A NEW DOUBLE $\chi$ SEQUENCE SPACE DEFINED BY A MODULUS FUNCTION

Abstract

The idea of single difference sequence space was introduced by Kizmaz[33] and this concept was generalized by various authors. In this chapter we define the double difference space $\chi^2 (\Delta^m, f, p, q, s)$ on a semi normed complex linear space by using modulus function and we give various properties and some inclusion relations on this space. Further more we study some of its properties solidity, etc.

8 Introduction

Let $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m,n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are found in Bromwich[7]. Later on, they were investigated by Hardy[27], Moricz[39], Moricz and Rhoades[40], Basarir and Solankan[4], Tripathy[61], Colak and Turkmenoglu[17], Turkmenoglu[63], and many others.

Let us define the following sets of double sequences:

$M_u (t) := \{ (x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \}$,

$C_p (t) := \{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \}$,

$C_0p (t) := \{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn}|^{t_{mn}} = 1 \}$,

$L_u (t) := \{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \}$,

$C_{bp} (t) := C_p (t) \cap M_u (t)$ and $C_{0bp} (t) = C_0p (t) \cap M_u (t)$;

where $t = (t_{mn})$ is the sequence of strictly positive reals $t_{mn}$ for all $m,n \in \mathbb{N}$ and $p - \lim_{m,n \to \infty}$ denotes the limit in the Pringsheim’s sense. In the case $t_{mn} = 1$ for all $m,n \in \mathbb{N}$; $M_u (t)$, $C_p (t)$, $C_0p (t)$, $L_u (t)$, $C_{bp} (t)$ and $C_{0bp} (t)$ reduce to the sets $M_u$, $C_p$, $C_0p$, $L_u$, $C_{bp}$ and $C_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [24,25] have proved that $M_u (t)$ and $C_p (t)$, $C_{bp} (t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $M_u (t)$ and $C_{bp} (t)$. Quite recently, in
her PhD thesis, Zelter [66] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [42] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences.

Next, Mursaleen [41] and Mursaleen and Edely [43] have defined the almost regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M$–core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M$–core of $x$. More recently, Altay and Basar [1] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha$– duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\theta)$ – duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [5] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space $\ell_q$ of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [54] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

We need the following inequality in the sequel of the chapter. For $a, b \geq 0$ and $0 < p < 1$, we have

$$ (a + b)^p \leq a^p + b^p \quad (110) $$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence $(s_{mn})$ is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \in \mathbb{N}$ (see[3]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double gai sequence if $(m+n)!|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by $\chi^2$. Let $\phi = \{all finite sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \mathbb{I}_{ij}$ for all $m, n \in \mathbb{N}$; where $\mathbb{I}_{ij}$ denotes the double sequence whose only non zero term is $1/(i+j)!$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)$X$ is said to have AK property if $(\mathbb{I}_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \to x$. 


An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings \( x = (x_k) \rightarrow (x_{mn})(m, n \in \mathbb{N}) \) are also continuous.

Orlicz [46] used the idea of Orlicz function to construct the space \( (L^M) \).

Lindenstrauss and Tzafriri [35] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space \( \ell_M \) contains a subspace isomorphic to \( \ell_p (1 \leq p < \infty) \). Subsequently, the different classes of sequence spaces were defined by Parashar and Choudhary [47], Mursaleen et al. [44], Bektas and Altin [6], Tripathy et al. [62], Rao and Subramanian [13], and many others. The Orlicz sequence space is the special case of Orlicz space studied in [34].

Recalling [46] and [34], an Orlicz function is a function \( M : [0, \infty) \rightarrow [0, \infty) \) which is continuous, non-decreasing, and convex with \( M(0) = 0 \), \( M(x) > 0 \), for \( x > 0 \) and \( M(x) \rightarrow \infty \) as \( x \rightarrow \infty \). If the convexity of Orlicz function \( M \) is replaced by subadditivity of \( M \), then this function is called modulus function, defined by Nakano [45] and further discussed by Ruckle [48] and Maddox [37], and many others.

An Orlicz function \( M \) is said to satisfy the \( \Delta_2 \)- condition for all values of \( u \) if there exists a constant \( K > 0 \) such that \( M(2u) \leq KM(u) (u \geq 0) \).

