Chapter - I

Introduction
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INTRODUCTION

In this chapter the progress of the field, fuzzy graph is presented. Organization of the thesis and the necessary basic definitions required for the development of the thesis are given.

1.1. Fuzzy Graphs

Graph theory is proved to be tremendously useful in modeling the essential features of systems with finite components. Graphical models are used to represent telephone network, railway network, communication problems, traffic network etc. Graph theoretic models can sometimes provide a useful structure upon which analytic techniques can be used. A graph is also used to model a relationship between a given set of objects. Each object is represented by a vertex and the relationship between them is represented by an edge if the relationship is unordered and by means of a directed edge if the objects have an ordered relation between them. Relationship among the objects need not always be precisely defined criteria; when we think of an imprecise concept, the fuzziness arises.

In 1965, L.A. Zadeh [68] introduced a mathematical frame work to explain the concept of uncertainty in real life through the publication of a seminal paper. A fuzzy set is defined mathematically by assigning to each possible individual in the universe of discourse a value, representing its grade
of membership, which corresponds to the degree, to which that individual is
similar or compatible with the concept represented by the fuzzy set. The fuzzy
graph introduced by A. Rosenfeld [56] using fuzzy relation, represents the
relationship between the objects by precisely indicating the level of the
relationship between the objects of the given set. Also he coined many fuzzy
analogous graph theoretic concepts like bridge, cut vertex and tree. Fuzzy
graphs have many more applications in modeling real time systems where the
level of information inherent in the system varies with different levels of
precision.

1.2. Review of Literature

of fuzzy graphs and introduced the notion of eccentricity and center in fuzzy
graphs. K.R. Bhutani and others discussed about Strong Arcs in fuzzy graphs,
M-strong fuzzy graphs, and fuzzy end nodes in fuzzy graphs in [9,10].
fuzzy line graph, J.N. Mordeson and C.S. Peng [29] dealt with the operations of
cycles and cocycles and in [31] gave a definition of complement of a fuzzy
graph. M.S. Sunitha and A. Vijayakumar [62] defined the complement of a
fuzzy graph in another way which gives a better understanding about that
concept. M.S. Sunitha and A. Vijayakumar also studied some properties of
fuzzy bridges, fuzzy cut vertex and fuzzy tree in [63] and obtained many results on metric spaces in [64]. A. Somasundaram and S. Somasundaram [60] introduced domination in fuzzy graphs in terms of effective edges.

A. Nagoorgani and V.T. Chandrasekaran [38] introduced domination using strong arcs. Also µ-complement, busy nodes, and free nodes of a fuzzy graph were defined and discussed by them. A. Nagoorgani and K. Radha [45,46] studied the concepts of Lower and Upper Truncations of a Fuzzy Graph, and Complement and Conjunction of Truncations of Fuzzy Graph.


1.3. Basic Definitions

Definition 1.3.1[56]

A fuzzy subset of a non-empty set S is a mapping \( \sigma : S \rightarrow [0,1] \) which assigns to each element ‘x’ in S a degree of membership, \( 0 \leq \sigma(x) \leq 1 \).

Definition 1.3.2[56]

A fuzzy relation on S is a fuzzy subset of \( S \times S \). A fuzzy relation \( \mu \) on S is a fuzzy relation on the fuzzy subset \( \sigma \) if \( \mu(x, y) \leq \sigma(x) \wedge \sigma(y) \) for all x, y in S where \( \wedge \) stands for minimum. A fuzzy relation on the fuzzy subset \( \sigma \) is reflexive if \( \mu(x, x) = \sigma(x) \) for all \( x \in S \). A fuzzy relation \( \mu \) on S is said to be
symmetric if $\mu(x, y) = \mu(y, x)$ for all $x, y \in S$.

$\sigma^* = \text{supp}(\sigma) = \{u \in S / \sigma(u) > 0\}$. $\mu^* = \text{supp}(\mu) = \{(u, v) \in S \times S / \mu(u, v) > 0\}$.

