \sum_{\ell_2+\ell_3+\ldots+k_2=i_1-i_2}^{(i_1-i_2)m} \gamma_{\mu,j_1-k_1+\ell_2+\ell_3+\ldots+k_2}(u)$$

$$q_{i_1\ell_2}\, q_{i_1-1\ell_3}\cdots q_{i_2j_2}\, e^{-\mu(t-u)}\, dF(u)$$

for $t\geqslant 0$ and $(i_1,j_1,k_1),(i_2,j_2,k_2) \neq (o,o,o)$

and

$$Q_1\{(o,o,o),(o,o,o),t\} = \sum_{j_1=1}^{m} \int_o^t dF(u)\, q_{1j_1}\, \gamma_{\mu,j_1}(t-u)$$

The Markov renewal function is given by

$$R_1\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \sum_{n=o}^{\infty} Q^{*n}\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\},\ t\geqslant$$

and $(i_1,j_1,k_1),(i_2,j_2,k_2) \in E$.

System Size Probabilities

Initially at time $T_o=0$, we assume that an arrival is just taking place so that the state of the system is (i_o,j_o,k_o) (fixed) As before defining $L(t) = \{(X(t),Y(t),Z(t))\}$, the stochastic process $\{L(t),t\geqslant 0\}$ is the semi-regenerative process defined over the state space E. In this case also it is readily seen that

$$P^{(1)}_{(i_o,j_o,k_o)}((i,j,k),t) = Pr\{L(t)=(i,j,k)\,|\,L(o)=(i_o,j_o,k_o)\}$$

satisfies the Markov renewal equation so that the solution is given by

(i) for $(i,j,k) \neq (o,o,o)$

$$P^{(1)}_{(i_o,j_o,k_o)}((i,j,k),t) = \int_o^t \sum_{i_1 \geq i} \sum_{j_1=1}^{m} \sum_{k_1=o}^{j_1-1} R_1\{(i_o,j_o,k_o),(i_1,j_1,k_1),du\}\, K^{(1)}_{(i_1,j_1,k_1)}((i,j,k),t-u$$

where

$$K^{(1)}_{(i_1,j_1,k_1)}((i,j,k),t) = Pr\{L(t)=(i,j,k),T_1>t\,|\,L(o)=(i_1,j_1,k_1)\}$$

$$= \begin{cases} [1-F(t)] \displaystyle\int_o^t \sum_{\ell_2+\ell_3+\ldots+k=i-i_1}^{(i-i_1)m} \gamma_{\mu,j_1-k_1+\ell_2+\ell_3+\ldots+k}(u)\, e^{-\mu(t-u)}\, du, & \text{for } (i_1,j_1,k_1) \neq (i,j,k) \\ [1-F(t)]e^{-\mu t} & \text{for } (i_1,j_1,k_1)=(i,j,k) \end{cases}$$

(ii) The probability that the system size is zero at time t is given by

$$P^{(1)}_{(i_o,j_o,k_o)}((o,o,o),t) = \int_o^t \sum_{i_1=o}^{b} \sum_{j_1=1}^{m} \sum_{k_1=o}^{j_1-1} R_1\{(i_o,j_o,k_o),(i_1,j_1,k_1),du\} K^{(1)}_{(i_1,j_1,k_1)}((o,o,o),$$

where

$$K^{(1)}_{(i_1,j_1,k_1)}((o,o,o),t) = [1-F(t)] \int_o^t \sum_{\ell_2+\ell_3+\ldots+\ell_{i_1}=i_1}^{i_1 m} \gamma_{\mu,j_1-k_1+\ell_2+\ldots+\ell_{i_1}}(u)du$$

Limiting Behaviour

For this model also, given q_{ij}'s, i=1,2,....,b and j=1,2,...,m, the one-step transition probabilities corresponding to the Markov chain $\{(X,Y,Z)\}$ are computed as in the case of Model-I. Also the stationary distributions $\pi^{(1)}(i,j,k)$, $(i,j,k) \in E$ are also evaluated as before. The mean sojourn time in any state is

$$m^{(1)}(i,j,k) = \int_o^\infty [1-F(t)]e^{-\mu t}\, dt, \quad i>0 \quad \text{and}$$

$$m^{(1)}(o,o,o) = \int_o^\infty [1-F(t)]dt = a$$

Let $\lim_{t \to \infty} P^{(1)}_{(i_o,j_o,k_o)}((i,j,k),t) = \underline{P}^{(1)}(i,j,k)$.