The \( \Delta_2 \)- condition is equivalent to \( M(\ell u) \leq K\ell M(u) \), for all values of \( u \) and for \( \ell > 1 \).

Lindenstrauss and Tzafriri [35] used the idea of Orlicz function to construct Orlicz sequence space

\[
\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},
\]

The space \( \ell_M \) with the norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},
\]

becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p (1 \leq p < \infty) \), the spaces \( \ell_M \) coincide with the classical sequence space \( \ell_p \).

If \( X \) is a sequence space, we give the following definitions:

(i) \( X' = \) the continuous dual of \( X \);

(ii) \( X^\alpha = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \right\} \);
(iii) $X^\beta = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} is\ convergent,\ for\ each\ x \in X \}$;  

(iv) $X^\gamma = \{ a = (a_{mn}) : \sup_{m,n} \sum_{m,n=1}^{M,N} a_{mn} x_{mn} < \infty,\ for\ each\ x \in X \}$;  

(v) Let $X$ be an $FK$–space $\supset \phi$; then $X^f = \{ f(\mathcal{S}_{mn}) : f \in X^f \}$;  

(vi) $X^\delta = \{ a = (a_{mn}) : \sup_{m,n} |a_{mn} x_{mn}|^{1/m+n} < \infty,\ for\ each\ x \in X \}$;  

$X^\alpha, X^\beta, X^\gamma$ are called $\alpha$–($or\ K\"{o}the$–Toeplitz)$dual of $X, \beta$–($or\ generalized$–K\"{o}the$–Toeplitz)$dual of $X, \gamma$ – dual of $X$ respectively. $X^\alpha$ is defined by Gupta and Kamptan [32]. It is clear that $X^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\alpha \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [33] as follows  

$$Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}$$  

for $Z = c, c_0$ and $\ell_\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here $w, c, c_0$ and $\ell_\infty$ denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by  

$$||x|| = |x_1| + \sup_{k \geq 1} |\Delta x_k|$$  

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by  

$$Z(\Delta) = \{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \}$$  

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. Therefore $\Delta^0 x = (x_{mn})$; $\Delta^1 x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1})$. Hence $\Delta^m x_{mn}$  

$$\Delta^{m-1} x_{mn} = (\Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1}) - (\Delta^{m-1} x_{m+1n} - \Delta^{m-1} x_{m+1n+1})$$.

### 8.1 Definitions and Preliminaries

Let $w^2$ denote the spaces of all double sequences. $\chi^2_M$ and $\Lambda^2_M$ denote the Pringscheims sense of double Orlicz space of gai sequences and Pringscheims sense of double Orlicz space of bounded sequences respectively.

The notion of a modulus function was introduced by Nakano [45]. We recall
that a modulus \( f \) is a function from \([0, \infty) \rightarrow [0, \infty)\), such that

1. \( f(x) = 0 \) if and only if \( x = 0 \)
2. \( f(x + y) \leq f(x) + f(y) \), for all \( x \geq 0, y \geq 0 \),
3. \( f \) is increasing,
4. \( f \) is continuous from the right at 0. Since \( |f(x) - f(y)| \leq f(|x - y|) \), it follows from condition (iv) that \( f \) is continuous on \([0, \infty)\).

Let \( p = (p_{mn}) \) be a sequence of strictly positive real numbers and \( s \geq 0 \). Let \( X \) be semi normed space over the field \( \mathbb{C} \) of complex numbers with the semi norm \( q \). The symbol \( w^2(X) \) denotes the space of all sequences defined over \( X \) such that \( p_{mn} > 0 \) for all \( m, n \) and \( \sup_{mn} p_{mn} = H < \infty \).