**Definition 1.3.3[56]**

A fuzzy graph is a pair $G: (\sigma, \mu)$ where $\sigma$ is a fuzzy subset of $S$, $\mu$ is a symmetric fuzzy relation on $\sigma$. The elements of $S$ are called the nodes or vertices of $G$ and the pair of vertices as edges in $G$. The underlying crisp graph of the fuzzy graph $G: (\sigma, \mu)$ is denoted as $G^*: (S, E)$ where $E \subseteq S \times S$. The crisp graph $(S, E)$ is a special case of the fuzzy graph $G$ with each vertex and edge of $(S, E)$ having degree of membership 1.

$(\sigma', \mu')$ is a **fuzzy sub graph** or a partial fuzzy sub graph of $(\sigma, \mu)$ if $\sigma' \subseteq \sigma$ and $\mu' \subseteq \mu$; that is if $\sigma'(u) \leq \sigma(u)$ for every $u \in S$ and $\mu'(e) \leq \mu(e)$ for every $e \in E$.

$(\sigma', \mu')$ is a **fuzzy spanning sub graph** of $(\sigma, \mu)$ if $\sigma' = \sigma$ and $\mu' \subseteq \mu$; that is if $\sigma'(u) = \sigma(u)$ for every $u \in S$ and $\mu'(e) \leq \mu(e)$ for every $e \in E$.

For any fuzzy subset $\nu$ of $S$ such that $\nu \subseteq \sigma$, the **fuzzy sub graph of** $(\sigma, \mu)$ **induced by** $\nu$ is the maximal fuzzy sub graph of $(\sigma, \mu)$, that has fuzzy vertex set $\nu$ and it is the fuzzy sub graph $(\nu, \tau)$ where

$$\tau(u, v) = \nu(u) \land \nu(v) \land \mu(u, v)$$

for all $u, v$ in $S$. 

4
Example

**Fig 1.1 G: \((\sigma, \mu)\)**

G: \((\sigma, \mu)\) is a fuzzy graph with the underlying set \(S = \{a, b, c, d, e\}\) where \(\sigma: S \rightarrow [0,1], \mu: S \times S \rightarrow [0,1]\), are defined as \(\sigma(a) = .5, \sigma(b) = .6, \sigma(c) = 1, \sigma(d) = .8, \sigma(e) = .4; \mu(a, b) = .2, \mu(b, c) = .6, \mu(b, d) = .4, \mu(d, e) = .1, \mu(a, e) = .3\).

**Fig 1.2 H: \((\sigma', \mu')\)**

H: \((\sigma', \mu')\) is a fuzzy sub graph of the fuzzy graph G: \((\sigma, \mu)\).

Throughout the thesis the underlying set of the fuzzy graph is taken to be a finite non-empty set, and the underlying graph to be simple.

**Definition 1.3.4 [34]**

Given a fuzzy graph G: \((\sigma, \mu)\), with the underlying set S, the order of G is defined and denoted as \(p = \sum_{x \in S} \sigma(x)\) and size of G is defined and denoted as \(q = \sum_{x, y \in S} \mu(x, y)\).
Definition 1.3.5 [31]

Let $G : (\sigma, \mu)$ be a fuzzy graph. The degree of a vertex ‘$u’$ is defined as
\[ d(u) = \sum_{v \in S} \mu(u, v). \]
It is also denoted as $d_G(u)$. A fuzzy graph is said to be regular if every vertex is of same degree.

Definition 1.3.6[28]

An edge $(x, y)$ in $\mu^*$ is an effective edge if $\mu(x, y) = \sigma(x) \land \sigma(y)$. A fuzzy graph $G$ is said to be a strong fuzzy graph if $\mu(x, y) = \sigma(x) \land \sigma(y)$ for all $(x, y)$ in $\mu^*$.

Definition 1.3.7[8]

A fuzzy graph $G$ is said to be a complete fuzzy graph if $\mu(x, y)$ is $\sigma(x) \land \sigma(y)$ for all $x, y$ in $\sigma^*$.