Then the limiting probabilities are now computed as, for all $(i,j,k) \in E$,

$$\underline{P}^{(1)}(i,j,k) = \frac{1}{m^{(1)}(i,j,k)} \Big\{ \pi^{(1)}(i,j,k)\, n^{(1)}((i,j,k),(i,j,k)) + \sum_{i_1=i+1}^{b} \sum_{j_1=1}^{m} \sum_{k_1=o}^{j_1-1} \pi^{(1)}(i_1,j_1,k_1)\, n^{(1)}((i_1,j_1,k_1),(i,j,k))$$

where $n^{(1)}(i_1,j_1,k_1),(i_2,j_2,k_2) = \int_o^{\infty} K^{(1)}_{(i_1,j_1,k_1)}((i_2,j_2,k_2),t)\,dt$,

for $(i_1,j_1,k_1),(i_2,j_2,k_2) \in E$

5.4. Numerical Illustrations

1. Consider the case when b=2 and m=2. For Model I, let the parameter λ of the arrival process be equal to 1. Also assume that the service time in each stage follows exponential distribution with parameter $\mu=1$. Let the probabilities determining the number of stages be $q_{11} = .3$ and $q_{12} = .7$.

From the given q_{ij}'s, the transition probability matrix $\mathbb{P}$ associated with the Markov chain $\{(X,Y,Z)\}$ defined on the state space $\{(0,0,0),\ (1,1,0),\ (1,2,0),\ (1,2,1),\ (2,2,1)\}$ is computed as

$$\mathbb{P} = \begin{bmatrix} 0.15 & 0.045 & 0.105 & 0.35 & 0.35 \\ 0.5 & 0.15 & 0.35 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.15 & 0.35 & 0 & 0 \\ 0 & 0.3 & 0.7 & 0 & 0 \end{bmatrix}$$

The stationary probabilities are obtained as

$$\pi = \{\pi(0,0,0)= 0.109,\ \pi(1,1,0)=0.129,\ \pi(1,2,0)=.232,\ \pi(1,2,1)= 0.265,\ \pi(2,2,1)=0.265\}$$

The limiting probabilities are given by

$$\underline{P}(0,0,0) = 0.109$$
$$\underline{P}(1,1,0) = 0.1707$$
$$\underline{P}(1,2,0) = 0.1923$$
$$\underline{P}(1,2,1) = 0.1325$$
$$\underline{P}(2,2,1) = 0.3975$$

2. Considering Model II let the given set of probabilities q_{ij}, $i=1,2,3$ be $q_{11}=0.1$, $q_{12}=0.2$; $q_{21}=0.2$; $q_{22}=0.1$; $q_{31}=0.2$; $q_{32}=0.2$ for $b=3$ and the maximum number of stages offered $m=2$.

For simplicity assume that the interarrival times follow exponential distribution with parameter $\lambda=1$ and the service rate μ in each stage be also equal to 1. The transition probability matrix associated with the Markov chain $\{(X,Y,Z)\}$ defined over the state space $\{(0,0,0), (1,1,0), (1,2,0), (1,2,1), (2,1,0), (2,2,0), (2,2,1), (3,1,0), (3,2,0), (3,2,1)\}$ computed from the given q_{ij}'s is as follows.

$$P^{(1)}=\begin{bmatrix} .5 & .1 & .2 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ .25 & .05 & .1 & .1 & .5 & 0 & 0 & 0 & 0 & 0 \\ .05 & .025 & .05 & .05 & 0 & .5 & .325 & 0 & 0 & 0 \\ .25 & .05 & .1 & .1 & 0 & 0 & .5 & 0 & 0 & 0 \\ .2375 & .0125 & .025 & .025 & .1 & .05 & .05 & .5 & 0 & 0 \\ .0125 & .0125 & .025 & .025 & .05 & .025 & .025 & 0 & .5 & .325 \\ .2375 & .0125 & .025 & .025 & .1 & .05 & .05 & 0 & 0 & .5 \\ .475 & .025 & .05 & .05 & .2 & .1 & .1 & 0 & 0 & 0 \\ .025 & .0125 & .025 & .025 & .1 & .05 & .05 & 0 & 0 & .7125 \\ .475 & .025 & .05 & .05 & .2 & .1 & .1 & 0 & 0 & 0 \end{bmatrix}$$