Define the sets:

\[
\chi_M^2 = \{ x \in w^2 : (M (\frac{((m+n)!|x_{mn}|)^{1/m+n}}{p})) \to 0 \text{ as } m, n \to \infty \text{ for some } \rho > 0 \} \quad \text{and}
\]

\[
\Lambda_M^2 = \{ x \in w^2 : \sup_{mn} p_{mn} \geq 1 \left( M \left( \frac{|x_{mn}|^{1/m+n}}{p} \right) \right) < \infty \text{ for some } \rho > 0 \}.
\]

The space \( \Lambda_M^2 \) is a metric space with the metric

\[
d(x, y) = \inf \left\{ \rho > 0 : \sup_{mn} p_{mn} \geq 1 \left( M \left( \frac{|x_{mn}-y_{mn}|}{\rho} \right) \right)^{1/m+n} \leq 1 \right\}.
\]

The space \( \chi_M^2 \) is a metric space with the metric

\[
\tilde{d}(x, y) = \inf \left\{ \rho > 0 : \sup_{mn} p_{mn} \geq 1 \left( M \left( \frac{(m+n)!|x_{mn}-y_{mn}|}{p} \right) \right)^{1/m+n} \leq 1 \right\}.
\]

We define the following sequence spaces as follows:

\[
\chi^2(\Delta^m, f, p, q, s) = \{ x \in w^2(X) : (mn)^{-s} \left( f \left( q \left( (m+n)! \Delta^m x_{mn} \right)^{1/m+n} \right) \right)^{p_{mn}} \to 0 \text{ as } (m, n) \to \infty, s \geq 0 \},
\]

\[
\Lambda^2(\Delta^m, f, p, q, s) = \{ x \in w^2(X) : \sup_{mn} (mn)^{-s} \left( f \left( (\Delta^m x_{mn})^{1/m+n} \right) \right)^{p_{mn}} < \infty, s \geq 0 \},
\]

where \( f \) is a modulus function. The following inequality will be used throughout this chapter. Let \( p = (p_{mn}) \) be a sequence of positive real numbers with \( 0 < p_{mn} \leq \sup_{mn} p_{mn} = H, D = \max \left( 1, 2^H \right) \). Then, for \( a_{mn}, b_{mn} \in \mathbb{C} \), we have

\[
|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}} \leq D \left\{ |a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}} \right\} \quad (111)
\]

### 8.2 Definitions

#### 8.3 Definition

Let \( p, q \) be semi norms on a vector space \( X \). \( p \) is said to be stronger than \( q \) if there exists a sequence \( (x_{mn}) \) such that \( p(x_{mn}) \to 0 \), then \( q(x_{mn}) \to 0 \). If
each is stronger than the other, then \( p \) and \( q \) are said to be equivalent.

### 8.4 Lemma

Let \( p \) and \( q \) be semi norms on a linear space \( X \). Then \( p \) is stronger than \( q \) if and only if there exists a constant \( M \) such that \( q(x) \leq Mp(x) \) for all \( x \in X \).

### 8.5 Definition

1. A sequence space \( X \) is said to be solid or normal if \((\alpha_{mn}x_{mn}) \in X\) whenever \((x_{mn}) \in X\) and for all sequences of scalars \((\alpha_{mn})\) with \(|\alpha_{mn}| \leq 1\), for all \( m,n \in \mathbb{N} \).
2. **Symmetric**: If \((x_{mn}) \in X\) implies \((x_\pi(mn)) \in X\), where \( \pi(mn) \) is a permutation of \( \mathbb{N} \times \mathbb{N} \);
3. **Sequence algebra**: If \( x \cdot y \in X\) whenever \( x,y \in X \)

### 8.6 Definition

A sequence space \( X \) is said to be monotone if it contains the canonical pre-images of all its step spaces.

### 8.7 Remark

From the two above definitions it is clear that a sequence space \( X \) is solid implies that \( X \) is monotone.

### 8.8 Definition

A sequence \( X \) is said to be convergence free if \((y_{mn}) \in X\) whenever \((x_{mn}) \in X\) and \( x_{mn} = 0 \) implies that \( y_{mn} = 0 \).

### 8.9 Main Results

In this section we will give some results on the sequence space \( \chi^2(\Delta^m, f, p, q, s) \) using which we characterize the structure of the space \( \chi^2(\Delta^m, f, p, q, s) \).