Definition 1.3.8[56]

If $\mu(x, y) > 0$ then $x$ and $y$ are called neighbours, $x$ and $y$ are said to lie on the edge $e = (x, y)$. A path $p$ in a fuzzy graph $G : (\sigma, \mu)$ is a sequence of distinct nodes $v_0, v_1, v_2, \ldots, v_n$ such that $\mu(v_{i-1}, v_i) > 0$, $1 \leq i \leq n$. Here ‘$n$’ is called the length of the path. The consecutive pairs $(v_{i-1}, v_i)$ are called arcs of the path.
**Definition 1.3.9**[56]

If \( u, v \) are nodes in \( G \) and if they are connected by means of a path then the strength of that path is defined as \( \wedge_{i=1}^{n} \mu(v_{i-1}, v_{i}) \) i.e., it is the strength of the weakest arc. If \( u, v \) are connected by means of paths of length ‘k’ then \( \mu^{k}(u, v) \) is defined as \( \mu^{k}(u, v) = \sup\{ \mu(u, v_{1}) \wedge \mu(v_{1}, v_{2}) \wedge \mu(v_{2}, v_{3}) \ldots \wedge \mu(v_{k-1}, v) / u, v_{1}, v_{2}, \ldots, v_{k-1}, v \in S \} \). If \( u, v \in S \) the strength of connectedness between \( u \) and \( v \) is, \( \mu^{\infty}(u, v) = \sup\{ \mu^{k}(u, v) / k = 1, 2, 3, \ldots \} \).

**Definition 1.3.10** [56]

A fuzzy graph \( G \) is connected if \( \mu^{\infty}(u, v) > 0 \) for all \( u, v \in \sigma^{*} \). An arc \((x, y)\) is said to be a strong arc if \( \mu(x, y) \geq \mu^{\infty}(x, y) \). A node \( x \), is said to be an isolated node if \( \mu(x, y) = 0 \ \forall \ y \neq x \).

**Definition 1.3.11**[31]

\( G: (\sigma, \mu) \) is a fuzzy cycle iff \( (\sigma^{*}, \mu^{*}) \) is a cycle and there does not exist a unique \((x, y)\) \( \in \mu^{*} \) such that \( \mu(x, y) = \wedge \{ \mu(u, v) / (u, v) \in \mu^{*} \} \).

**Definition 1.3.12**[62]

Let \( G: (\sigma, \mu) \) be a fuzzy graph. The complement of \( G \) is defined as \( G: (\sigma, \overline{\mu}) \) where \( \overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \ \forall \ x, y \in S \).
Definition 1.3.13 [40]

The $\mu$-complement of $G$ is denoted as $G^\mu : (\sigma^\mu, \mu^\mu)$ where $\sigma^\mu = \sigma$ and $\mu^\mu(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^\mu(u, v) = \sigma(u) \land \sigma(v) - \mu(u, v)$ if $\mu(u, v) > 0$.

Definition 1.3.14 [40]

The busy value of a node ‘$v$’ in $G$ is $D(v) = \sum_i \sigma(v) \land \sigma(v_i)$ where $v_i$ are the neighbours of $v$ and the busy value of $G$ is $D(G) = \sum_i D(v_i)$ where $v_i$ are the nodes of $G$.

Definition 1.3.15 [40]

A node in $G$ is a busy node if $\sigma(v) \leq d(v)$, otherwise it is called a free node.

Definition 1.3.16 [40]

A node $v$ of a fuzzy graph $G$ is said to be a

i. partial free node if it is a free node in both $G$ and $G^\mu$.

ii. fully free node if it is free in $G$ but busy in $G^\mu$.

iii. partial busy node if it is a busy node in both $G$ and $G^\mu$.

iv. fully busy node if it is busy in $G$ but free in $G^\mu$.

