CHAPTER III

Q-FUZZY SUBGROUPS OF β-FUZZY CONGRUENCE RELATIONS
ON A GROUP

Introduction: The concept of fuzzy sets was first introduced by Zadeh in 1965 and since then there has been a tremendous interest in the subject due to diverse applications ranging from engineering and computer science to social behavior studies. The concept of fuzzy relation on a set was defined by Zadeh [1965] and other authors like Rosenfeld [1971], Tamura et.al. [1971], and Yeh and Bang [1975] considered it further. The notion of fuzzy congruence on a group was introduced by Kuroki [1992] and that the universal algebra was studied by Filep and Maurer [1989], and Murali [1991].

The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971. Several mathematicians have followed the Rosenfeld approach in investigating the fuzzy subgroup theory. Fuzzy normal subgroups were studied by Wu [1981], Dib [1988], Kumar et.al. [1992] and Mukherjee [1984]. The concept of fuzzy quotient group was studied by Kuroki [1992]. In this study, we define some new special fuzzy equivalence relations and derive some simple consequences. Then using those relations we define suitable Q-fuzzy subgroupoids and Q-fuzzy quotient subgroup of $G / H$ differently.
3.2 Section II - Preliminaries

In this section, we shall formulate the preliminary definitions, and results that that are required in more contents of this section are contained in the literature. Let X be a nonempty set and I be the unit interval. A fuzzy binary relation on X is a fuzzy subset A on X×X. By a fuzzy relation, we mean a fuzzy binary relation given by A: X × X → I. All fuzzy subsets considered here are assumed to take values in I.

3.2.1 Definition: (i) A fuzzy relation A on X is said to be reflexive if A(x, x) = 1 for all x ∈ X and said to be symmetric if A(x, y) = A(y, x) for all x, y in X. (ii) If A_1 and A_2 are two relations on X, then their max-product composition denoted by A_1 ° A_2 is defined as A_1 ° A_2 (x, y) = max { A_1(x, z), A_2(z, y)}. (iii) If A_1 = A_2 = A and A ° A ≤ A, then the fuzzy relation A is called transitive.

3.2.2 Definition: A fuzzy binary relation A in X is called similarity relation if A is reflexive, symmetric and transitive.

Example: Let G = {1,w,w^2} be the group with respect to the usual multiplication, where w denotes the cube root of unity. Define λ,μ : G → [0,1] by

\[ \lambda(x) = \begin{cases} 1 & \text{if } x = 1; \\ 0.6 & \text{if } x = w; \\ 0.5 & \text{if } x = w^2 \end{cases} \]

and \[ \mu(x) = \begin{cases} 0.5 & \text{if } x = 1; \\ 0.4 & \text{if } x = w; \\ 0.3 & \text{if } x = w^2 \end{cases} \]

if x = w^2. It can be found that for every x ∈ G, R_{µ∩λ}(x,x) = (µ∩λ)(xx^{-1}) = (µ∩λ)(1) = 0.5. Hence R_{µ∩λ} is not reflexive and not a similarity relation on the group G.

3.2.3 Definition: Let S be a semi group. A fuzzy binary relation A on S is called fuzzy left (right) compatible if and only if A(x,y) ≤ A(tx,ty) for all x, y, t ∈ S (A(x,y) ≤ A(xt,yt) for all x, y, t ∈ S.

3.2.4 Definition: A fuzzy binary relation A on a semi group S is called fuzzy compatible if and only if min{ A(a, b), A(c, d) } ≤ A(ac, bd) for all a, b, c, d ∈ S.

3.2.5 Definition: Fuzzy compatible similarity relation on a semi group S is called fuzzy congruence.
3.2.6 Definition: A Q-fuzzy subgroup $A_H$ of $G$ is called a Q-fuzzy normal subgroup of $G$ if $A_H(xy, q) = A_H(yx, q)$ for all $x, y$ of $G$ and $q \in Q$.

3.3 Section III: $\beta_q$ – Fuzzy relation and Fuzzy congruence

In this section, we shall define some special fuzzy relation and give some its results. We need to define a special relation $\beta_q$ as follows.

3.3.1 Definition: Let $G$ be a group with identity $e$ and $A_H$ be a Q-fuzzy subgroup of $G$. A fuzzy relation $\beta_q$ can be defined on $G$ by $\beta_q(a, b) = \min \{ A_H(a, q), A_H(b, q) \}$, if $(a, q) \neq (b, q)$ $A_H(e, q)$ if $(a, q) = (b, q)$. Now we can show some properties of $\beta_q$.

In the chapter, $\beta_q$-fuzzy congruence by using special fuzzy equivalence relation of Q-fuzzy subgroups which is defined in this study and we define suitable Q-fuzzy subgroupoids and Q-fuzzy quotient subgroup of finite group $G / H$ differently then we investigate some basic properties.

The following propositions are proved

3.3.1 Proposition: Let $G$ be a group with identity $e$ and $A_H$ be a Q-fuzzy subgroup of a group $G$. Then the relation $\beta_q$ defined on $G$ is Q-similarity relation on $G$.