### 8.10 Theorem

The sequence space \( \chi^2(\Delta^m, f, p, q, s) \) is a linear space over \( \mathbb{C} \)

**Proof:** Let \( x, y \in \chi^2(\Delta^m, f, p, q, s) \). For \( \lambda, \mu \in \mathbb{C} \), there exist positive integers \( M_\lambda \) and \( N_\mu \), such that \(|\lambda| \leq M_\lambda\) and \(|\mu| \leq N_\mu\). Since \( f \) is subadditive, \( q \) is a seminorm, and \( \Delta^m \) is linear.
\[(mn)^{-s} \left[ f \left( q \left( (m + n)! \left| \Delta^m (\lambda x_{mn} + \mu y_{mn}) \right| \right)^{\frac{1}{m+n}} \right) \right]^{p_{mn}} \leq D \left( \max \left( 1, |M_\lambda|^H \right) \right) (mn)^{-s} \left[ f \left( q \left( (m + n)! \left| \Delta^m x_{mn} \right| \right)^{\frac{1}{m+n}} \right) \right]^{p_{mn}} + D \left( \max \left( 1, |N_\mu|^H \right) \right) (mn)^{-s} \left[ f \left( q \left( (m + n)! \left| \Delta^m y_{mn} \right| \right)^{\frac{1}{m+n}} \right) \right]^{p_{mn}} \rightarrow 0 \text{ as } m, n \rightarrow \infty. \]

Hence, \( \chi^2 (\Delta^m, f, p, q, s) \) is a linear space.

### 8.11 Theorem

The space \( \chi^2 (\Delta^m, f, p, q, s) \) is a paranormed space (not totally paranormed) paranormed by

\[ g_\Delta (x) = \left\{ (mn)^{-s} \left[ f \left( q \left( (m + n)! \left| \Delta^m x_{mn} \right| \right)^{\frac{1}{m+n}} \right) \right]^{p_{mn}} \rightarrow 0 \text{ as } m, n \rightarrow \infty \right\}^{1/M} \]

where \( H = \text{supp}_{mn} < \infty \) and \( M = \text{Max} (1, H) \)

**Proof:** Clearly \( g_\Delta (0) = 0 \) and \( g_\Delta (x) = g_\Delta (-x) \),

where \( \overline{\theta} = \left( \theta, \theta, \ldots, \theta, \theta, \ldots \right) \) and \( \theta \) is the zero element of \( X \). It follows from equation (111), Minkowski’s inequality and the definition of \( f \) that \( g_\Delta \) is sub additive. Now for a complex number \( \lambda \), by inequality

\[ |\lambda|^{p_{mn}} \leq \max \left( 1, |\lambda|^H \right) \]

and the definition of modulus \( f \), we have

\[ g_\Delta (x) = \left\{ (mn)^{-s} \left[ f \left( q \left( (m + n)! \left| \Delta^m x_{mn} \right| \right)^{\frac{1}{m+n}} \right) \right]^{p_{mn}} \rightarrow 0 \text{ as } m, n \rightarrow \infty \right\}^{1/M} \]

\[ \leq (1 + |\lambda|)^{\frac{1}{M}} \cdot g_\Delta (x). \]

where \( |\lambda| \) denotes the integer part of \( \lambda \), hence \( \lambda \rightarrow 0 \), \( x \rightarrow \theta \) imply \( \lambda x \rightarrow \theta \) and also \( x \rightarrow \theta \), \( \lambda \) fixed imply \( \lambda x \rightarrow \theta \).

Now suppose \( \lambda_n \rightarrow 0 \) and \( x \) is a fixed point in \( \chi^2 (\Delta^m, f, p, q, s) \). Given \( \epsilon > 0 \), let \( m = M + 1 \) and \( n = N + 1 \) be such that

\[ \left\{ (mn)^{-s} \left[ f \left( q \left( (m + n)! \left| \Delta^m x_{mn} \right| \right)^{\frac{1}{m+n}} \right) \right]^{p_{mn}} \rightarrow 0 \text{ as } m = M + 1 \text{ and } n = N + 1 \rightarrow \infty \right\} < \left( \frac{\epsilon}{2} \right)^M. \]