Definition 1.3.17[31]

Let $G : (\sigma, \mu)$ be a fuzzy graph with the underlying graph $(V, E)$. The fuzzy line graph of $G$ is $L(G) : (\lambda, \omega)$ with the underlying graph $(Z, W)$ where
\( Z = \{S_x = \{x\} \cup \{u_x, v_x\} \mid x \in E, u_x, v_x \in V, x = (u_x, v_x)\} \) with \( \lambda(S_x) = \mu(x), \forall S_x \in Z \) and \( W = \{(S_x, S_y) \mid S_x \cap S_y \neq \emptyset, x, y \in E, x \neq y\} \) with \( \omega(S_x, S_y) = \lambda(S_x) \land \lambda(S_y) = \mu(x) \land \mu(y), \forall (S_x, S_y) \in W. \)

**Definition 1.3.18 [56]**

The \( \mu \)-distance \( \delta(u, v) \) is the smallest \( \mu \)-length of any \( u-v \) path, where the \( \mu \)-length of a path \( \rho : v_0, v_1, v_2, \ldots, v_n \) is \( l(\rho) = \sum_{i=1}^{n} \frac{1}{\mu(u_{i-1}, u_i)} \).

**Definition 1.3.19 [7]**

The eccentricity of a node \( v \) is defined as \( e(v) = \max_u (\delta(u,v)) \). The diameter \( \text{diam}(G) = \vee \{e(v) \mid v \in S\} \), radius \( r(G) = \wedge \{e(v) \mid v \in S\} \). A node whose eccentricity is minimum in a connected fuzzy graph is called a central node. A connected fuzzy graph is called self-centered if each node is a central node.

**Definition 1.3.20 [30]**

Let \( G_i : (\sigma_i, \mu_i) \) be fuzzy graphs, where \( \sigma_i \) is a fuzzy subset of a non-empty set \( V_i \) and \( \mu_i \) is a fuzzy relation on \( \sigma_i \) for \( i = 1, 2 \). Let their underlying crisp graphs be \( G_i^c : (V_i, E_i) \). The union \( G_1 \cup G_2 : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2) \) is a fuzzy graph defined on the underlying set \( V_1 \cup V_2 \), as

\[
(\sigma_1 \cup \sigma_2)(u) = \sigma_1(u) \text{ if } u \in V_1 - V_2
\]

\[
= \sigma_2(u) \text{ if } u \in V_2 - V_1
\]

\[
= \sigma_1(u) \lor \sigma_2(u) \text{ if } u \in V_1 \cap V_2.
\]
\[(\mu_1 \cup \mu_2)(u, v) = \mu_1(u, v) \text{ if } (u, v) \in E_1\]
\[= \mu_2(u, v) \text{ if } (u, v) \in E_2\]
\[= \mu_1(u, v) \lor \mu_2(u, v) \text{ if } (u, v) \in E_1 \cap E_2\]

**Definition 1.3.21** [30]

The sets \(\sigma^t = \{u \in S / \sigma(u) > t\}\) for \(0 \leq t \leq 1\) are called level sets or t-cuts of \(\sigma\). Let \(\sigma\) be a fuzzy subset of \(S\). The lower truncation of \(\sigma\) at level \(t\), \(0 < t \leq 1\), is the fuzzy subset \(\sigma_{(0)}\), defined as

\[\sigma_{(0)}(u) = \sigma(u) \text{ if } u \in \sigma^t\]
\[= 0 \text{ if } u \notin \sigma^t.\]

The upper truncation of \(\sigma\) at level \(t\), \(0 < t \leq 1\), is the fuzzy subset \(\sigma^{(0)}\), defined as

\[\sigma^{(0)}(u) = t \text{ if } u \in \sigma^t\]
\[= \sigma(u) \text{ if } u \notin \sigma^t.\]

**Definition 1.3.22** [46]

Taking \(V_{(0)} = \sigma^t, E_{(0)} = \mu^t, G_{(0)} : (\sigma_{(0)}, \mu_{(0)})\) is a fuzzy graph with underlying crisp graph, \(G_{(0)^*} : (V_{(0)}, E_{(0)})\). This is called the lower truncation of the fuzzy graph \(G\) at level \(t\). Taking \(V^{(0)} = V, E^{(0)} = E, G^{(0)} : (\sigma^{(0)}, \mu^{(0)})\) is a fuzzy graph with \(G^{(0)^*} : (V^{(0)}, E^{(0)})\). This is called the upper truncation of the fuzzy graph \(G\) at level \(t\).
**Definition 1.3.23** [38]

Two nodes of a fuzzy graph are said to be fuzzy independent if there is no strong arc between them.