Proof: $\beta_q$ is reflexive, for each $a \in G$ and $q \in Q$, $\beta_q(a, a) = A_H(e, q) = 1$

$\beta_q$ is symmetric. $\beta_q(a, b) = \min \{ A_H(a, q), A_H(b, q) \}$

\[ = \min \{ A_H(b, q), A_H(a, q) \} \]

\[ = \beta_q(b, a) \text{, for } a, b \in G. \]
β_q is transitive.

\[ \beta_q(a, c) = \max_{b \in G} \{ \beta_q(a, b), \beta_q(b, c) \} \]

\[ = \max_{b \in G} \{ \min_{b \in G} \{ A_H(a, q), A_H(b, q) \}, \min_{b \in G} \{ A_H(b, q), A_H(c, q) \} \} \]

\[ \leq \max_{b \in G} \{ \min_{b \in G} A_H(a, q), A_H(b, q) \} \max_{b \in G} \{ \min_{b \in G} A_H(b, q), A_H(c, q) \} \]

\[ \leq \max_{b \in G} A_H(a, q) \max_{b \in G} A_H(c, q) \]

\[ = \beta_q(a, c), \text{ for all } a, c \in G. \]

Therefore β_q is a Q- similarity relation.

3.3.2 Corollary: \( \beta_q(x^{-1}, y^{-1}) = \beta_q(x, y) \) for all \( x, y \in G, q \in Q. \)

Proof: \( A_H \) is a Q- fuzzy subgroup of \( G. \) It gives that

\[ \beta_q(x^{-1}, y^{-1}) = \min_{b \in G} \{ A_H(x^{-1}, q), A_H(y^{-1}, q) \} = \min_{b \in G} A_H(x, q), A_H(y, q) \] = \( \beta_q(x, y). \)

3.3.3 Proposition: The fuzzy relation \( \beta_q \) defined on \( G \) is Q- fuzzy compatible.

Proof: By using the definition of Q- fuzzy compatible and the definition of \( \beta_q \)

\[ \beta_q(ac, bd) = \min_{b \in G} \{ A_H(ac, q), A_H(bd, q) \} \]

\[ \geq \min_{b \in G} \{ \min A_H(a, q), A_H(c, q) \}, \min_{b \in G} \{ A_H(b, q), A_H(d, q) \} \]

\[ = \min_{b \in G} \{ \min A_H(a, q), A_H(b, q) , A_H(c, q), A_H(d, q) \} \]

\[ = \min_{b \in G} \{ A_H(a, q), A_H(b, q) \}, \min A_H(c, q), A_H(d, q) \} \]

\[ = \min_{b \in G} \{ \beta_q(a, b), \beta_q(c, d) \}. \text{ This completes the proof.} \]

3.3.4 Proposition: The fuzzy relation \( \beta_q \) defined on \( G \) is a Q- fuzzy congruence.
Proof: \( \beta_q \) is Q- fuzzy compatible is proved in Proposition (3.3.3). Therefore \( \beta_q \) is a Q- fuzzy congruence.

3.3.2 Definition: If a Q- fuzzy set is a Q-fuzzy subgroup of \( G / H \), then it is called Q-fuzzy quotient subgroup. Similarly, if it is a Q- fuzzy normal subgroup of \( G / H \), then it is called Q- fuzzy quotient normal subgroup.

By using the Q- fuzzy congruence \( \beta_q \), we define a special function \( N \) as follows.

3.3.3 Definition: Let \( G \) be group and \( A_H \) be Q-fuzzy normal subgroup of \( G \). \( N : G / H \times Q \rightarrow [0, 1] \) can be defined by \( N(xH, q) = \beta_q(x, h) \) for all \( h \in H \) and \( q \in Q \).

Now some algebraic properties of \( N \) are investigated.

3.3.5 Proposition: The defined fuzzy set \( N \) is a Q- fuzzy quotient subgroup of \( G / H \).

Proof: We have to show that the \( N \) is a Q- fuzzy subgroup of \( G / H \).

\( A_H \) is a Q- fuzzy subgroup of \( G \). Using this, for every \( xH, yH \in G / H \), we get

\[
N(xHyH, q) = \beta_q(xy, h) = \min \{ A_H(xy, q), A_H(h, q) \} \\
= A_H(xy, q) \\
\geq \min \{ A_H(x, q), A_H(y, q) \} \\
= \min \{ \min \{ A_H(x, q), A_H(h, q) \}, \min \{ A_H(y, q), A_H(h, q) \} \} \\
= \min \{ \beta_q(x, h), \beta_q(y, h) \} \\
= \min \{ N(xH, q), N(yH, q) \}.
\]

and \( N(x^{-1}H, q) = \beta_q(x^{-1}, h) \)

\[
= \min \{ A_H(x^{-1}, q), A_H(h, q) \} \\
\geq \min \{ A_H(x, q), A_H(h, q) \}
\]
\[ = \beta_q(x, h) \]
\[ = N(xH, q). \text{Thus } N \text{ is a } Q-\text{fuzzy quotient subgroup of } G/H. \]

**3.3.6 Proposition:** The defined fuzzy set \( N \) is a \( Q-\)fuzzy quotient normal subgroup of \( G/H \).

**Proof:** Here we have to prove that the \( N \) is a \( Q-\)fuzzy normal subgroup of \( G/H \).
Since \( A_H \) is \( Q-\)fuzzy normal subgroup of \( G \), it gives that

\[
N(xHyH, q) = \beta_q(xy, h) = \min \{ A_H(xy, q), A_H(h, q) \}
\geq \min \{ A_H(yx, q), A_H(h, q) \}
= \beta_q(yx, q)
\geq N(yHxH, q). \text{Hence } N \text{ is a } Q-\text{fuzzy normal subgroup of } G/H.
\]

**3.3.7 Proposition:** If \( N \) is a \( Q-\)fuzzy quotient subgroupoid of finite group \( G/H \), then \( N \) is a \( Q\)-fuzzy subgroup.