Hence we have

\[ \left\{ (mn)^{-s} \left[ f \left( q \left( (m + n)! \left| \Delta^m x_{mn} \right| \right)^{\frac{1}{m+n}} \right) \right]^{p_{mn}} \rightarrow 0 \text{ as } m = M + 1 \text{ and } n = N + 1 \rightarrow \infty \right\} < \left( \frac{\epsilon}{2} \right). \]

Since \( f \) is continuous on \([0, \infty] \).

\[ h(t) = \left\{ (mn)^{-s} \left[ f \left( q \left( (m + n)! \left| \Delta^m t_{mn} \right| \right)^{\frac{1}{m+n}} \right) \right]^{p_{mn}} \rightarrow 0 \text{ as } m = M \text{ and } n = N \rightarrow \infty \right\} \]

is continuous at zero. Therefore, there exists \( 0 < \delta \rightarrow 1 \) such that \( |\lambda_n| < \delta \) implies
\begin{align*}
\{ (mn)^{-s} \left[ f \left( q \left( \lambda_n \left( (m + n)! |\Delta^m x_{mn} | \right)^{1/m+n} \right) \right] \}_n^{p_m} \rightarrow 0 \text{ as } m = M \text{ and } n = N \rightarrow \infty \} < \\
\bigg\{ (mn)^{-s} \left[ f \left( q \left( \lambda_n \left( (m + n)! |\Delta^m x_{mn} | \right)^{1/m+n} \right) \right] \bigg\}^{p_m} \rightarrow 0 \text{ as } m, n \rightarrow \infty \bigg\} \frac{1}{\epsilon} < \\
\epsilon, \text{ for } n > N. \text{ Therefore } g(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0. \text{ This completes the proof.}
\end{align*}

8.12 Theorem

Let \( f, f_1 \) and \( f_2 \) be modulus functions, \( q, q_1 \) and \( q_2 \) be seminorms, and \( s, s_1 \) and \( s_2 \geq 0 \) real numbers.

1. \( \chi^2 (\Delta^m, f_1, p, q, s) \subseteq \chi^2 (\Delta^m, f \circ f_1, p, q, s) \),
2. \( \chi^2 (\Delta^m, f_2, p, q, s) \subseteq \chi^2 (\Delta^m, f, f_2, p, q, s) \),
3. \( \chi^2 (\Delta^m, f, p, q_1, s) \subseteq \chi^2 (\Delta^m, f, p, q_2, s) \),
4. If \( q_1 \) is stronger than \( q_2 \), then \( \chi^2 (\Delta^m, f, p, q_1, s) \subseteq \chi^2 (\Delta^m, f, p, q_2, s) \),
5. If \( s_1 \leq s_2 \), then \( \chi^2 (\Delta^m, f, p, q, s_1) \subseteq \chi^2 (\Delta^m, f, p, q, s_2) \).

**Proof(1):** Let \( (x_{mn}) \in \chi^2 (\Delta^m, f_1, p, q, s) \). Let \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( f(t) < \epsilon \) for \( 0 \leq t \leq \delta \). Write

\[
t_{mn} = \left\{ \left[ f_1 \left( q \left( \left( (m + n)! |\Delta^m x_{mn} | \right)^{1/m+n} \right) \right] \right) \right\}^{p_m} \rightarrow 0 \text{ as } m, n \rightarrow \infty \}
\]

and consider \( (mn)^{-s} \left[ f (t_{mn}) \right]^{p_m} \rightarrow 0 \text{ as } m, n \rightarrow \infty \) is stronger than \( (mn)^{-s} \left[ f (t_{mn}) \right]^{p_m} \rightarrow 0 \text{ as } m, n \rightarrow \infty \) + \( (mn)^{-s} \left[ f (t_{mn}) \right]^{p_m} \rightarrow 0 \text{ as } m, n \rightarrow \infty \), where the first term is over \( t_{mn} \leq \delta \) and the second term is over \( t_{mn} > \delta \). Since \( f \) is continuous, we have

\[
(mn)^{-s} \left[ f (t_{mn}) \right]^{p_m} \rightarrow 0 \text{ as } m, n \rightarrow \infty < \max \left( (1, \epsilon) \right) \left( (mn)^{-s} \rightarrow 0 \text{ as } m, n \rightarrow \infty \right)
\]

and for \( t_{mn} > \delta \) we use the fact that

\[
t_{mn} < \frac{t_{mn}}{\delta} < 1 + \left[ \left( \frac{t_{mn}}{\delta} \right) \right]
\]