**Definition 1.3.24** [38]

A subset $S'$ of $S$ is said to be fuzzy independent if any two nodes of $S'$ are fuzzy independent.

**Definition 1.3.25** [38]

A fuzzy graph $G$ is said to be fuzzy bipartite if the node set $S$ can be partitioned into two subsets $S_1, S_2$ such that $S_1$ and $S_2$ are fuzzy independent sets. These sets are called fuzzy bipartition of $S$.

**Definition 1.3.26** [52]

A fuzzy matrix is the matrix whose elements are taking their values from $[0, 1]$.

**Definition 1.3.27** [52]

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are fuzzy matrices of order $n \times p$, their addition is defined as $A + B = [a_{ij} \lor b_{ij}]$ where $\lor$ stands for the maximum. Multiplication of fuzzy matrices is done by the maxmin operation. If $A = [a_{ij}]$ is a $n \times p$ fuzzy matrix and $B = [b_{ij}]$ is a $p \times m$ fuzzy matrix then $C = AB$ is the product of the $A$ and $B$ matrices, of order $n \times m$ where $c_{ij} = \lor_{k=1}^{p} \land_{k=1}^{p} (a_{ik}, b_{kj})$, which is also a fuzzy matrix.
Definition 1.3.28 [4]

A permutation matrix is a square binary matrix that has exactly one ‘1’ in each row and column.

Definition 1.3.29 [4]

Two matrices A and B are said to be isomorphic if there is permutation matrix P such that BP = PA = B = PAP⁻¹.

Definition 1.3.30 [52]

The determinant of an n × n fuzzy matrix A, is

\[ |A| = \sum_{\sigma \in S_n} a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)} \]

where \( S_n \) denotes the symmetric group of all permutations of the indices (1, 2, ..., n).

1.4. Scope and Organization of the Thesis

In this thesis we introduce few new concepts like, subdivision fuzzy graph, square fuzzy graph, total fuzzy graph, middle fuzzy graph and discussed few properties of them. Also some relation between them is established using isomorphism on fuzzy graphs.

Chapter I presents the basic definitions required for the development of the thesis.

Chapter II deals with order, size and degree of the nodes of the isomorphic fuzzy graphs. Nature of the isomorphism relation and weak isomorphism relation between fuzzy graphs are discussed. Self weak
complementary fuzzy graph is introduced and some properties of self complementary and self weak complementary fuzzy graphs are discussed.

**Chapter III** discusses some properties of isomorphic fuzzy graphs with reference to strong arcs in fuzzy graphs, strong fuzzy graphs and also about complement of a fuzzy graph. In crisp case the bipartite nature is preserved under isomorphism. As an analogous to this, a result is proved for fuzzy case also. The association of $\mu$-complement with morphism properties is studied in this chapter.

**Chapter IV** defines and discusses subdivision fuzzy graph, square fuzzy graph, and total fuzzy graph of a given fuzzy graph. In this chapter middle fuzzy graph of a fuzzy graph is defined and some properties of it are also discussed.

**Chapter V** defines and deals with the antipodal fuzzy graph and eccentric fuzzy graph of the given fuzzy graph. Relation between them is also dealt here.

**Chapter VI** is meant for the study of upper and lower truncations on total fuzzy graph, middle fuzzy graph and antipodal fuzzy graph.

**Chapter VII** is designed for the study of neighbourhood fuzzy graph of a fuzzy graph and some of its properties. Also few sections are left for the discussion of nature of the fuzzy matrices on fuzzy graphs. The relation between the incidence matrix of the fuzzy graph and the matrix representation of its fuzzy line graph are also discussed.