**Proof:** Let \( xH \in G/H \). Since \( G/H \) is finite, \( xH \) has finite order, say \( r \). Then \( (xH)^r = x^rH = H \), where \( H \) is identity of \( G/H \). Thus \( (xH)^{-1} = x^{-1}H = x^{r-1}H \). Now using the definition of a \( Q-\)fuzzy subgroupoid repeatedly, it follows that

\[
N(x^{-1}H, q) = N(x^{r-1}H, q) = \beta_q(x^{r-1}, h)
\geq \min \{ A_H(x^{r-2}, q), A_H(x, q) \}
\geq A_H(x, q)
= \beta_q(x, h) = N(xH, q).
\]

80
Interchanging $xH$ with $x^{-1}H$, then $N(xH, q) \geq N(x^{-1}H, q)$. Hence $N$ is a $Q$-fuzzy quotient subgroup.

**3.3.8 Proposition:** Let $N$ be a $Q$-fuzzy quotient subgroup of a group $G / H$ and let $xH \in G / H$. Then $N(xHyH, q) = N(yH, q)$, for all $yH \in G / H \leftrightarrow N(xH, q) = N(H, q)$.

**Proof:** Suppose that $N(xHyH, q) = N(yH, q)$ for all $yH \in G / H$. Then by choosing $yH = H$, we obtain $N(xH, q) = N(H, q)$.

Conversely, suppose that $N(xH, q) = N(H, q)$. Since $N$ is a $Q$-fuzzy subgroup of $G / H$ and $A_H$ be $Q$-fuzzy subgroup of $G$, it implies that

$$N(xHyH, q) \geq \min \{N(xH, q), N(yH, q)\}$$

$$= \min \{N(H, q), N(yH, q)\}$$

$$= \min \{\beta_q(e, h), \beta_q(y, h)\}$$

$$= \min \{\min \{N_H(h, q), N_H(y, q)\}, \min \{N_H(y, q), N_H(h, q)\}\}$$

$$= \{N_H(h, q), N_H(y, q)\} = \beta_q(y, h) = N(yH, q).$$

Interchanging $xHyH$ with $yH$, we get $N(yH, q) \geq N(xHyH, q)$.

**3.3.9 Proposition:** Let $N$ and $R$ be two $Q$-fuzzy quotient subgroups of $G / H$. Then $N \cap R$ is a $Q$-fuzzy quotient normal subgroup of $G / H$.

**Proof:** For every $xH, yH \in G / H$ and $q \in Q$, the observation is that

$$N \cap R (xHyH, q) = \min \{N(xHyH, q), R(xHyH, q)\}$$

$$\geq \min \{\min \{N(xH, q), N(yH, q)\}, \min \{R(xH, q), R(yH, q)\}\}$$

$$= \min \{\min \{N(xH, q), R(xH, q)\}, \min \{N(yH, q), R(yH, q)\}\}$$

$$= \min \{N \cap R(xH, q), N \cap R(yH, q)\}$$

81
and

\[ N \cap R \left( x^{-1}, q \right) = \min \{ N(x^{-1}H, q), R(x^{-1}H, q) \} \]
\[ = \min \{ N(xH, q), R(xH, q) \} \]
\[ = N \cap R \left( xH, q \right). \]

Interchanging now \( xH \) with \( x^{-1}H \), it makes that \( N \cap R \left( xH, q \right) \leq N \cap R \left( x^{-1}H, q \right) \).

Hence \( N \cap R \) is a Q-fuzzy subgroup of \( G/H \).

\[ N \cap R(xHyH, q) = \min \{ N(xHyH, q), R(xHyH, q) \} \]
\[ = \min \{ N(yHxH, q), R(yHxH, q) \} \]
\[ \leq N \cap R \left( yHxH, q \right). \] Hence \( N \cap R \) is Q-fuzzy normal subgroup of \( G/H \).

By using the Q-fuzzy normal quotient subgroup \( N \), Q-fuzzy relation \( \mu_N \) is defined as follows.

3.3.4 Definition: For all \((xH, yH) \in G/H \times G/H\), the Q-fuzzy relation \( \mu_N \) on \( G/H \) is defined by \( \mu_N (xH, yH) = N(xH^{-1}yH, q) \) where \( q \in Q \).