By the definition of \( f \) we have for \( t_{mn} > \delta \)

\[
f (t_{mn}) < f (1) \left[ 1 + \left( \frac{t_{mn}}{\delta} \right) \right] \leq 2 f (1) \frac{t_{mn}}{\delta}
\]

\[
(mn)^{-s} \left[ f (t_{mn}) \right]^{p_m} \rightarrow 0 \text{ as } m, n \rightarrow \infty \leq \max \left( (1, \left( \frac{2 f (1)}{\delta} \right) \right) \left( (mn)^{-s} \left[ f (t_{mn}) \right]^{p_m} \right) \rightarrow 0 \text{ as } m, n \rightarrow \infty \)
\]

By (112) and last equation we have \( \chi^2 (\Delta^m, f_1, p, q, s) \subseteq \chi^2 (\Delta^m, f \circ f_1, p, q, s) \),

**Proof(2):** Let \( x = (x_{mn}) \in \chi^2 (\Delta^m, f_1, p, q, s) \cap \chi^2 (\Delta^m, f_2, p, q, s) \) then using (111) it can be shown that \( (x_{mn}) \in \chi^2 (\Delta^m, f_1 + f_2, p, q, s) \). Hence \( \chi^2 (\Delta^m, f_1, p, q, s) \cap \chi^2 (\Delta^m, f_2, p, q, s) \subseteq \chi^2 (\Delta^m, f_1 + f_2, p, q, s) \),

**Proof(3):** The proof of (3) is similar to the proof of (2), by using the inequality \( (mn)^{-s} \left[ f (q_1 + q_2) \left( (m + n)! |\Delta^m x_{mn} | \right)^{1/m+n} \right]^{p_m} \) \( \leq \)

\[
C \left( (mn)^{-s} \left[ f (q_1) \left( (m + n)! |\Delta^m x_{mn} | \right)^{1/m+n} \right]^{p_m} \right) + \]
\[
C \left( (mn)^{-s} \left[ f \left( q (m+n) \right) \right] \xi_m x_{mn} \right)^{1/m+n} \right)^{p_m n} , \text{ where } C = \max \left( 1, 2^{H-1} \right).
\]

Proof (4) and (5) follows easily. We get the following sequence spaces from \( \chi^2 (\Delta^m, f, p, q, s) \) by choosing some of the special \( p, f \) and \( s \).

For \( f(x) = (x_{mn}) \) we get

\[
\chi^2 (\Delta^m, f, p, q, s) = \left\{ x \in w^2 (X) : (mn)^{-s} \left( (q ((m+n)! \Delta^m x_{mn})^{1/m+n} \right) \right\}^{p_m n} \to 0 (m, n \to \infty) , s \geq 0 \};
\]

for \( p_m n = 1, \) for all \( m, n \) we get \( \chi^2 (\Delta^m, f, q, s) = \left\{ x \in w^2 (X) : (mn)^{-s} \left( f \left( q ((m+n)! \Delta^m x_{mn})^{1/m+n} \right) \right) \right\}^{p_m n} \to 0 (m, n \to \infty) , s \geq 0 \};

for \( s = 0 \) we get

\[
\chi^2 (\Delta^m, f, p, q) = \left\{ x \in w^2 (X) : \left( f \left( q ((m+n)! \Delta^m x_{mn})^{1/m+n} \right) \right) \right\}^{p_m n} \to 0 (m, n \to \infty) \};
\]

for \( f(x) = (x_{mn}) \) and \( s = 0 \) we get

\[
\chi^2 (\Delta^m, p, q) = \left\{ x \in w^2 (X) : \left( q ((m+n)! \Delta^m x_{mn})^{1/m+n} \right) \right\}^{p_m n} \to 0 (m, n \to \infty) \};
\]

for \( p_m n = 1, \) for all \( m, n \) and \( s = 0 \) we get

\[
\chi^2 (\Delta^m, f, q) = \left\{ x \in w^2 (X) : \left( f \left( q ((m+n)! \Delta^m x_{mn})^{1/m+n} \right) \right) \right\}^{p_m n} \to 0 (m, n \to \infty) \};
\]

for \( f(x) = x_{mn}, p_m n = 1, \) for all \( m, n \) and \( s = 0 \) we get

\[
\chi^2 (\Delta^m, q) = \left\{ x \in w^2 (X) : \left( q ((m+n)! \Delta^m x_{mn})^{1/m+n} \right) \right\}^{p_m n} \to 0 (m, n \to \infty) \};
\]