The stationary distributions are given by

$\pi^{(1)}=\{\pi^{(1)}(0,0,0)=0.32,\ \pi^{(1)}(1,1,0)=.056,\ \pi^{(1)}(1,2,0)=0.11,$
$\pi^{(1)}(1,2,1)=0.13,\ \pi^{(1)}(2,1,0) = .009,\ \pi^{(1)}(2,2,0)=0.08,$
$\pi^{(1)}(2,2,1)=0.13,\ \pi^{(1)}(3,1,0)=0.004,\ \pi^{(1)}(3,2,0)=0.04,\ \pi^{(1)}(3,2,1)=$

The mean sojourn time in any state

$$m^{(1)}(i,j,k) = \frac{1}{\lambda+\mu}, \quad i > 0 \text{ and}$$
$$m^{(1)}(0,0,0) = \frac{1}{\lambda}, \quad i = 0$$

The limiting probabilities are computed and tabulated as follows:

$n^{(1)}((i_1,j_1,k_1)(i,j,k))$			$\underline{P}^{(1)}(i,j,k)$
$n^{(1)}((0,0,0),\ (0,0,0))$	=	1	$\underline{P}(0,0,0) = 0.32$
$n^{(1)}((1,1,0),\ (1,1,0))$	=	0.5	
$n^{(1)}((2,1,0),\ (1,1,0))$	=	0.009	
$n^{(1)}((2,2,0),\ (1,1,0))$	=	0.0125	
$n^{(1)}((2,2,1),\ (1,1,0))$	=	0.05	$\underline{P}^{(1)}(1,1,0) = 0.06$
$n^{(1)}((3,1,0),\ (1,1,0))$	=	0.003	
$n^{(1)}((3,2,0),\ (1,1,0))$	=	0.002	
$n^{(1)}((3,2,1),\ (1,1,0))$	=	0.003	
$n^{(1)}((1,2,0),\ (1,2,0))$	=	0.5	
$n^{(1)}((2,1,0),\ (1,2,0))$	=	0.1	
$n^{(1)}((2,2,0),\ (1,2,0))$	=	0.025	
$n^{(1)}((2,2,1),\ (1,2,0))$	=	0.1	$\underline{P}^{(1)}(1,2,0) = 0.14$
$n^{(1)}((3,1,0),\ (1,2,0))$	=	0.006	
$n^{(1)}((3,2,0),\ (1,2,0))$	=	0.003	
$n^{(1)}((3,2,1),\ (1,2,0))$	=	0.006	

$n^{(1)}((1,2,1)(1,2,1))$	$= 0.5$	
$n^{(1)}((2,1,0)(1,2,1))$	$= 0.0125$	
$n^{(1)}((2,2,0)(1,2,1))$	$= 0.0125$	
$n^{(1)}((2,2,1)(1,2,1))$	$= 0.025$	$\underline{P}(1,2,1) = 0.13$
$n^{(1)}((3,1,0)(1,2,1))$	$= 0.003$	
$n^{(1)}((3,2,0)(1,2,1))$	$= 0.001$	
$n^{(1)}((3,2,1)(1,2,1))$	$= 0.003$	
$n^{(1)}((2,1,0)(2,1,0))$	$= 0.5$	
$n^{(1)}((3,1,0)(2,1,0))$	$= 0.1$	$\underline{P}(2,1,0) = .02$
$n^{(1)}((3,2,0)(2,1,0))$	$= 0.025$	
$n^{(1)}((3,2,1)(2,1,0))$	$= 0.1$	
$n^{(1)}((2,2,0)(2,2,0))$	$= 0.5$	
$n^{(1)}((3,1,0)(2,2,0))$	$= 0.05$	$\underline{P}(2,2,0) = 0.08$
$n^{(1)}((3,2,0)(2,2,0))$	$= 0.0125$	
$n^{(1)}((3,2,1)(2,2,0))$	$= 0.05$	
$n^{(1)}((2,2,1)(2,2,1))$	$= 0.5$	
$n^{(1)}((3,1,0)(2,2,1))$	$= 0.0125$	$\underline{P}(2,2,1) = 0.12$
$n^{(1)}((3,2,0)(2,2,1))$	$= 0.0125$	
$n^{(1)}((3,2,1)(2,2,1))$	$= 0.006$	
$n^{(1)}((3,1,0)\ (3,1,0))$	$= 0.5$	$\underline{P}(3,1,0) = 0.004$
$n^{(1)}((3,2,0)(3,2,0))$	$= 0.5$	$\underline{P}(3,2,0) = 0.03$
$n^{(1)}((3,2,1)(3,2,1))$	$= 0.5$	$\underline{P}(3,2,1) = 0.11$