3.3.10 Proposition: The Q-fuzzy relation \( \mu_N \) is a Q-fuzzy congruence on \( G/H \).

Proof: Let \( xH, yH \) be any element of \( G/H \). Then \( \mu_N \) is fuzzy reflexive, since \( \mu_N (xH, xH) = N(xH^{-1}xH, q) = N(H, q) = 1 \). \( \mu_N \) is a fuzzy symmetric, since

\[ \mu_N(xH, yH) = N(xH^{-1}yH, q) \]
\[ = N((yx^{-1})^{-1}H, q) \]
\[ = N(yx^{-1}H, q) \]
\[ = N(yHx^{-1}H, q) \]
\[ = \mu_N(yH, xH). \]
Let \( xH, yH \) be elements of \( G/H \) and \( N_H \) be a Q-fuzzy normal subgroup of \( G \).

Then \( N_H \) is Q-fuzzy transitive, since

\[
\mu_N \circ \mu_N(xH, yH) = \max \{ \mu_N(xH, zH), \mu_N(zH, yH) \}
\]

\[
= \max \{ N(xHz^{-1}H, q) N(zHy^{-1}H, q) \}
\]

\[
= \max \{ N(xz^{-1}H, q) N(zy^{-1}H, q) \}
\]

\[
= \max \{ \beta_q(xz^{-1}, h) \beta_q(zy^{-1}, h) \}
\]

\[
= \max \{ \min \{ N_H(xz^{-1}, q), N_H(h, q) \} \min \{ N_H(zy^{-1}, q), N_H(h, q) \} \}
\]

\[
\leq \max \{ \min \{ \min \{ N_H(xy^{-1}, q), N_H(h, q) \} \} \}
\]

\[
= \mu_N(xH, yH).
\]

Thus \( \mu_N \) is Q-fuzzy compatible, since

\[
\min \{ \mu_N(xH, yH), \mu_N(zH, wH) \} = \min \{ N(xH^{-1}H, q), N(zHw^{-1}H, q) \}
\]

\[
= \min \{ N(xy^{-1}H, q), N(zw^{-1}H, q) \}
\]

\[
= \min \{ \beta_q(xy^{-1}, h), \beta_q(zw^{-1}, h) \}
\]

\[
= \min \{ \min \{ N_H(xy^{-1}, q), N_H(h, q) \}, \min \{ N_H(zw^{-1}, q), N_H(h, q) \} \}
\]

\[
= \min \{ N_H(xy^{-1}, q), N_H(zw^{-1}, q) \}
\]

\[
= \min \{ N_H(y^{-1}x, q), N_H(zw^{-1}, q) \}
\]

Since \( N_H \) is a Q-fuzzy normal subgroup of \( G \)

\[
\leq N_H(y^{-1}xzw^{-1}, q) = N_H(xzw^{-1}y^{-1}, q).
\]
Since $N_H$ is a $Q$-fuzzy normal subgroup of $G$

$$= \min\{ N_H(xz(yw)^{-1}, q), N_H(h, q) \}$$

$$= \beta_q (xz(yw)^{-1}, h)$$

$$= N (xzH(yw)^{-1}H, q)$$

$$= \mu_N (xzH, ywH),$$

So it is $Q$-fuzzy congruence on $G/H$. This completes the proof.

**Conclusion:** W.M. Wu [1981] and A. Rosenfeld [1971] introduced the concept of fuzzy normal subgroups and fuzzy groups. We investigate the concept of special fuzzy relations of $Q$-fuzzy group and derive some simple consequences.

**3.4 Section IV: S-Product of S-anti-Fuzzy right R-subgroup of near rings**

**Introduction:** Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. B. Schweitzer and A. Sklar [1963] introduce the notions of Triangular norm (t-norm) and Triangular co-norm (S-norm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators respectively. First, Abu. Osman [1987] introduced the notion of fuzzy subgroup with respect to t-norm.

In this section, we redefine anti-fuzzy right R- subgroups of a near-ring ‘R’ with respect to a S-norm and investigate it is related properties.

3.4.2 Preliminaries: A semi ring S is a system consisting of a non-empty set S together with two binary operations on S called addition and multiplication such that

   (i) S together with addition is a semi group.
   (ii) S together with multiplication is a semi group.
   (iii) a( b + c ) = ab + ac and (a + b)c = ac + bc for all a, b, c ∈ S.

A semi ring S is said to be additively commutative if a + b = b + a for all a,b ∈ S. A zero element of a semi ring S is an element 0 such that 0. x = x. 0 = 0 and 0 + x = x + 0 = x for all x ∈ S. By a near-ring we mean a non-empty set ‘R’ with two binary operations + and · satisfies the following axioms

   (i) (R,+) is a group.
   (ii) (R, ·) is a semi group.
   (iii) ( b + c ) a = ba + ca for all a, b, c ∈ R.

Precisely speaking it is a right near-ring because it satisfies the right distribution law.

Note that x0= 0 and x(-y) = -(xy) but in general 0x ≠ 0 for some x ∈R. A two sided R- subgroup of a near- ring ‘R’ is a subset N of R such that

   (i) (N, +) is a subgroup of (R, +).
   (ii) RN ⊂ N
   (iii) NR ⊂ N.

If N satisfies (i) and (ii), then it is called a right R subgroup of R.