8.13 Corollary

(1) If \( s > 1 \) then for any modulus \( f \) we have \( \chi^2 (\Delta^m, p, q, s) \subseteq \chi^2 (\Delta^m, f, p, q, s) \);

(2) If \( q_1 \) and \( q_2 \) are equivalent semi norms then \( \chi^2 (\Delta^m, f, p, q_1, s) \subseteq \chi^2 (\Delta^m, f, p, q_2, s) \);

(3) \( \chi^2 (\Delta^m, f, p, q) \subseteq \chi^2 (\Delta^m, f, p, q, s) \);

(4) \( \chi^2 (\Delta^m, p, q) \subseteq \chi^2 (\Delta^m, p, q, s) \);

(5) \( \chi^2 (\Delta^m, f, q) \subseteq \chi^2 (\Delta^m, f, q, s) \)

Proof(1):If \( f_1 (t) = t \) in Theorem 8.12(1), then the result follows easily.

Proof(2): It follows from Theorem 8.12(4).

Proof(3): If we take \( s_1 = 0 \) and \( s_2 = s \) in the Theorem 8.12(5), then we get
\( \chi^2(\Delta^m, f, p, q) \subseteq \chi^2(\Delta^m, f, p, q, s) \)

**Proof (4):** If we take \( s_1 = 0, s_2 = s \) and \( f(t) = t \) in Theorem 8.12(5), then we get \( \chi^2(\Delta^m, p, q) \subseteq \chi^2(\Delta^m, p, q, s) \)

**Proof (5):** If we take \( s_1 = 0, s_2 = s \), and \( p_{mn} = 1 \) for all \( m, n \), in Theorem 8.12(5), then \( \chi^2(\Delta^m, f, q) \subseteq \chi^2(\Delta^m, f, q, s) \)

### 8.14 Theorem

\( \chi^2(\Delta^{m-1}, f, p, q) \subseteq \chi^2(\Delta^m, f, q, s) \) for \( m \geq 1 \) and the inclusion is strict

**Proof:** Let \( x \in \chi^2(\Delta^{m-1}, f, q, s) \). Then we have

\[
(mn)^{-s} \left( f \left( q \left( (m + n)! \left| \Delta^{m-1} x_{mn} \right| \right)^{1/m+n} \right) \right) \rightarrow 0 \quad (m, n \rightarrow \infty) \tag{113}
\]

Since \( (m + 1, n + 1) - s \leq (mn)^{-s} \leq 2^{s} (m + 1, n + 1)^{-s} \), for all \( m, n \in \mathbb{N} \) we get the following inequality

\[
(mn)^{-s} \left( f \left( q \left( (m + n)! \left| \Delta^{m-1} x_{m+1,n+1} \right| \right)^{1/m+n} \right) \right) \leq 2^{s} (m + 1, n + 1)^{-s} \left( f \left( q \left( (m + n)! \left| \Delta^{m-1} x_{m+1,n+1} \right| \right)^{1/m+n} \right) \right)
\]

From (113) and last equation together imply that

\[
(mn)^{-s} \left( f \left( q \left( (m + n)! \left| \Delta^{m-1} x_{m+1,n+1} \right| \right)^{1/m+n} \right) \right) \rightarrow 0 \quad (m, n \rightarrow \infty) \tag{114}
\]

Since \( f \) is increasing, \( f(x + y) \leq f(x) + f(y) \) and \( q \) is a semi norm, from (113) and (114) we get

\[
(mn)^{-s} \left( f \left( q \left( (m + n)! \left| \Delta^{m-1} x_{mn} \right| \right)^{1/m+n} \right) \right) =
\]

\[
\leq (mn)^{-s} \left( f \left( q \left( (m + n)! \left| \Delta^{m-1} x_{mn} \right| \right)^{1/m+n} \right) \right) + (mn)^{-s} \left( f \left( q \left( (m + n)! \left| \Delta^{m-1} x_{m+1,n+1} \right| \right)^{1/m+n} \right) \right) \]

Thus \( \chi^2(\Delta^{m-1}, f, q, s) \subseteq \chi^2(\Delta^m, f, q, s) \). This completes the proof.