Some fuzzy logic concepts are given.
A fuzzy set $\mu$ in a set $R$ is a function $\mu: R \rightarrow [0, 1]$. Let $\text{Im}(\mu)$ denote the image set of $\mu$. Let $\mu$ be a fuzzy set in $R$. For $t \in [0, 1]$, the set $L(\mu; \alpha) = \{ x \in R / \mu(x) \leq \alpha \}$ is called a lower level subset of $\mu$. Let $R$ be a near-ring and let $\mu$ be a fuzzy set in $R$. We say that $\mu$ is a fuzzy sub near-ring of $R$ if for all $x, y \in R$, (FS1) $\mu(x - y) \geq \min \{ \mu(x), \mu(y) \}$ and (FS2) $\mu(xy) \geq \min \{ \mu(x), \mu(y) \}$.

If a fuzzy set $\mu$ in a near-ring $R$ satisfies the property (FS1), then $\mu(0) \geq \mu(x)$ for all $x \in R$.

3.4.1 Definition: By a s-norm $S$, we mean a function $S: [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

- $(S1)$ $S(x, 0) = x$
- $(S2)$ $S(x, y) \leq S(x, z)$ if $y \leq z$
- $(S3)$ $S(x, y) = S(y, x)$
- $(S4)$ $S(x, S(y, z)) = S(S(x, y), z)$, for all $x, y, z \in [0, 1]$.

Replacing 0 by 1 in condition $S1$ we obtain the concept of t-norm $T$.

3.4.2 Definition: For an $S$-norm, $S(x, y) \geq \max \{ x, y \}$ holds for all $x, y \in [0, 1]$.

3.4.3 Definition: Let $S$ be a s-norm. A fuzzy set $\mu$ in $R$ is said to be sensible with respect to $S$ if $\text{Im}(\mu) \subset \Delta_s$, where $\Delta_s = \{ s(\alpha, \alpha) = \alpha / \alpha \in [0, 1] \}$.

3.4.4 Definition: Let $(R, +, \cdot)$ be a near-ring. A fuzzy set $\mu$ in $R$ is called an anti fuzzy right (resp. left) $R$-subgroup of $R$ if (AF1) $\mu(x - y) \leq \max \{ \mu(x), \mu(y) \}$, for all $x, y \in R$; (AF2) $\mu(xy) \leq \mu(y)$ for all $r, x \in R$.

3.4.5 Definition: Let $(R, +, \cdot)$ be a near-ring. A fuzzy set $\mu$ in $R$ is called a fuzzy right (resp. left) $R$-subgroup of $R$ if (FR1) $\mu$ is a fuzzy subgroup of $(R, +)$; (FR2) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$), for all $r, x \in R$.

Example: Let $R = \{ a, b, c, d \}$ be a near-ring. A fuzzy set $\mu: R \rightarrow [0,1]$ is defined by $\mu(c) = \mu(d) < \mu(b) < \mu(a)$. Then $\mu$ is a fuzzy right $R$-subgroup of $R$.
3.4.6 Definition: Let $S$ be a $s$- norm. A function $\mu: \mathbb{R} \rightarrow [0, 1]$ is called a fuzzy right (resp. left) R- subgroup of $\mathbb{R}$ with respect to $S$ if

(C1) $\mu (x - y) \leq S (\mu(x), \mu(y))$

(C2) $\mu (xr) \leq \mu (x)$ (resp. $\mu (rx) \leq \mu(x)$) for all $r, x \in \mathbb{R}$.

If a fuzzy R-subgroup $\mu$ of $\mathbb{R}$ with respect to $S$ is sensible, we say that $\mu$ is a sensible fuzzy R-subgroup of $\mathbb{R}$ with respect to $S$.

3.4.7 Example: Let $K$ be the set natural numbers including 0 and $K$ is a R-subgroup with usual addition and multiplication.

3.4.8 Definition: A fuzzy subset $\mu: \mathbb{R} \rightarrow [0,1]$ by $\mu (x) = 0$ if $x$ is even; $= 1$, otherwise.

Let $S_m: [0, 1] \rightarrow [0, 1]$ by a function defined by $S_m (\alpha, \beta) = \min \{ x + y , 1 \}$ for all $x, y \in [0, 1]$. Then $S_m$ is a t-norm. Here $\mu$ is sensible R-fuzzy subgroup of $\mathbb{R}$.

In this section, we introduce the notion of $S$-anti-fuzzy right R-subgroups of near-rings and its basic properties are investigated. We also study the homomorphic image and pre image of $S$-anti-fuzzy right R-subgroups. Using $S$-norm, we introduce the notion on sensible anti-fuzzy right R-subgroups in near-rings and some related properties of near-rings ‘R’ are discussed.

The following are the properties of anti-fuzzy R subgroups.

3.4.1 Proposition: Let $S$ be a $s$-norm. Then every sensible $S$-anti fuzzy right R- subgroups $\mu$ of $\mathbb{R}$ is anti-fuzzy R- subgroups of $\mathbb{R}$.

Proof: Assume that $\mu$ is a sensible $S$- anti fuzzy right R-subgroups of $\mathbb{R}$. Then (AF1) $\mu (x - y) \leq S (\mu(x), \mu(y))$ and (AF2) $\mu (xr) \leq \mu(x)$ for all $x, y \in S$. Since ‘$\mu$’ is sensible, we have $\max \{ \mu(x) , \mu(y) \} = S(\min \{\mu(x), \mu(y)\} , \min \{ \mu(x), \mu(y) \})$

and so $S (\mu(x), \mu(y)) = \max \{ \mu(x), \mu(y) \}$.