### 8.15 Theorem

(1) Let \( 0 \leq t_{mn} \leq r_{mn} < \infty \) for each \( m, n \in \mathbb{N} \). Then \( \chi^2(\Delta^m, f, t, q) \subseteq \chi^2(\Delta^m, f, t, q, s) \)

(2) \( \chi^2(\Delta^m, f, q) \subseteq \chi^2(\Delta^m, f, q, s) \)

(3) \( \chi^2(\Delta^m, f, t, p) \subseteq \chi^2(\Delta^m, f, t, q, s) \)

**Proof (1):** If \( x \in \chi^2(\Delta^m, f, t, q) \) then for all sufficiently large \( m, n \),

\[
\left( f \left( q \left( (m + n)! \left| \Delta^{m} x_{mn} \right| \right)^{1/m+n} \right) \right)^{t_{mn}} \leq 1
\]

and so

\[
\left( f \left( q \left( (m + n)! \left| \Delta^{m} x_{mn} \right| \right)^{1/m+n} \right) \right)^{r_{mn}} \leq \left( f \left( q \left( (m + n)! \left| \Delta^{m} x_{mn} \right| \right)^{1/m+n} \right) \right)^{t_{mn}}
\]

This completes the proof.
8.16 Theorem

(1) If $0 < p_{mn} \leq 1$ for each $m, n \in \mathbb{N}$, then $\chi^2(\Delta^m, f, p, q) \subseteq \chi^2(\Delta^m, f, q)$;
(2) If $p_{mn} \geq 1$ for all $m, n \in \mathbb{N}$, then $\chi^2(\Delta^m, f, q) \subseteq \chi^2(\Delta^m, f, p, q)$

Proof(1): If we take $p_{mn} = t_{mn}$ and $r_{mn} = 1$ for all $m, n \in \mathbb{N}$, in Theorem 8.15 (1), then

$$\chi^2(\Delta^m, f, p, q) \subseteq \chi^2(\Delta^m, f, p, q)$$

Proof(2): If we take $p_{mn} = r_{mn}$ and $t_{mn} = 1$ for all $m, n \in \mathbb{N}$, in Theorem 8.15 (2), then

$$\chi^2(\Delta^m, f, q) \subseteq \chi^2(\Delta^m, f, p, q);$$

This completes the proof

8.17 Theorem

$\chi^2(\Delta^m, f, p, q, s)$ is not solid for $m > 0$

Proof: To prove the space is not solid in general, consider the following example.

Example: Let $X = \mathbb{C}$, $f(x) = x, q(x) = |x|, \alpha_{mn} = (-1)^{m+n}, s = 0, p_{mn} = 1$ for all $m, n \in \mathbb{N}$. Then

$$((m+n)!|x_{mn}|)^{\frac{1}{m+n}} = (mn) \in \chi^2(\Delta^m, f, p, q, s),$$

but $(\alpha_{mn} x_{mn}) \notin \chi^2(\Delta^m, f, p, q, s)$.

8.18 Theorem

$\chi^2(\Delta^m, f, p, q, s)$ is not sequence algebra

Proof: Let $q(x) = |x|, f(x) = x, s = 0, p_{mn} = 1$ for all $m, n \in \mathbb{N}.$

Consider $((m+n)!|x_{mn}|)^{\frac{1}{m+n}} = (mn)^{m-1}$ and $((m+n)!|y_{mn}|)^{\frac{1}{m+n}} = (mn)^{m-1}$,
then $x, y \in \chi^2(\Delta^m, f, p, q, s)$ and $x \cdot y \notin \chi^2(\Delta^m, f, p, q, s)$.