It follows that $\mu(x-y) \leq S(\mu(x), \mu(y)) = \max \{ \mu(x), \mu(y) \}$ for all $x, y$ in $\mathbb{R}$.

Clearly $\mu(xr) \leq \mu(x)$ for all $r, x$ in $\mathbb{R}$. So $\mu$ is an anti-fuzzy R- subgroup of $\mathbb{R}$.
3.4.2 **Proposition:** If $\mu$ is S-anti fuzzy right R-subgroups of a near ring R and $\theta$ is an endomorphism of R, then $\mu[\theta]$ is S-anti fuzzy right R-subgroups of R.

**Proof:** For any $x, y \in R$, we have

(i) $\mu[\theta](x - y) = \mu(\theta(x-y))$

$$= \mu(\theta(x) - \theta(y))$$

$$\leq S(\mu[\theta](x), \mu[\theta](y))$$

(ii) $\mu[\theta](xr) = \mu(\theta(xr))$

$$= \mu(\theta(x)r)$$

$$\leq \mu(\theta(x))$$

$$\leq \mu[\theta](x).$$  Hence $\mu[\theta]$ is a S-anti fuzzy right R-subgroups of R.

3.4.9 **Definition:** Let $f$ be a mapping defined on R. If $\psi$ is a fuzzy subset in $f(R)$, then the fuzzy subset $\mu = \psi o f$ in R. $\mu(x) = \psi(f(x))$ for all $x$ in R is called the pre image of ‘$\psi$’ under $f$.

3.4.3 **Proposition:** An onto homomorphic pre image of a S-anti fuzzy right R-subgroups of a near-ring is S-anti fuzzy right R-subgroups.

**Proof:** Let $f: R \rightarrow R^1$ be an onto homomorphism of near-ring and let $\psi$ be an S-anti fuzzy right R-subgroups of R and $\mu$ the pre image of $\psi$ under $f$. Then it follows that

(i) $\mu(x-y) = \psi(f(x-y))$

$$= \psi(f(x)-f(y))$$

$$\leq S(\psi(f(x)), \psi(f(y)))$$

$$= S(\mu(x), \mu(y))$$

(ii) $\mu(xr) = \psi(f(xr))$

$$= \psi(f(x)r)$$

$$\leq \psi(f(x))$$

$$= \mu(x).$$

Hence $\mu$ is a S-anti fuzzy right R-subgroups of R.
3.4.4 Proposition: An onto homomorphic image of a anti fuzzy right R- subgroups with inf property is anti-fuzzy right R- subgroups.

Proof: Let \( f: R \rightarrow R^1 \) be an onto homomorphism of near-ring and let \( \mu \) be an S-anti fuzzy right R-subgroup of R with inf property. Given \( x, y \in R \), \( x_o \in f^{-1}(x^1) \), and \( y_o \in f^{-1}(y^1) \) be such that

\[
\mu(x_o) = \inf_{h \in f^{-1}(x^1)} \mu(h), \quad \mu(y_o) = \inf_{h \in f^{-1}(y^1)} \mu(h)
\]

respectively. Then it can deduce that

\[
\mu^f(x^1 - y^1) = \inf_{z \in f^{-1}(x^1 - y^1)} [\mu(z)] \leq \max \{ \mu(x_o), \mu(y_o) \} = \max \{ \inf_{h \in f^{-1}(x^1)} \mu(h), \inf_{h \in f^{-1}(y^1)} \mu(h) \} = \max \{ \mu^f(x^1), \mu^f(y^1) \}
\]

\[
\mu^f(xr) = \inf_{z \in f^{-1}(x^1 r^1)} [\mu(z)] \leq \mu(y_o) = \inf_{h \in f^{-1}(y^1)} \mu(h) = \mu^f(y^1)
\]

Hence \( \mu^f \) is anti fuzzy right R- subgroups of R.

The above proposition can be further strengthened; we first give the following definitions.

3.4.10 Definition: A S-norm \( S \) on \([0, 1]\) is called a continuous function from \([0, 1] \times [0, 1] \rightarrow [0, 1]\) with respect to the usual topology. We observe that the function max is always a continuous S-norm.
3.4.5 Proposition: Let \( f: R \rightarrow R^1 \) be a homomorphism of near-rings. If \( \mu \) be \( S \)-anti fuzzy right \( R \)-subgroups of \( R^1 \), then \( \mu^f \) is \( S \)-anti fuzzy right \( R \)-subgroup of \( R \).

Proof: Suppose \( \mu \) is \( S \)-anti fuzzy right \( R \)-subgroups of \( R^1 \).

Then

(i) For all \( x, y \in R \), it gives that
\[
\mu^f(x-y) = \mu f(x-y)
\]
\[
\leq S(\mu f(x), \mu f(y))
\]
\[
\leq S(\mu^f(x), \mu^f(y))
\]

(ii) For all \( x, y \in R \), it makes
\[
\mu^f(xr) = \mu f(xr)
\]
\[
= \mu(f(x), r)
\]
\[
\leq \mu(f(x))
\]
\[
\leq \mu^f(x)
\]

Hence \( \mu^f \) is a \( S \)-anti fuzzy right \( R \)-subgroup of \( R \).

3.4.6 Proposition: Let \( f: R \rightarrow R^1 \) be a homomorphism of near-rings. If \( \mu^f \) is a \( S \)-anti fuzzy right \( R \)-subgroups of \( R \), then \( \mu \) is \( S \)-anti fuzzy right \( R \)-subgroup \( R^1 \).

Proof: Let \( x^1, y^1 \in R^1 \), there exists \( x, y \in R \), such that \( f(x) = x^1 \) and \( f(y) = y^1 \),

It follows that
\[
\mu(x^1 - y^1) = \mu(f(x) - f(y))
\]
\[
= \mu(f(x - y))
\]
\[
= \mu^f(x - y)
\]
\[
\leq S(\mu^f(x), \mu^f(y))
\]
\[
= S(\mu(f(x), \mu f(y))
\]
\[
= S(\mu(x^1), \mu(y^1))
\]
(iii) Let \(x^1, y^1 \in \mathbb{R}\), there exists \(x, r \in \mathbb{R}\) such that \(f(x) = x^1\), \(f(y) = y^1\), We have
\[
\mu(x^1 r^1) = \mu(f(x), f(y)) = \mu(f(xr)) \leq \mu^f(x) \leq \mu(f(x) \leq \mu(x^1).
\]

3.4.7 Proposition: Let \(S\) be a continuous \(S\)-norm and let \(f\) be a homomorphism on a near-ring \(R\). If \(\mu\) is \(S\)-anti fuzzy right \(R\)-subgroups of \(R\), then \(\mu^f\) is \(S\)-anti fuzzy right \(R\)-subgroups of \(f(R)\).

Proof: Let \(A_1 = f^{-1}(y_1)\), \(A_2 = f^{-1}(y_2)\) and \(A_{12} = f^{-1}(y_1-y_2)\), where \(y_1-y_2 \in f(R)\). Consider the set \(A_1 - A_2 = \{x \in R / x = a_1-a_2\text{ for some }a_1 \in A_1, a_2 \in A_2\}\). If \(x \in A_1 - A_2\), then \(x = x_1 - x_2\) for some \(x_1 \in A_1\) and \(x_2 \in A_2\) so that \(f(x) = f(x_1) - f(x_2) = y_1 - y_2\).

So \(x \in f^{-1}(y_1-y_2) = A_{12}\). Thus \(A_1 - A_2 \subseteq A_{12}\).

It follows that
\[
\mu^f(y_1-y_2) = \inf \{ \mu(x) / x \in f^{-1}(x_1-x_2) \}
= \inf \{ \mu(x) / x \in A_{12} \}
\leq \inf \{ \mu(x) / x \in A_1-A_2 \}
\leq \inf \{ \mu(x_1-x_2) / x_1 \in A_1, x_2 \in A_2 \}
\leq \inf \{ S(\mu(x_1), \mu(x_2)) / x_1 \in A_1, x_2 \in A_2 \}
\]

Since \(S\) is continuous for every \(\varepsilon > 0\), we see that \(\inf \{ \mu(x_1) / x_1 \in A_1 \}=x_1^* \leq \delta\) and \(\inf \{ \mu(x_2) / x_2 \in A_2 \}=x_2^* \leq \delta\).

Then \(S(\inf \{ \mu(x_1) / x_1 \in A_1 \}, \inf \{ \mu(x_2) / x_2 \in A_2 \})-S(x_1^*,x_2^*) \leq \varepsilon\).

Choose \(a_1 \in A_1\), and \(a_2 \in A_2\), such that
\[
\inf \{ \mu(x_1) / x_1 \in A_1 \} - \mu(a_1) \leq \delta \text{ and } \inf \{ \mu(x_2) / x_2 \in A_2 \} - \mu(a_2) \leq \delta.
\]

Then \(S(\inf \{ \mu(x_1) / x_1 \in A_1 \}, \inf \{ \mu(x_2) / x_2 \in A_2 \})-S(\mu(a_1),\mu(a_2)) \leq \varepsilon\).

Thus we have
\[
(i) \mu^f(y_1-y_2) \leq \inf \{ S(\mu(x_1), \mu(x_2)) / x_1 \in A_1, x_2 \in A_2 \}
= S(\inf \{ \mu(x_1) / x_1 \in A_1 \}, \inf \{ \mu(x_2) / x_2 \in A_2 \})
= S(\mu^f(y_1), \mu^f(y_2)).
(ii) Similarly, we can prove that \(\mu^f(xr) \leq \mu^f(x)\).
Hence $\mu^f$ is a $S$-anti fuzzy right $R$-subgroups of $f(R)$.

3.4.8 Lemma: Let $T$ be a $t$-norm. Then $t$-co norm $S$ is $S(x, y) = 1 - T(1 - x, 1 - y)$.

Proof: straightforward.

3.4.9 Proposition: A fuzzy subset $\mu$ of $R$ is a $T$-anti fuzzy right $R$-subgroups if and only if $\mu^c$ is a $S$-anti fuzzy right $R$-subgroup of $R$.

Proof: Let $\mu$ be a $T$-anti fuzzy right $R$-subgroups of $R$. for all $x, y \in R$. We have

$$
(i) \mu^c(x-y) = 1 - \mu(x-y) \leq 1 - T(\mu(x), \mu(y)) = 1 - T(1 - \mu^c(x), 1 - \mu^c(y)) = S(\mu^c(x), \mu^c(y))$

(ii) $\mu^c(xr) = 1 - \mu(xr) \leq 1 - \mu(x) = \mu^c(x)$

$\mu^c$ is a a anti fuzzy right $R$-subgroups of $R$.

3.4.11 Definition: A fuzzy relation on any set $X$ is a fuzzy set $\mu$: $X \times X \rightarrow [0, 1]$.

3.4.12 Definition: Let $S$ be a $s$-norm. If $\mu$ is a fuzzy relation on a set $R$ and $\chi$ be fuzzy set in $R$, Then $\mu$ is a $S$-fuzzy relation on $\chi$ if $\mu_\chi(x, y) \geq S(\chi(x), \chi(y))$ for all $x, y \in R$.

3.4.13 Definition: Let $S$ be a $s$-norm. let $\mu$ and $\chi$ be a fuzzy subset of $R$. Then direct $S$-product of $\mu$ and $\chi$ is defined as $(\mu \times \chi)(x, y) = S(\mu(x), \chi(y))$, for all $x, y \in R$.

3.4.10 Lemma: Let $S$ be a $s$-norm. let $\mu$ and $\chi$ be a fuzzy set of $R$. Then

(i) $\mu \times \chi$ is a $S$-fuzzy relation on $S$.

(ii) $L(\mu \times \chi; t) = L(\mu; t) \times L(\chi; t)$ for all $t \in [0, 1]$.

Proof: It is obvious.

3.4.14 Definition: Let $S$ be a $s$-norm. let $\mu$ be a fuzzy subset of $R$. Then $\mu$ is called strongest $S$-fuzzy relation on $R$ if $\mu_\chi(x, y) \geq S(\chi(x), \chi(y))$ for all $x, y \in R$.

3.4.11 Proposition: Let $S$ be a $s$-norm. let $\mu$ and $\chi$ be a $S$-anti fuzzy right $R$-subgroup of $R$. Then $\mu \times \chi$ is a anti fuzzy right $R$-subgroup of $R$.

Proof: (i) $(\mu \times \chi)(x-y) = (\mu \times \chi)((x_1, x_2) - (y_1, y_2))$
\[
= (\mu \times \chi) \left( (x_1-y_1) , (x_2-y_2) \right)
\]
\[
= S(\mu(x_1-y_1) , \chi(x_2-y_2))
\]
\[
\leq S (S(\mu(x_1) , \mu(y_1)) , S(\chi(x_2) , \chi(y_2)))
\]
\[
= S (S(\mu(x_1) , \chi(x_2)) , S(\mu(y_1) , \chi(y_2)) )
\]
\[
= S (((\mu \times \chi)(x_1,x_2)) , ((\mu \times \chi)(y_1,y_2))
\]
\[
= S ((\mu \times \chi)(x_1,x_2) , (\mu \times \chi)(y_1,y_2))
\]
\[
= S (((\mu \times \chi)(x) , (\mu \times \chi)(y))
\]
\[
(iii) \quad (\mu \times \chi) (xr) \quad = (\mu \times \chi) ((x_1,x_2)(r_1,r_2))
\]
\[
= (\mu \times \chi) (x_1r_1, x_2r_2)
\]
\[
= S(\mu(x_1) , \chi(x_2))
\]
\[
= (\mu \times \chi)(x_1,x_2)
\]
\[
= (\mu \times \chi)(x).
\]

\textbf{3.4.12 Proposition:} Let $\mu$ and $\chi$ be sensible S- anti fuzzy right R- subgroups of a near- ring $R$. Then $\mu \times \chi$ is a sensible S- anti fuzzy right R- subgroup of $R \times R$.

\textbf{Proof:} By proposition 3.4.11, we have $\mu \times \chi$ is S- anti fuzzy right R- subgroup of $R \times R$.

Let $x = (x_1, x_2)$ be any element of $S \times S$. Then
\[
S((\mu \times \chi)(x) , (\mu \times \chi)(x) ) = S((\mu \times \chi)(x_1,x_2) , (\mu \times \chi)(x_1,x_2))
\]
\[
= S( S(\mu(x_1) , \chi(x_2)) , S(\mu(x_1) , \chi(x_2)) )
\]
\[
= S(S(\mu(x_1) , \mu(x_1)) , S(\chi(x_2) , \chi(x_2)))
\]
\[
= S(\mu(x_1) , \chi(x_2))
\]
\[
= (\mu \times \chi)(x_1,x_2) = (\mu \times \chi)(x).
\]

\textbf{3.4.13 Remark:} If $\mu \times \chi$ is a sensible S- anti fuzzy right R- subgroup of $R \times R$, then $\mu \times \chi$ need not be sensible S- anti fuzzy right R- subgroup of $R$. 

93